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GALOIS GROUPS OF RECIPROCAL SEXTIC POLYNOMIALS

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Abstract

Let F be a subfield of the complex numbers and $f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in F[x]$ an irreducible polynomial. We give an elementary characterisation of the Galois group of $f(x)$ as a transitive subgroup of S_6. The method involves determining whether three expressions involving a, b and c are perfect squares in F and whether a related quartic polynomial has a linear factor. As an application, we produce one-parameter families of reciprocal sextic polynomials with a specified Galois group.

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1. Introduction

Let F be a subfield of the complex numbers and $f(x) \in F[x]$ an irreducible polynomial of degree n. Identifying the Galois group, $\mathrm{Gal}(f)$, of $f(x)$ as a transitive subgroup of S_n is a fundamental problem in computational algebra. In general, this is a difficult task; most modern approaches are based on [9, 10]. However, when $f(x)$ has a special form, the computation can be more straightforward.

For example, Galois groups of even quartic polynomials ($x^4 + ax^2 + b$), even sextic polynomials ($x^6 + ax^4 + bx^2 + c$) and doubly even octic polynomials ($x^8 + ax^4 + b$) have elementary characterisations (see for example [1, 2]). In each case, the characterisation leverages information about the index-2 subfield of the field defined by the polynomial.

A natural extension of this technique is to irreducible reciprocal polynomials, which are polynomials satisfying $f(x) = x^n \cdot f(1/x)$, since the field extension defined by such a polynomial also has an index-2 subfield (see Theorem 2.1). Note that if $f(x) = \sum_{i=0}^{n} f_i x^i$ is a reciprocal polynomial, then $f_i = f_{n-i}$; that is, the sequence of coefficients $\{f_i\}$ forms a palindrome.

In this setting, some similar characterisations of $\mathrm{Gal}(f)$ are known. The following classical result of Dickson determines the Galois group of an irreducible reciprocal quartic polynomial by testing the squareness of two elements of F (see [4]). In the theorem, as in the rest of the paper, we will use the following standard convention for describing groups: C_n denotes the cyclic group of order n, D_n the dihedral group of order $2n$, and A_n and S_n the alternating and symmetric groups on n letters, respectively. We also use $\times$ to denote a direct product, $\rtimes$ a semidirect product and $\wr$ a wreath product.

THEOREM 1.1 (Dickson). *Let* $x^4 + ax^3 + bx^2 + ax + 1 \in F[x]$ *be irreducible. Then* $\mathrm{Gal}(f)$ *is isomorphic to:*

- $C_2 \times C_2$ *if and only if* $(b + 2)^2 - 4a^2$ *is a square in F;*
- C_4 *if and only if* $((b + 2)^2 - 4a^2)(a^2 - 4b + 8)$ *is a square in F;*
- D_4 *if and only if neither* $(b + 2)^2 - 4a^2$ *nor* $((b + 2)^2 - 4a^2)(a^2 - 4b + 8)$ *is a square in F.*

Since reciprocal polynomials of odd degree are reducible (they have -1 as a root), the next logical case to consider is Galois groups of irreducible reciprocal sextic polynomials; that is, those of the form $x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in F[x]$. The purpose of this paper is to provide a similar characterisation, reflecting the spirit of Theorem 1.1. In doing so, we will generalise the results in [6, 7], which verify that certain families of irreducible reciprocal sextic polynomials with rational coefficients have Galois group isomorphic to either S_3, D_6 or $S_4 \times C_2$.

The remainder of the paper is organised as follows. In Section 2, we collect several results concerning field extensions defined by reciprocal polynomials and their Galois groups; these will be used to prove our main theorem in Section 3. Our main result, Theorem 3.5, gives a characterisation of the Galois group of $x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1$ that depends only on the squareness of three elements whose expressions involve a, b and c, along with whether or not a related quartic polynomial has a linear factor. As an application, we provide one-parameter families of reciprocal sextics for each possible Galois group (see Theorem 3.7).

2. Preliminary results

In this section, we let $f(x) \in F[x]$ be a monic irreducible reciprocal polynomial of degree $n = 2m$, and we let $L = F(\alpha)$ where $f(\alpha) = 0$. We collect results about L and $\mathrm{Gal}(f)$ that are used later in the paper.

The reader is referred to [8] for an elementary overview of several standard facts about reciprocal polynomials. One such result shows there exists a polynomial $g(x)$ of degree m such that $f(x) = x^m \cdot g(x + 1/x)$ (see [8, Proposition 2.0.16]). It turns out that $g(x)$ is the minimal polynomial of $\alpha + 1/\alpha$. This result is straightforward to establish and shows L has an index-2 subfield. For convenience, we include a proof.

THEOREM 2.1. *The minimal polynomial,* $g(x)$*, of* $\beta = \alpha + 1/\alpha$ *has degree m. Thus, L has a subfield,* $K = F(\beta)$*, of index* 2*.*

PROOF. Let the roots of $f(x)$ be $\{\alpha = r_1, 1/r_1, \ldots, r_m, 1/r_m\}$, and let $g(x)$ be the polynomial whose roots are $\{r_i + 1/r_i\}$ for $1 \le i \le m$. Thus, the degree of $g(x)$ is m and $g(\beta) = 0$. Let $h(x) = x^m \cdot g(x + 1/x)$. It follows that $f(x) = h(x)$ since both polynomials

TABLE 1. Possible Galois groups of irreducible reciprocal sextic polynomials. Size gives the order of the group.

T	Name	Size	Generators
6T1	C_6	6	(164253)
6T2	S_3	6	(16)(25)(34), (145)(236)
6T3	D_6	12	(36)(45), (135246)
6T4	A_4	12	(154)(263), (34)(56)
6T6	$A_4 \times C_2$	24	(34)(56), (164253)
6T7	S_4^+	24	(35)(46), (154)(263)
6T8	S_4^-	24	(154)(263), (12)(36)(45)
6T11	$S_4 \times C_2$	48	(12)(3645), (146235)

are monic, are of the same degree and have the same roots. If $g(x)$ were reducible, say $g(x) = k(x) \cdot l(x)$, then this would imply $f(x)$ is reducible since

$$f(x) = h(x) = x^{\deg(k)} k(x + 1/x) \cdot x^{\deg(l)} l(x + 1/x).$$

However, this contradicts the irreducibility of $f(x)$. Thus, $g(x)$ is irreducible and is therefore the minimal polynomial of β. □

Since subfields of L correspond to block systems of $\mathrm{Gal}(f)$ (see [5, Section 1.6] for more information about block systems), it follows that $\mathrm{Gal}(f)$ can be embedded in a suitable wreath product. In particular, the following is an immediate consequence of Theorem 2.1 and [5, Theorem 2.6 A].

COROLLARY 2.2. *We have* $\mathrm{Gal}(f)$ *is a subgroup of* $C_2 \wr \mathrm{Gal}(g) \simeq C_2^m \rtimes \mathrm{Gal}(g)$. *Thus,* $2^m \cdot m!$ *is an upper bound for* $|\mathrm{Gal}(f)|$.

3. Galois groups of reciprocal sextics

For the rest of the paper, we let $f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in F[x]$ be an irreducible reciprocal polynomial, α a root of $f(x)$ and $L = F(\alpha)$. Let $g(x)$ denote the minimal polynomial of $\alpha + 1/\alpha$. Thus, we have the following result as a consequence of Theorem 2.1.

COROLLARY 3.1. *The polynomial* $g(x) = x^3 + ax^2 + (b-3)x - 2a + c$ *is irreducible and defines a cubic subfield* K *of* L.

Since $f(x)$ is irreducible of degree 6, it follows that $\mathrm{Gal}(f)$ is a transitive subgroup of S_6 that is also a subgroup of $C_2 \wr S_3 \simeq S_4 \times C_2$, by Corollary 2.2. Among the 16 transitive subgroups of S_6, only 8 are subgroups of $S_4 \times C_2$. Table 1 gives information about these 8 groups: their transitive numbers (as given in [3]), their orders, descriptive names and generators for (one representative of the conjugacy class of) each group.

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TABLE 2. One sample reciprocal sextic Polynomial in $\mathbb{Q}[x]$ for each possible Galois group.

T	Name	Polynomial
6T1	C_6	$x^6 + x^3 + 1$
6T2	S_3	$x^6 + 3x^4 + 4x^3 + 3x^2 + 1$
6T3	D_6	$x^6 + 3x^3 + 1$
6T4	A_4	$x^6 + 4x^5 - x^4 - x^2 + 4x + 1$
6T6	$A_4 \times C_2$	$x^6 + x^5 - 3x^4 - 5x^3 - 3x^2 + x + 1$
6T7	S_4^+	$x^6 + x^4 + 5x^3 + x^2 + 1$
6T8	S_4^-	$x^6 + 5x^4 + 2x^3 + 5x^2 + 1$
6T11	$S_4 \times C_2$	$x^6 + x^4 + 2x^3 + x^2 + 1$

We point out that 6T7 and 6T8 are isomorphic copies of S_4 that are distinguished by their parity; that is, 6T7 contains only even permutations while 6T8 does not; this is reflected in the table by the respective superscripts of each group's 'Name'. Note that each of these eight groups does appear as a Galois group over $\mathbb{Q}$. See Table 2, which gives one sample irreducible reciprocal sextic polynomial in $\mathbb{Q}[x]$ for each possible Galois group.

We can determine properties of $\mathrm{Gal}(f)$ from properties of $\mathrm{Gal}(g)$, as our next general result shows.

THEOREM 3.2. *Let $\phi(x) \in F[x]$ be irreducible of degree 6 and ρ be a root of ϕ. Suppose that $F(\rho)$ has a subfield M of degree 3 defined by $\gamma(x) \in F[x]$. Then:*

(1) $\mathrm{Disc}(\phi)$ *is a perfect square in F if and only if* $\mathrm{Gal}(\phi)$ *is either A_4 or S_4^+;*
(2) $\mathrm{Disc}(\gamma)$ *is a perfect square in F if and only if* $\mathrm{Gal}(\phi)$ *is either C_6, A_4 or $A_4 \times C_2$;*
(3) $\mathrm{Disc}(\phi) \cdot \mathrm{Disc}(\gamma)$ *is a perfect square in F if and only if* $\mathrm{Gal}(\phi)$ *is either S_3, A_4 or S_4^-.*

PROOF. A standard result in Galois theory shows that $\mathrm{Gal}(\phi)$ is a subgroup of A_6 if and only if $\mathrm{Disc}(\phi)$ is a square in F. Of the eight possibilities, only A_4 and S_4^+ are subgroups of A_6, proving item (1).

Since γ is a cubic polynomial, another standard result in Galois theory shows $\mathrm{Gal}(\gamma) \simeq C_3$ if and only if $\mathrm{Disc}(\gamma)$ is a perfect square. Under the Galois correspondence, M corresponds to an index 3 subgroup H of $\mathrm{Gal}(\phi)$ containing the stabiliser of ρ. Let N be the normal core of H in $\mathrm{Gal}(\phi)$; that is, the largest normal subgroup of $\mathrm{Gal}(\phi)$ contained in H. Therefore, $\mathrm{Gal}(\gamma)$ is isomorphic to $\mathrm{Gal}(\phi)/N$. Using [11] to perform group computations, we see that each of the eight possibilities has a unique such subgroup H of index 3, up to conjugation. This means M/F is the unique cubic subfield of $F(\rho)/F$, up to isomorphism. Further group computations show that in the cases of C_6, A_4 and $A_4 \times C_2$, $\mathrm{Gal}(\phi)/N$ is isomorphic to C_3; in all other cases it is isomorphic to S_3, proving item (2).

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If both $\mathrm{Disc}(\phi)$ and $\mathrm{Disc}(\gamma)$ are perfect squares, then $\mathrm{Disc}(\phi) \cdot \mathrm{Disc}(\gamma)$ is a perfect square. According to the previous paragraphs, there is only one group among the eight where this occurs; namely, A_4. For the remainder of the proof, we suppose neither $\mathrm{Disc}(\phi)$ nor $\mathrm{Disc}(\gamma)$ is a perfect square. Thus, the polynomials $x^2 - \mathrm{Disc}(\phi)$ and $x^2 - \mathrm{Disc}(\gamma)$ define quadratic subfields of the splitting field of $\phi(x)$. By the Galois correspondence, $F(\sqrt{\mathrm{Disc}(\phi)})$ corresponds to $H_1 = A_6 \cap \mathrm{Gal}(\phi)$. Similarly, if M' is the normal closure of $\gamma(x)$, then the subgroup fixing M' is N. Thus, $F(\sqrt{\mathrm{Disc}(\gamma)})$ corresponds to the unique subgroup H_2 of $\mathrm{Gal}(\phi)$ of index 2 that contains N. It follows that $\mathrm{Disc}(\phi) \cdot \mathrm{Disc}(\gamma)$ is a perfect square if and only if $H_1 = H_2$. Among the four remaining possible Galois groups, direct computation shows S_3 and S_4^- have $H_1 = H_2$. The groups D_6 and $S_4 \times C_2$ have $H_1 \neq H_2$, proving item (3). □

Our next result is an immediate consequence of Theorem 3.2 and the fact that $\mathrm{Disc}(f) = ((2a + c)^2 - (2b + 2)^2) \cdot \mathrm{Disc}(g)^2$.

COROLLARY 3.3. *We have the following:*

(1) $(2a + c)^2 - (2b + 2)^2$ *is a perfect square in F if and only if* $\mathrm{Gal}(f)$ *is either A_4 or S_4^+;*
(2) $\mathrm{Disc}(g)$ *is a perfect square in F if and only if* $\mathrm{Gal}(f)$ *is either C_6, A_4 or $A_4 \times C_2$;*
(3) $((2a + c)^2 - (2b + 2)^2) \cdot \mathrm{Disc}(g)$ *is a perfect square in F if and only if* $\mathrm{Gal}(f)$ *is either S_3, A_4 or S_4^-.*

Next, we introduce a degree 4 resolvent polynomial that is helpful in determining $\mathrm{Gal}(f)$.

THEOREM 3.4. *Let $h(x) = x^4 + Ax^3 + Bx^2 + Cx + D$, where*

- $A = -4(a^2 - 2b - 6)$;
- $B = 2(3a^4 - 4a^2(3b + 5) + 8(ac + b^2 + 4b + 9))$;
- $C = -4(a^4(a^2 - 6b - 2) + 8a^2(ac + b^2 + 5) + 16(ac - 2b^2 - b(ac + 2) - 4))$;
- $D = (a^4 - 4a^2(b - 1) + 8(ac - 2b))^2$.

Then, $h(x)$ is separable and has a linear factor if and only if $\mathrm{Gal}(f)$ *is either C_6, S_3 or D_6.*

PROOF. Let $\{\alpha = r, 1/r, s, 1/s, t, 1/t\}$ be the roots of $f(x)$. The roots of $h(x)$ are $\{(r - 1/r) \pm (s - 1/s) \pm (t - 1/t)\}$, which can be verified by using the theory of elementary symmetric functions to express the coefficients of $h(x)$ in terms of a, b and c.

If $h(x)$ were not separable, then two roots would be equal. There are six cases:

(1) $(r - 1/r) + (s - 1/s) + (t - 1/t) = (r - 1/r) + (s - 1/s) - (t - 1/t)$;
(2) $(r - 1/r) + (s - 1/s) + (t - 1/t) = (r - 1/r) - (s - 1/s) + (t - 1/t)$;
(3) $(r - 1/r) + (s - 1/s) + (t - 1/t) = (r - 1/r) - (s - 1/s) - (t - 1/t)$;
(4) $(r - 1/r) + (s - 1/s) - (t - 1/t) = (r - 1/r) - (s - 1/s) + (t - 1/t)$;
(5) $(r - 1/r) + (s - 1/s) - (t - 1/t) = (r - 1/r) - (s - 1/s) - (t - 1/t)$;
(6) $(r - 1/r) - (s - 1/s) + (t - 1/t) = (r - 1/r) - (s - 1/s) - (t - 1/t)$.

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TABLE 3. Let $f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in F[x]$ be irreducible, $g(x) = x^3 + ax^2 + (b-3)x - 2a + c$ and $h(x)$ as defined in Theorem 3.4. The table lists whether $\mathrm{Disc}(f)$, $\mathrm{Disc}(g)$ and $\mathrm{Disc}(f) \cdot \mathrm{Disc}(g)$ are perfect squares in F, and whether $h(x)$ has a linear factor, according to $\mathrm{Gal}(f)$.

T	Name	Disc(f)	Disc(g)	Disc(f)·Disc(g)	Linear
1	C_6	no	yes	no	yes
2	S_3	no	no	yes	yes
3	D_6	no	no	no	yes
4	A_4	yes	yes	yes	no
6	$A_4 \times C_2$	no	yes	no	no
7	S_4^+	yes	no	no	no
8	S_4^-	no	no	yes	no
11	$S_4 \times C_2$	no	no	no	no

We will show each case leads to a contradiction. Cases 2 and 5 imply $s = 1/s$ and Cases 1 and 6 imply $t = 1/t$; these contradict the fact that $f(x)$ is irreducible and thus separable. Case 4 implies $st(s-t) = -(s-t)$. Since $f(x)$ is separable, this implies $st = -1$. Thus, $s = -1/t$ and $1/s = -t$. Therefore, $-a = r + 1/r + s + 1/s + t + 1/t = r + 1/r$, which is rational. However, this contradicts Theorem 2.1, which shows $r + 1/r$ is not rational. Similarly, Case 3 implies $st(s+t) = s + t$. If $s + t \neq 0$, then $st = 1$; which contradicts the separability of $f(x)$. If $s = -t$, then $1/s = -1/t$. We again reach the contradiction $-a = r + 1/r$. Thus, $h(x)$ is separable.

To prove the rest of the theorem, let $G = S_4 \times C_2$ and $H = D_6$ be the subgroups of S_6 as given in Table 1. Then, a complete set of a right coset representatives of G/H is $\{\mathrm{id}, (34), (56), (34)(56)\}$. Further, the only block system of G is $R = \{\{1,2\},\{3,4\},\{5,6\}\}$. We identify r as root 1, $1/r$ as root 2, s as root 3, $1/s$ as root 4, t as root 5 and $1/t$ as root 6.

A multivariable function stabilised by H is $T(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - x_2 + x_3 - x_4 + x_5 - x_6$; the action on T is via subscripts. We form the resolvent polynomial corresponding to G, H and T (see [9]); this produces the polynomial $h(x)$. By the theory of resolvent polynomials, the factorisation of $h(x)$ corresponds to the orbits of $\mathrm{Gal}(f)$ acting on the cosets G/H. Direct computation on each possibility for $\mathrm{Gal}(f)$ shows that in the cases of C_6, S_3 and D_6, there is an orbit of length 1 and an orbit of length 3. In the other five cases, there is a single orbit of length 4. This means $h(x)$ factors as a linear times a cubic polynomial in those three cases and remains irreducible in the other five cases, proving the theorem. □

We can now state our main result, which gives an elementary characterisation of $\mathrm{Gal}(f)$. This is an immediate consequence of Corollary 3.3 and Theorem 3.4. For convenience, Table 3 summarises this characterisation.

THEOREM 3.5. *Let $f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in F[x]$ be irreducible, $g(x) = x^3 + ax^2 + (b-3)x - 2a + c$ and $h(x)$ as defined in Theorem 3.4.*

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TABLE 4. One-parameter families of reciprocal sextic polynomials with specified Galois group over **Q**.

T	G	Polynomials
1	C_6	$(2t^2-2t+13)x^6+(-4t+2)x^5+(-2t^2+2t+19)x^4+(8t-4)x^3+(-2t^2+2t+19)x^2+(-4t+2)x+(2t^2-2t+13)$
2	S_3	$(3t^2+1)x^6+(18t^2-6)x^5+(45t^2+15)x^4+(60t^2-20)x^3+(45t^2+15)x^2+(18t^2-6)x+(3t^2+1)$
3	D_6	$(t-16)x^6+6tx^5+15tx^4+(20t+32)x^3+15tx^2+6tx+(t-16)$
4	A_4	$(2t-3)x^6-18x^5+(-2t+3)x^4-28x^3+(-2t+3)x^2-18x+(2t-3)$
6	$A_4\times C_2$	$(t^3+3t^2-1)x^6+(6t^3+6t^2+6)x^5+(15t^3-3t^2-15)x^4+(20t^3-12t^2+20)x^3+(15t^3-3t^2-15)x^2+(6t^3+6t^2+6)x+(t^3+3t^2-1)$
7	S_4^+	$tx^6+(-2t-12)x^5-tx^4+(4t-40)x^3-tx^2+(-2t-12)x+t$
8	S_4^-	$(3t^2+2)x^6+(18t^2+24)x^5+(45t^2+78)x^4+(60t^2+48)x^3+(45t^2+78)x^2+(18t^2+24)x+(3t^2+2)$
11	$S_4\times C_2$	$(t+2)x^6+(6t+24)x^5+(15t+78)x^4+(20t+48)x^3+(15t+78)x^2+(6t+24)x+(t+2)$

(1) *If* Disc(f) *is a perfect square in F, then* Gal(f) *is* A_4 *if* Disc(g) *is a square and is* S_4^+ *otherwise.*
(2) *If* Disc(f) *is not a square and* Disc(g) *is a square, then* Gal(f) *is* C_6 *if* $h(x)$ *has a linear factor and is* $A_4\times C_2$ *otherwise.*
(3) *If* Disc(f) *and* Disc(g) *are not squares and* Disc$(f)\cdot$Disc(g) *is a square, then* Gal(f) *is* S_3 *if* $h(x)$ *has a linear factor and is* S_4^- *otherwise.*
(4) *If none of* Disc(f), Disc(g) *and* Disc$(f)\cdot$Disc(g) *is a square, then* Gal(f) *is* D_6 *if* $h(x)$ *has a linear factor and is* $S_4\times C_2$ *otherwise.*

EXAMPLE 3.6. As an example, we use Theorem 3.5 to compute the Galois group of a family of sextic reciprocal polynomials. Take $t>-27/4$ and suppose the polynomial $f(x)=x^6+3x^5+(t+6)x^4+(2t+7)x^3+(t+6)x^2+3x+1\in\mathbb{Q}[x]$ is irreducible. Then, Disc$(f)=-t^4(4t+27)^3$, which is not a square. We also have $g(x)=x^3+3x^2+(t+3)x+(2t+1)$. Then, Disc$(g)=-t^2(4t+27)$, which is also not a square. However, Disc$(f)\cdot$Disc(g) is a square. Further, $h(x)$ has $x+4t+27$ as a linear factor. By item (3) of Theorem 3.5, Gal(f) is S_3. Note, this also confirms item (3) of [6, Theorem 1].

As an application of Theorem 3.5, we give one-parameter families of reciprocal sextics defined over $\mathbb{Q}$ for each possible Galois group.

THEOREM 3.7. *The polynomials in Table 4 have the indicated Galois group over* $\mathbb{Q}$, *except for values of t that result in reducible polynomials.*

Verifying each family of polynomials in Table 4 has the indicated Galois group is a straightforward computation using Theorem 3.5 and a computer algebra system. For example, we can consider the polynomial in the first row of Table 4: $f(x)=$

$(2t^2 - 2t + 13)x^6 + (-4t + 2)x^5 + (-2t^2 + 2t + 19)x^4 + (8t - 4)x^3 + (-2t^2 + 2t + 19)x^2 + (-4t + 2)x + (2t^2 - 2t + 13)$. Then, $\mathrm{Disc}(f) = -(t^2 - t - 1)^4(t^2 - t + 7)^4$, which is not a square. We also have $g(x) = (2t^2 - 2t + 13)x^3 + (-4t + 2)x^2 + (-8t^2 + 8t - 20)x + (16t - 8)$. Then, $\mathrm{Disc}(g) = (t^2 - t - 1)^2(t^2 - t + 7)^2$, which is a square. Furthermore, $h(x)$ has $(t^4 - 2t^3 + 14t^2 - 13t + 169/4)x + 4t^4 - 8t^3 + 36t^2 - 32t + 64$ as a linear factor. This proves $\mathrm{Gal}(f) = C_6$, as claimed.

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