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ON THE REPRESENTATION OF MODULES BY SHEAVES OF FACTOR MODULES

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In memory of Jean-Marie Maranda

Throughout this paper we consider associative rings with unity elements. In §1 various results on the representation of rings by rings of sections of special rings are compared. In particular, it is shown that results enunciated by Dauns and Hofmann, Koh, and the present author may all be deduced from one statement, the proof of which appears in §3.

In §2 some of the usual commutative ideal theory is extended to "symmetric" ideals in noncommutative rings. These are ideals A for which $rst \in A$ implies $rts \in A$.

In §3 we study modules M_R with the property that $m^{-1}0=\{r \in R \mid mr=0\}$ is a symmetric ideal for each $m \in M$. We show that such a module may be represented by the module of sections of a sheaf whose stalks are factor modules of M_R. The argument is adapted from [6, Appendix 1], which was itself an adaptation from Grothendieck [4, Theorem 1.3.7].

1. **Variations on a theme.** A classical result by G. Birkhoff [1] asserts the following.

(A) Every ring R is a subdirect product of subdirectly irreducible rings.

Actually he obtained such a result not just for rings, but for the objects of any algebraic category. Here we shall only be interested in rings and, later, in modules.

To explain Birkhoff's theorem, we recall that a ring is called *subdirectly irreducible* if it has a smallest nonzero ideal. Let $\{0_P \mid P \in \Pi\}$ be the family of all those ideals 0_P for which $R/0_P$ is subdirectly irreducible. (For the sake of comparison with other results, we make use of the index set Π.)

The intersection of this family is zero. In other words, if $\pi_P : R \to R/0_P$ is the canonical surjection, and if $\hat{r}(P)=\pi_P(r)$ for all $r \in R$ and $P \in \Pi$, then the canonical mapping

$$r \to \hat{r} \in \prod_{P \in \Pi} R/0_P$$

is one-to-one. Note that $\hat{r}$ may be viewed as a function from Π to the disjoint union of the sets $R/0_P$.

For commutative rings a much sharper theorem was proved by Grothendieck [4, Theorem 1.3.7].

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(B) Every commutative ring is isomorphic to the ring of sections of a sheaf of local rings.

We shall explain what this means by and by. At the moment let us only point out that the local rings which appear as stalks of this sheaf are not factor rings of the represented ring R, although they are epimorphic images in the category of rings.

In [6, Appendix 1] a simplified version of (B) was presented. For the purpose of stating this in a neat form here, let us call a commutative ring R *prime-torsion free* if it contains a prime ideal P which contains all zero-divisors of R. We note that if P is a minimal prime ideal then R is usually called "primary".

(C) Every commutative ring is isomorphic to the ring of sections of a sheaf of prime-torsion free rings.

We shall now explain what this means with some indication of a proof.

Let Π be the set of all prime ideals of the commutative ring R, and for each $P \in \Pi$ put

$$0_P = \{r \in R \mid \exists_{s \notin P} rs = 0\}.$$

Then again the canonical mapping $r \to \hat{r}$ is one-to-one. In fact, this would be true even if Π was restricted to consist of maximal ideals only. However, we can say much more.

Take the Stone–Zariski topology on Π. It is also possible to topologize the disjoint union

$$\Sigma = \dot{\bigcup_{P \in \Pi}} R/0_P$$

in such a way that the stalks $R/0_P$ are discrete and the functions $\hat{r}: \Pi \to \Sigma$ are continuous. In particular, when $\hat{r}$ vanishes at a point, then it also vanishes on a neighborhood of that point. For reasons that need not concern us here, Σ is called a "sheaf".

A continuous function $f: \Pi \to \Sigma$ such that $f(P) \in R/0_P$ for all $P \in \Pi$ is called a *section*. Not only are the $\hat{r}$ sections, but every section has this form.

It seems desirable to generalize (B) and (C) to noncommutative rings. Similar results for noncommutative rings have been obtained by Dauns and Hofmann, Pierce, Koh, and the present author. We shall present and unify some of these results now.

A ring is called *biregular* if every principal two-sided ideal is generated by a central idempotent. Dauns and Hofmann [2] proved the following.

(D) Every biregular ring is isomorphic to the ring of sections of a sheaf of simple rings.

On the other hand, in [6, Appendix 1] the following result is stated as an exercise.

(E) Every ring is isomorphic to the ring of sections of a sheaf of rings of the form $R/0_P$, where P ranges over the prime ideals of the center C of R and

$$0_P = \{r \in R \mid \exists_{c \in C-P} rc = 0\}.$$

Clearly 0_P is a proper ideal of R. We shall take this opportunity to deduce (D) from (E) by showing that 0_P is a maximal ideal when R is biregular.

Suppose $r \notin 0_P$, then $rc=0 \Rightarrow c \in P$, for all $c \in C$. Now $(r)=(e)$, where e is a central idempotent, hence $r(1-e)=0$, and so $1-e \in P$. Thus $e \notin P$, but $(1-e)e=0$, hence $1-e \in 0_P$. Therefore $1 \in 0_P+(e)=0_P+(r)$, and so 0_P is maximal, as was to be proved.

Actually Dauns and Hofmann did not require their rings to have unity elements. However (E) may easily be adapted to rings without unity by considering only sections with compact support.

Recently Koh [5] obtained the following result.

(F) Every ring R without (nonzero) nilpotent elements is isomorphic to the ring of all sections of the sheaf of rings $R/0_P$, where P ranges over all prime ideals of R, and

$$0_P = \{r \in R \mid r^{-1}0 \nsubseteq P\}.$$

Since the class of rings without nilpotent elements does not include all commutative rings, we propose to consider a generalization. Let us call a ring R *symmetric* if, for all $r, s, t \in R$,

$$rst = 0 \Rightarrow rts = 0.$$

We assert the following.

(G) Every symmetric ring R is isomorphic to the ring of sections of a sheaf of rings of the form $R/0_P$, where P and 0_P are as above.

We shall deduce (F) from (G) by showing that every ring R without nilpotent elements is symmetric.

First, take the special case $r=1$, then from $st=0$ one deduces $(ts)^2=0$, hence $ts=0$.

Next, take any $r \in R$, and suppose that $rst=0$. From the special case we get $(st)r=0$, hence $strt=0$. Again from the special case we obtain $(rt)(st)=0$, and again $t(rts)=0$, so that $trtsr=0$. Once more from the special case we have $(rtsr)t=0$, hence $(rts)^2=0$, hence $rts=0$.

We should point out that the class of symmetric rings includes other rings than commutative rings and rings without nilpotent elements. For example, $R \times S$ is symmetric when R is the ring of integer quaternions and S is the ring of integers modulo 4.

Clearly both (E) and (G) are special cases of the following, which will be established in §3.

(H) Let R be a ring, C a subring of R such that $rcs=0 \Rightarrow rsc=0$ for all $r, s \in R$ and $c \in C$. Then R is isomorphic to the ring of sections of a sheaf of rings $R/0_P$, where P ranges over the prime ideals of C and 0_P is as in (E).

The obvious implications and the implications derived so far are summarized in the following diagram:

$$\begin{tikzcd} & & \text{(H)} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\ & \text{(E)} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{(G)} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \text{(D)} & & \text{(C)} & & \text{(F)} \end{tikzcd}$$

2. **Symmetric ideals.** Let R be any associative ring with 1. We shall call the right ideal A *symmetric* if

$$(*) \qquad rst \in A \Rightarrow rts \in A$$

for all $r, s, t \in R$.

PROPOSITION 1. *Let A be a symmetric right ideal of R. Then the following statements are true*:

(1) *A is an ideal.*

(2) *For each $u \in R$, the right ideal $u^{-1}A = \{v \in R \mid uv \in A\}$ is also symmetric.*

(3) *If $r_1r_2 \ldots r_n \in A$ then, for any permutation p of the set $\{1, 2, \ldots, n\}$, also $r_{p(1)}r_{p(2)} \ldots r_{p(n)} \in A$.*

Proof. (1) Suppose $s \in A$ and $t \in R$, then $st \in A$, hence $ts \in A$, by (*).

(2) Suppose $rst \in u^{-1}A$, then $urst \in A$. By (*), $(ur)ts \in A$, hence $rts \in u^{-1}A$.

(3) It suffices to show that

$$rstuv \in A \Rightarrow rutsv \in A;$$

for any permutation is a product of transpositions. From $rstuv \in A$, we obtain by successive applications of (*) that $r(uv)(st)$, $(ru)t(vs)$, and $(rut)sv$ all belong to A.

According to McCoy [9], an ideal P of R is called *prime* if it is proper, that is, $1 \notin P$, and if

$$r \notin P \ \&\ s \notin P \Rightarrow \exists_t\, rts \notin P,$$

for all $r, s \in R$. When P is a symmetric ideal, this condition simplifies to

$$r \notin P \ \&\ s \notin P \Rightarrow rs \notin P,$$

as in the commutative case. Indeed, if $rs \in P$, then $rst \in P$, hence $rts \in P$ by (*).

A subset Δ of R is called *multiplicative* if it is closed under finite products. In particular, this definition is supposed to imply that $1 \in \Delta$. For any subset A of R we write

$$\Delta^{-1}A = \bigcup_{\delta \in \Delta} \delta^{-1}A = \{r \in R \mid \exists_{\delta \in \Delta} \delta r \in A\}.$$

PROPOSITION 2. *Let Δ be a multiplicative subset of R, A a symmetric ideal not meeting Δ. Then the set of all proper symmetric ideals B containing A such that $\Delta^{-1}B \subseteq B$ contains maximal elements and these are prime ideals.*

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Proof. First we check that $\Delta^{-1}A$ is closed under addition. Suppose r_1 and $r_2 \in \Delta^{-1}A$, then there exist $\delta_i \in \Delta$ such that $\delta_i r_i \in A$. Then $\delta_1 r_1 \delta_2$ and $\delta_2 r_2 \delta_1 \in A$. Using the symmetry of A, we deduce that $\delta_1\delta_2 r_1$ and $\delta_1\delta_2 r_2 \in A$, hence $\delta_1\delta_2(r_1+r_2) \in A$, so that $r_1+r_2 \in \Delta^{-1}A$.

Thus $\Delta^{-1}A$ is a right ideal. Being the union of symmetric ideals, it is also symmetric, hence an ideal by Proposition 1. Moreover $\Delta^{-1}(\Delta^{-1}A)\subseteq\Delta^{-1}A$, $A\subseteq\Delta^{-1}A$, and $1 \notin \Delta^{-1}A$, hence the set of all ideals B under consideration is nonempty. By Zorn's lemma, this set is easily seen to contain at least one maximal element P. We claim that P is prime.

Indeed, suppose $s \notin P$. Then $s^{-1}P$ is a proper symmetric ideal containing P. Moreover, for any $\delta \in \Delta$, we shall prove that $\delta^{-1}s^{-1}P\subseteq s^{-1}P$. For if $r \in \delta^{-1}s^{-1}P$, then $s\delta r \in P$, hence $\delta sr \in P$, hence $sr \in P$, that is, $r \in s^{-1}P$. Thus $\Delta^{-1}s^{-1}P\subseteq s^{-1}P$, and so $s^{-1}P$ is in the set under consideration. By maximality of P, $s^{-1}P\subseteq P$. This means that for any $t \notin P$ also $st \notin P$, hence P is prime, as was to be proved.

Proposition 3. *Let A be a proper symmetric ideal. Then the following three sets are equal:*

(a) *the intersection of all prime ideals containing A,*

(b) *the intersection of all symmetric prime ideals containing A,*

(c) *the set of all $r \in R$ for which there is a natural number n such that $r^n \in A$.*

We call this set the radical *of A.*

Proof. Since clearly $(a)\subseteq(b)$, we shall prove that $(b)\subseteq(c)\subseteq(a)$.

(b)$\subseteq$(c). Assume that, for all natural numbers n, $r^n \notin A$. Then the multiplicative set $\Delta=\{1, r, r^2,\ldots\}$ does not meet A. By Proposition 2, there exists a symmetric prime ideal P containing A such that $\Delta^{-1}P\subseteq P$. It follows that $r \notin P$, for otherwise $1 \in \Delta^{-1}P\subseteq P$.

(c)$\subseteq$(a). Suppose P is a prime ideal containing A and $r \notin P$. Then there exists $t_1 \in R$ such that $rt_1r \notin P$. Continuing in this manner, we obtain a sequence of elements $t_1, t_2, \ldots$ such that

$$rt_1rt_2r\ldots rt_{n-1}r \notin P$$

for each natural number $n\geq 1$. Therefore

$$rt_1rt_2r\ldots rt_{n-1}r \notin A,$$

hence

$$r^nt_1t_2\ldots t_{n-1} \notin A,$$

since A is symmetric. Thus $r^n \notin A$ for each n.

3. **Symmetric modules.** Given any right ideal P of R and any right R-module M_R, we put

$$(1)\qquad T_P(M) = \{m \in M \mid \forall_{r\in R}\forall_{s\notin P} r^{-1}(m^{-1}0) \nsubseteq s^{-1}P\}.$$

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This is of interest because it is the torsion submodule of M_R relative to the largest torsion theory for which R_R/P is torsion free [7, §0].

We call M_R *symmetric* if

$$mrs = 0 \Rightarrow msr = 0,$$

for all $m \in M$ and $r, s \in R$. Then $m^{-1}0$ is an ideal and $r^{-1}(m^{-1}0)$ contains $m^{-1}0$. Formula (1) then simplifies to

$$(2) \qquad T_P(M) = \{m \in M \mid \forall_{s \notin P} m^{-1}0 \nsubseteq s^{-1}P\}.$$

Assuming that P is a prime ideal, this simplifies even further to

$$(3) \qquad T_P(M) = \{m \in M \mid m^{-1}0 \nsubseteq P\}.$$

In the special case $M_R = R_R$, $T_P(M)$ is usually called the "P-component of 0" and denoted 0_P.

The set Π of prime ideals of R is topologized as usual by declaring as open sets all sets of the form

$$\Gamma(A) = \{P \in \Pi \mid A \nsubseteq P\},$$

where A is any ideal of R. Then Π becomes a compact topological space.

We let

$$\Sigma = \dot{\bigcup_{P \in \Pi}} M/T_P(M)$$

be the disjoint union of the sets $M/T_P(M)$. Strictly speaking, an element of Σ is a pair (P, x), where $P \in \Pi$ and $x \in M/T_P(M)$. If $\pi_P : M \to M/T_P(M)$ is the canonical surjection, we define

$$\hat{m}(P) = (P, \pi_P(m)),$$

for each $m \in M$, thus $\hat{m} : \Pi \to \Sigma$.

We note that the mapping $m \to \hat{m}$ is one-to-one. For if $\hat{m}(P) = 0$ for all prime ideals P, then, according to (3), $m^{-1}0$ is not contained in any prime ideal, hence $1 \in m^{-1}0$, that is, $m = 0$. Thus we have represented M_R as a subdirect product of the modules $M/T_P(M)$.

Remark. Let M_R be a symmetric module and $m \in M$. If $\hat{m}$ vanishes at P then it vanishes on an open neighborhood of P.

Proof. Suppose $\hat{m}(P) = 0$, then $m \in T_P(M)$, that is, $m^{-1}0 \nsubseteq P$, according to (3). Here $m^{-1}0$ is an ideal, and so P belongs to the open set $\Gamma(m^{-1}0)$. Take any $P' \in \Gamma(m^{-1}0)$ and reverse this argument. It follows that $\hat{m}(P') = 0$.

This remark suggests that we can topologize Σ in such a way that the mappings $\hat{m}$ become continuous while the stalks $M/T_P(M)$ remain discrete.

We introduce a topology on Σ by taking as basic open sets the sets

$$\hat{m}(\Gamma(A)) = \{\hat{m}(P) \mid A \nsubseteq P\},$$

where m is any element of M and A is any ideal of R. We note that

$$\hat{m}_1(\Gamma(A_1)) \cap \hat{m}_2(\Gamma(A_2)) = \hat{m}_1(\Gamma(A_1 \cap A_2 \cap m^{-1}0)),$$

where $m=m_1-m_2$, as is checked by a straightforward calculation.

An easy computation also shows that

$$\hat{m}_1^{-1}(\hat{m}_2(\Gamma(A)) = \Gamma(A \cap m^{-1}0),$$

where $m=m_1-m_2$, hence the $\hat{m}$ are continuous. For reasons that we shall not go into, Σ is called a "sheaf". We note that the stalks of Σ are discrete.

To describe the stalks of this sheaf we shall call a symmetric module M_R *prime-torsion free* if there is a prime ideal P of R such that $T_P(M)=0$, that is, such that P contains all annihilators of nonzero elements of M.

LEMMA 1. *If M_R is a symmetric module and P is a prime ideal of R, then $M/T_P(M)$ is symmetric and prime-torsion free.*

Proof. To see that $M/T_P(M)$ is symmetric, suppose that $mrs \in T_P(M)$, that is, $(mrs)^{-1}0 \nsubseteq P$. This means that, for some $t \notin P$, $rst \in m^{-1}0$, hence $srt \in m^{-1}0$, since $m^{-1}0$ is a symmetric ideal. Reversing the argument, we infer that $msr \in T_P(M)$.

To see that $M/T_P(M)$ is prime-torsion free, assume that $mr \in T_P(M)$ but $m \notin T_P(M)$. Then $mrs=0$, for some $s \notin P$. We claim that $r \in P$. Otherwise there exists t such that $rts \notin P$. Since $m^{-1}0$ is a symmetric ideal, we have $mrts=0$, hence $m \in T_P(M)$, a contradiction. (This can also be deduced from the fact that $T_P(M)$ is the torsion submodule of M_R for the largest torsion theory for which R/P is torsion free.)

LEMMA 2. *If M_R is a symmetric module, m is any element of M, and A is any ideal of R, then $\hat{m}$ vanishes on the open set $\Gamma(A)$ if and only if*

$$\forall_{a \in A} \exists_n \, ma^n = 0.$$

Proof. $\hat{m}$ vanishes on $\Gamma(A)$ if and only if

$$\forall_{P \in \Pi} A \nsubseteq P \Rightarrow m^{-1}0 \nsubseteq P,$$

that is to say, A is contained in the radical of $m^{-1}0$. By Proposition 3, this is equivalent to saying that for each $a \in A$ there is a natural number n such that $a^n \in m^{-1}0$.

THEOREM 1. *A module is symmetric if and only if it is isomorphic to the module of sections of a sheaf of prime-torsion free symmetric modules.*

Proof. Let M_R be a symmetric module. As we have seen, to each element $m \in M$ there corresponds a unique section $\hat{m}$ of the sheaf $\Sigma = \dot{\bigcup}_{P \in \Pi} M/T_P(M)$. Now let f be any section of Σ, we shall prove that $f=\hat{m}$ for some $m \in M$.

For each $P \in \Pi$, $f(P)$ is a pair (P, x_P), where $x_P \in M/T_P(M)$, say $x_P=\pi_P(m_P)$,

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$m_P \in M$. Thus $f-\hat{m}_P$ vanishes at P. Since the stalk $M/T_P(M)$ is discrete, the set whose only element is $(P, 0)$ is open, hence $f-\hat{m}_P$ vanishes on a basic open set

$$\Gamma(r_P) = \{P' \in \Pi \mid r_P \notin P'\},$$

where $r_P \in R$.

By compactness of Π, the neighborhoods of a finite collection of prime ideals $P_1, \ldots, P_k$ already cover Π. We replace the subscript P_i by the subscript i. Thus $f-\hat{m}_i$ vanishes on $\Gamma(r_i)$, and so $\hat{m}_i-\hat{m}_j$ vanishes on

$$\Gamma(r_i) \cap \Gamma(r_j) = \Gamma(r_i R r_j),$$

which certainly contains $\Gamma(r_i r_j)$.

By Lemma 2, we have

$$(m_i-m_j)(r_i r_j)^{n(i,j)} = 0$$

for some natural number $n(i, j)$. Let n be the maximum of the $n(i, j)$, then

(i) $$(m_i-m_j)r_i^n r_j^n = 0$$

in view of the fact that M_R is symmetric.

Observing that a prime ideal contains the principal ideal $(r_i) = Rr_iR$ if and only if it contains $(r_i)^n$, we deduce that

$$\Pi = \bigcup_{i=1}^{k} \Gamma(r_i) = \bigcup_{i=1}^{k} \Gamma((r_i)^n) = \Gamma\left(\sum_{i=1}^{k} (r_i)^n\right).$$

Therefore $1 \in R = \sum_{i=1}^{k} (r_i)^n$, and so

(ii) $$1 = \sum_{i=1}^{k} s_i,$$

where s_i is a sum of elements of the form

$$t_0 r_i t_1 r_i \ldots r_i t_n.$$

From (i) and the fact that M_R is symmetric, we obtain

$$(m_i-m_j)t_0 r_i t_1 r_i \ldots r_i t_n r_j^n = 0.$$

By summation, this yields

(iii) $$(m_i-m_j)s_i r_j^n = 0.$$

Now put

$$m = \sum_{i=1}^{k} m_i s_i,$$

then

$$\begin{aligned} mr_j^n &= \sum_{i=1}^{k} m_i s_i r_j^n \\ &= \sum_{i=1}^{k} m_j s_i r_j^n, && \text{by (iii),} \\ &= m_j r_j^n, && \text{by (ii).} \end{aligned}$$

Therefore $\hat{m}-\hat{m}_j$ vanishes on $\Gamma(r_j)$, by Lemma 2. But so does $f-\hat{m}_j$, hence also $f-\hat{m}$. Since the $\Gamma(r_j)$ cover Π, it follows that $f=\hat{m}$, as was to be shown.

In the converse direction, let there be given a sheaf of symmetric prime-torsion free modules. The sections of a sheaf of modules also form a module, as is well known. In the present situation we have a submodule of a direct product of symmetric modules, hence also a symmetric module. The proof of the theorem is now complete.

It is easily seen that the right module R_R is symmetric if and only if the left module $_RR$ is. We may as well call R a *symmetric* ring in that case. Moreover R_R is then prime-torsion free if and only if $_RR$ is, hence we may as well call the ring *prime-torsion free*. When R is commutative this reduces to the concept introduced in § 1.

COROLLARY 1. *A ring is symmetric if and only if it is isomorphic to the ring of sections of a sheaf of prime-torsion free symmetric rings.*

Proof. It need only be pointed out that $T_P(R)$ is an ideal, hence $R/T_P(R)$ is a ring. It is symmetric and prime-torsion free as an R-module by the theorem. An easy calculation shows that it is also prime-torsion free for the prime ideal $P/T_P(R)$, hence as a ring.

Actually $T_P(R)=0_P$, the ideal which appeared in (F) and (G) of §1. Thus (G) follows from Corollary 1. It may also be easily shown [8] that $T_P(R)$ is the annihilator of the injective hull of R_R/P.

Better is the following result, which finally implies (H). Given a ring M and a homomorphism $R \to M$, one often calls M an *extension* of R. We shall say that such an extension is *right symmetric* if

$$mrm' = 0 \Rightarrow mm'r = 0,$$

for all $m, m' \in M$ and $r \in R$. In particular, taking $m' \in R$, we see that M_R is a symmetric module.

COROLLARY 2. *A ring extension M of R is right symmetric if and only if it is isomorphic to the ring of sections of a sheaf of right symmetric extensions of R which are prime-torsion free R-modules.*

Proof. It suffices to show that $M/T_P(M)$ is an extension of R which is right symmetric. Since M_R is a symmetric module, $T_P(M)$ is a submodule. Moreover, it is clearly a left ideal. To see that it is a right ideal, assume $m \in T_P(M)$, then $mr=0$ for some $r \notin P$. Hence, for any $m' \in M$, also $mrm'=0$, and so $mm'r=0$, by right symmetry, whence $mm' \in T_P(M)$. Thus $T_P(M)$ is an ideal, and $R \to M \to M/T_P(M)$ is an extension of R.

To show that this extension is right symmetric, assume $mrm' \in T_P(M)$, that is, $mrm's = 0$ for some $s \notin P$. Since M is a right symmetric extension of R, $mm'rs=0$ and, for the same reason, $mm'sr=0$. Therefore $mm'r \in T_P(M)$, and our argument is complete.

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