Canad. Math. Bull. Vol. **53** (2), 2010 pp. 347–359
doi:10.4153/CMB-2010-030-4

Multiple Nontrivial Solutions for Doubly Resonant Periodic Problems

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Abstract. We consider semilinear periodic problems with the right-hand side nonlinearity satisfying a double resonance condition between two successive eigenvalues. Using a combination of variational and degree theoretic methods, we prove the existence of at least two nontrivial solutions.

1 Introduction

In this paper we consider the following periodic problem

$$\begin{cases} -x''(t) = f(t, x(t)) \text{ a.e. on } T := [0, b], \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{1.1}$$

The goal of this work is to establish the existence of multiple nontrivial solutions for problem (1.1). When the nonlinearity of f can grow linearly and asymptotically at $\pm\infty$, the *slope*, $\frac{f(z,x)}{x}$, stays between two successive distinct eigenvalues of the negative scalar Laplacian with periodic boundary conditions, and resonance is possible at both ends of the spectral interval (double resonance).

In the past the problem initially was investigated under uniform and nonuniform nonresonance conditions by Iannacci–Nkashama [9], Habets–Metzen [7], and Fonda–Mawhin [5]. The doubly resonant case was studied by Fabry–Fonda [4] and Omari–Zanolin [13]. Both works proved existence theorems, but did not address the question of existence of multiple solutions for the doubly resonant problem. We also mention the recent work of Kyritsi–Papageorgiou [10], which proved an existence theorem for a doubly resonant periodic problem driven by the scalar p-Laplacian and with a nonsmooth potential function.

In this paper, using a combination of variational and degree theoretic methods, we prove the existence of at least two nontrivial solutions for problem (1.1) under a double resonance condition.

Received by the editors October 9, 2009.
Published electronically December 4, 2009.
This research was partially supported by the Portuguese Foundation for Sciences and Technology (FCT) under the Sabbatical Fellowship SFRH/BSAB/794/2008.
AMS subject classification: **34B15**.
Keywords: double resonance, generalized LL-condition, Leray-Schauder degree, Cerami condition.

2 Background and Hypotheses

Consider the linear eigenvalue problem:

$$\begin{cases} -x''(t) = \lambda x(t) \text{ a.e. on } T := [0, b], \\ x(0) = x(b), x'(0) = x'(b). \end{cases} \tag{2.1}$$

It is well known that problem (2.1) has a nontrivial solution $x \in C^1(T)$ if and only if

$$\lambda = \lambda_k = \left(\frac{2\pi k}{b}\right)^2, \quad k \geq 0.$$

These are the eigenvalues of the negative scalar Laplacian under periodic boundary conditions, $(-\triangle, W^{1,2}_{\text{per}}(0, b))$ for short, where

$$W^{1,2}_{\text{per}}(0, b) := \{x \in W^{1,2}(0, b) : x(0) = x(b)\}.$$

By $E(\lambda_k)$ we denote the two-dimensional eigenspace corresponding to the eigenvalue λ_k. Of course, we have the orthogonal direct sum decomposition

$$W^{1,2}_{\text{per}}(0, b) = E(\lambda_k) \oplus V \text{ with } V = E(\lambda_k)^{\perp},$$

and so for every $x \in W^{1,2}_{\text{per}}(0, b)$ we can write, in a unique way,

$$x = x^0 + \widehat{x}, \text{ with } x^0 \in E(\lambda_k) \text{ and } \widehat{x} \in V.$$

If $g \in L^\infty(T)_+ := \{g \in L^\infty(T) : g(t) \geq 0 \text{ a.e. on } T\}$ and for some $k \geq 0$ we have $\lambda_k \leq g(t) \leq \lambda_{k+1}$ a.e. on T and the inequalities are strict on sets (not necessarily the same) of positive measure, then the linear problem

$$\begin{cases} -x''(t) = g(t)x(t) \text{ a.e. on } T := [0, b], \\ x(0) = x(b), x'(0) = x'(b). \end{cases}$$

has only the trivial solution. (For a more general result in this direction, see[1].)

The hypotheses on the nonlinearity $f(t, x)$ are the following:

(H_f) $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $t \to f(t, x)$ is measurable;

(ii) for almost all $t \in T$, $f(t, \cdot) \in C^1(T)$;

(iii) for every $M > 0$, there exists $a_M \in L^1(T)_+$ such that

$$|f(t, x)| \leq a_M(t) \text{ for a.a. } t \in T \text{ and all } |x| \leq M;$$

(iv) there exists $k \geq 0$ such that

$$\lambda_k \leq \liminf_{|x|\to\infty} \frac{f(t, x)}{x} \leq \limsup_{|x|\to\infty} \frac{f(t, x)}{x} \leq \lambda_{k+1}$$

uniformly for a.a. $t \in T$;

(v) when $x_n \in W^{1,2}_{\mathrm{per}}(0,b)$, $\|x_n\| \to \infty$, and $\frac{\|x_n^0\|}{\|x_n\|} \to 1$, we have

(a) if $x_n^0 \in E(\lambda_k)$, there exist $\gamma_1 > 0$ and $n_1 \geq 1$ such that

$$\int_0^b (f(t,x_n(t)) - \lambda_k x_n(t)) x_n^0(t)\, dt \geq \gamma_1 > 0 \text{ for all } n \geq n_1,$$

(b) if $x_n^0 \in E(\lambda_{k+1})$, there exist $\gamma_2 > 0$ and $n_2 \geq 1$ such that

$$\int_0^b (f(t,x_n(t)) - \lambda_{k+1} x_n(t)) x_n^0(t)\, dt \leq -\gamma_2 < 0 \text{ for all } n \geq n_2;$$

(vi) there exist $\delta > 0$ and $w_0 \in \mathbb{R} \setminus \{0\}$ such that, if $F(t,x) = \int_0^x f(t,r)dr$, then

$$F(t,x) \leq 0 \text{ for a.a. } t \in T \text{ and all } |x| \leq \delta$$

and

$$\int_0^b F(t,w_0)\, dt \geq 0.$$

Remark. Hypothesis (H_f)(iv) is the double resonance condition. Hypothesis (H_f)(v) is a generalization of the well-known Landesman–Lazer sufficiency conditions (LL-conditions for short) for the solvability of resonant problems (see [11, 12]). We find analogous conditions in the works of Fabry–Fonda [4] and Iannacci–Nkashama [9].

We consider now an example. For simplicity we drop the t-dependence on f and consider $f(x) = \lambda_k x + g(x)$, with $g \in C^1(\mathbb{R})$. Then

$$F(x) = \lambda_k x^2 + G(x), \text{ with } G(x) = \int_0^x g(s)\, ds.$$

Assume that near the origin $G(x) = x^4 - \sin x$ and for $|x|$ large, $G(x) = c|x|^{\frac{3}{2}}$, $c > 0$. Such nonlinearity $f(\cdot)$ satisfies hypotheses (H_f). Then the equation

$$\begin{aligned} &-x''(t) = \lambda_k x(t) + g(x(t)) \text{ a.e. on } T := [0,b], \\ &x(0) = x(b), x'(0) = x'(b). \end{aligned}$$

can serve as a model equation for our work.

The Euler functional $\varphi\colon W^{1,2}_{\mathrm{per}}(0,b) \to \mathbb{R}$ for problem (1.1) is defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \int_0^b F(t,x(t))\, dt, \text{ for all } x \in W^{1,2}_{\mathrm{per}}(0,b).$$

We have that $\varphi \in C^2(W^{1,2}_{\mathrm{per}}(0,b))$. In fact, if by $\langle \cdot , \cdot \rangle$ we denote the duality brackets for the pair $(W^{1,2}_{\mathrm{per}}(0,b), W^{1,2}_{\mathrm{per}}(0,b)^*)$, we have

$$\langle \varphi'(x), y \rangle = \int_0^b x'(t) y'(t)\, dt - \int_0^b f(t,x(t)) y(t)\, dt$$

and

$$\varphi''(x)(u,v) = \int_0^b u'(t)v'(t)\,dt - \int_0^b f'_x(t,x(t))u(t)v(t)\,dt,$$

for all $y, u, v \in W^{1,2}_{\mathrm{per}}(0,b)$.

Finally recall that if H is a Hilbert space and $\psi \in C^1(H)$, we say that ψ satisfies the Cerami condition (C-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq H$ such that $|\psi(x_n)| \leq M_1$ for some $M_1 > 0$, all $n \geq 1$, and $(1 + \|x_n\|)\psi'(x_n) \to 0$ in H^* has a strongly convergent subsequence. This is a compactness condition on ψ, weaker than the usual PS-condition. It was shown by Bartolo–Benci–Fortunato [3] that this condition is enough to prove a deformation theorem and from it derive minimax characterizations of the critical values of ψ (see also [6]).

3 Multiple Solutions

As we already mentioned, we will combine variational and degree theoretic techniques. For the implementation of the variational methods, we need the following proposition.

Proposition 3.1 *If hypotheses* (H_f) *hold, then* φ *satisfies the C-condition.*

Proof We consider a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\mathrm{per}}(0,b)$ such that $|\varphi(x_n)| \leq M_1$ for some $M_1 > 0$, all $n \geq 1$, and $(1 + \|x_n\|)\varphi'(x_n) \to 0$ in $W^{1,2}_{\mathrm{per}}(0,b)^*$ as $n \to \infty$. We will show that $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\mathrm{per}}(0,b)$ is bounded.

We argue indirectly. Suppose that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\mathrm{per}}(0,b)$ is unbounded. We may assume that $\|x_n\| \to \infty$. We set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we can say that

$$y_n \xrightarrow{w} y \text{ in } W^{1,2}_{\mathrm{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T) \text{ as } n \to \infty.$$

(Recall that $W^{1,2}_{\mathrm{per}}(0,b)$ is embedded compactly in $C(T)$). Hypotheses (H_f)(iii) and (iv) imply that

$$|f(t,x)| \leq a(t) + c|x| \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R},$$

with $a \in L^1(T)_+$ and $c > 0$. Hence we have

$$\frac{|f(t,x_n(t))|}{\|x_n\|} \leq \frac{a(t)}{\|x_n\|} + c|y_n(t)| \text{ for a.a. } t \in T, \tag{3.1}$$

therefore $\{f(\,\cdot\,,x_n(\,\cdot\,))/\|x_n\|\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable. By virtue of the Dunford–Pettis theorem, we can say that

$$\frac{f(\,\cdot\,,x_n(\,\cdot\,))}{\|x_n\|} \xrightarrow{w} h \text{ in } L^1(T) \text{ as } n \to \infty.$$

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For every $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$C^+_{\varepsilon,n} = \Big\{ t \in T : x_n(t) > 0, \lambda_k - \varepsilon \leq \frac{f(t, x_n(t))}{x_n(t)} \leq \lambda_{k+1} + \varepsilon \Big\}$$

and

$$C^-_{\varepsilon,n} = \Big\{ t \in T : x_n(t) < 0, \lambda_k - \varepsilon \leq \frac{f(t, x_n(t))}{x_n(t)} \leq \lambda_{k+1} + \varepsilon \Big\}.$$

Note that $x_n(t) \to \infty$ for all $t \in \{y > 0\}$ and $x_n(t) \to -\infty$ for all $t \in \{y < 0\}$ as $n \to \infty$. So by virtue of hypothesis (H_f)(iv), we have

$$\chi_{C^+_{\varepsilon,n}}(t) \to 1 \text{ a.e. on } \{y > 0\} \quad \text{and} \quad \chi_{C^-_{\varepsilon,n}}(t) \to 1 \text{ a.e. on } \{y < 0\}.$$

Using the dominated convergence theorem, we obtain

$$\Big\| (1 - \chi_{C^+_{\varepsilon,n}}) \frac{f(\,\cdot\,, x_n(\,\cdot\,))}{\|x_n\|} \Big\|_{L^1(\{y>0\})} \to 0$$

and

$$\Big\| (1 - \chi_{C^-_{\varepsilon,n}}) \frac{f(\,\cdot\,, x_n(\,\cdot\,))}{\|x_n\|} \Big\|_{L^1(\{y<0\})} \to 0.$$

It follows that

$$\chi_{C^+_{\varepsilon,n}}(\,\cdot\,) \frac{f(\,\cdot\,, x_n(\,\cdot\,))}{\|x_n\|} \xrightarrow{w} h \text{ in } L^1(\{y > 0\})$$

and

$$\chi_{C^-_{\varepsilon,n}}(\,\cdot\,) \frac{f(\,\cdot\,, x_n(\,\cdot\,))}{\|x_n\|} \xrightarrow{w} h \text{ in } L^1(\{y < 0\}).$$

From the definition of the sets $C^+_{\varepsilon,n}$ and $C^-_{\varepsilon,n}$, we have

$$\begin{aligned}(\lambda_k - \varepsilon) y_n(t) &\leq \frac{f(t, x_n(t))}{x_n(t)} y_n(t) \\ &= \frac{f(t, x_n(t))}{\|x_n\|} \leq (\lambda_{k+1} + \varepsilon) y_n(t) \text{ a.e. on } C^+_{\varepsilon,n}\end{aligned}$$

and

$$\begin{aligned}(\lambda_k - \varepsilon) y_n(t) &\geq \frac{f(t, x_n(t))}{x_n(t)} y_n(t) \\ &= \frac{f(t, x_n(t))}{\|x_n\|} \geq (\lambda_{k+1} + \varepsilon) y_n(t) \text{ a.e. on } C^-_{\varepsilon,n}.\end{aligned}$$

Passing to the limit as $n \to \infty$, using Mazur's lemma, and recalling that $\varepsilon > 0$ was arbitrary, we obtain

$$\lambda_k y(t) \leq h(t) \leq \lambda_{k+1} y(t) \text{ a.e. on } \{y > 0\} \tag{3.2}$$

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and

$$\lambda_k y(t) \geq h(t) \geq \lambda_{k+1} y(t) \text{ a.e. on } \{y < 0\}. \tag{3.3}$$

Moreover, from (3.1) it is clear that

$$h(t) = 0 \text{ a.e. on } \{y = 0\}. \tag{3.4}$$

Combining (3.2), (3.3), and (3.4), we see that $h(t) = g(t)y(t)$ a.e. on T, with $g \in L^\infty(T)_+$ such that $\lambda_k \leq g(t) \leq \lambda_{k+1}$ a.e. on T. Let $V\colon W^{1,2}_{\text{per}}(0,b) \to W^{1,2}_{\text{per}}(0,b)$ be the linear operator defined by

$$\langle V(x), y\rangle = \int_0^b x'(t)y'(t)\,dt \text{ for all } x, y \in W^{1,2}_{\text{per}}(0,b).$$

Clearly V is continuous, *i.e.,* $V \in \mathcal{L}(W^{1,2}_{\text{per}}(0,b), W^{1,2}_{\text{per}}(0,b)^*$. Also let $N\colon C(T) \to L^1(T)$ be the Nemitsky operator corresponding to the nonlinearity f, *i.e.,*

$$N(x)(.) = f(., x(.)) \text{ for all } x \in C(T).$$

Evidently N is bounded continuous. Moreover, because of the compact embedding of $W^{1,2}_{\text{per}}(0,b)$ into $C(T)$ and of $L^1(T)$ into $W^{1,2}_{\text{per}}(0,b)^*$, we see that N is completely continuous as a map from $W^{1,2}_{\text{per}}(0,b)$ into $W^{1,2}_{\text{per}}(0,b)^*$.

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0,b)$, we have

$$|\langle \varphi'(x_n), v\rangle| \leq \varepsilon_n \text{ for all } v \in W^{1,2}_{\text{per}}(0,b) \text{ and with } \varepsilon_n \downarrow 0.$$

We know that $\varphi'(x_n) = V(x_n) - N(x_n)$ for all $n \geq 1$. So

$$\Big| \langle V(y_n), v\rangle - \int_0^b \frac{N(x_n)}{\|x_n\|} v\,dt \Big| \leq \frac{\varepsilon_n}{\|x_n\|},\ n \geq 1. \tag{3.5}$$

Use as a test function $v = y_n - y \in W^{1,2}_{\text{per}}(0,b)$. Since

$$\int_0^b \frac{N(x_n)}{\|x_n\|}(y_n - y)\,dt \to \infty \text{ as } n \to \infty,$$

from (3.5) it follows that $\lim_{n\to\infty}\langle V(y_n), y_n - y\rangle = 0$. Recall that $V(y_n) \xrightarrow{w} V(y)$, so we have $\lim_{n\to\infty}\langle V(y_n), y_n\rangle = \langle V(y), y\rangle$. Hence $\|y_n'\|_2 \to \|y'\|_2$. Since $y_n' \xrightarrow{w} y'$ in $L^2(T)$, from the Kadec–Klee property of the Hilbert space $L^2(T)$, we infer that $y_n' \to y'$ in $L^2(T)$. This combined with the fact that $y_n \to y$ in $C(T)$, implies that $y_n \to y$ in $W^{1,2}_{\text{per}}(0,b)$ and so $\|y\| = 1$. Passing to the limit as $n \to \infty$ in (3.5), we obtain

$$\langle V(y), v\rangle = \int_0^b g(t)y(t)v(t)\,dt \text{ for all } v \in W^{1,2}_{\text{per}}(0,b).$$

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Hence

$$-y''(t) = g(t)y(t) \text{ a.e. on } T,\ y(0) = y(b),\ y'(0) = y'(b). \tag{3.6}$$

We consider three distinct cases for problem (3.6), depending on the position of the weight function $g \in L^\infty(T)_+$ in the spectral interval $[\lambda_k, \lambda_{k+1}]$.

Case 1: $g(t) = \lambda_k$ a.e. on T. From (3.6) it follows that $y = y^0 \in E(\lambda_k)$. So we have

$$\frac{\|x_n^0\|}{\|x_n\|} \to 1 \tag{3.7}$$

(recall $x_n = x_n^0 + \widehat{x}_n$, with $x_n^0 \in E(\lambda_k)$, $\widehat{x}_n \in V$, $n \geq 1$). From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0,b)$, we have

$$\Big| \langle V(x_n), x_n^0\rangle - \int_0^b N(x_n)x_n^0\,dt \Big| \leq \varepsilon_n.$$

Hence

$$\Big| \|(x_n^0)'\|_2^2 - \int_0^b N(x_n)x_n^0\,dt \Big| \leq \varepsilon_n,$$

and since $x_n^0 \in E(\lambda_k)$, we get

$$\Big| \lambda_k\|x_n^0\|_2^2 - \int_0^b N(x_n)x_n^0\,dt \Big| \leq \varepsilon_n.$$

Therefore,

$$\int_0^b (f(t,x_n(t)) - \lambda_k x_n(t))x_n^0(t)\,dt \leq \varepsilon_n. \tag{3.8}$$

Here we have used the orthogonality of the component spaces. But because of (3.7), inequality (3.8) contradicts hypothesis (H_f)(v) (the generalized LL-condition).

Case 2: $g(t) = \lambda_{k+1}$ a.e. on T. This case is treated similarly to Case 1, using this time the second half of hypothesis (H_f)(v).

Case 3: $\lambda_k \leq g(t) \leq \lambda_{k+1}$ a.e. on T, $g \neq \lambda_k$, $g \neq \lambda_{k+1}$. As we already mentioned in Section 2, from (3.6) it follows that $y = 0$, a contradiction to the fact that $\|y\| = 1$.

So in all three cases, we have reached a contradiction. This means that $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0,b)$ is bounded and so we may assume that

$$x_n \xrightarrow{w} x \text{ in } W^{1,2}_{\text{per}}(0,b) \quad \text{and} \quad x_n \to x \text{ in } C(T).$$

Then reasoning as earlier in this proof via the continuity of the linear operator V and the Kadec–Klee property of $L^2(T)$, we conclude that $x_n \to x$ in $W^{1,2}_{\text{per}}(0,b)$, which proves that φ satisfies the C-condition. ∎

Proposition 3.2 *If hypotheses* (H_f) *hold, then the origin is a local minimizer of* φ.

Proof Since $W^{1,2}_{\mathrm{per}}(0,b)$ is embedded continuously (in fact compactly) into $C(T)$, we can find $c_0 > 0$ such that $\|x\|_\infty \le c_0\|x\|$ for all $x \in W^{1,2}_{\mathrm{per}}(0,b)$. If $\delta > 0$ is as in hypothesis (H_f)(vi) and we set $\delta_0 = \frac{\delta}{c_0}$, then for all $x \in W^{1,2}_{\mathrm{per}}(0,b)$ with $\|x\| \le \delta_0$, we have $|x(t)| \le c_0\|x\| \le \delta$ for all $t \in T$. Thus, because of hypothesis (H_f)(vi), we have $F(t,x(t)) \le 0$ a.e. on T. Therefore, for all $x \in W^{1,2}_{\mathrm{per}}(0,b)$ with $\|x\| \le \delta_0$, we have

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \int_0^b F(t,x(t))\,dt \ge 0 = \varphi(0).$$

Hence the origin is a local minimizer of φ. ∎

From this proposition we have that the origin is a critical point of φ. We will assume that it is an isolated critical point of φ. Otherwise, we have a sequence of nontrivial critical points of φ, hence a sequence of nontrivial solutions for problem (1.1) and so we are done.

Let $f_1(t,x) = f(t,x) + x$. By the Riesz representation theorem, we can find a continuous $G_1 \colon W^{1,2}_{\mathrm{per}}(0,b) \to W^{1,2}_{\mathrm{per}}(0,b)$, such that

$$(G_1(u), v)_{W^{1,2}_{\mathrm{per}}(0,b)} = \int_0^b f_1(t,u(t))v(t)\,dt \text{ for all } u, v \in W^{1,2}_{\mathrm{per}}(0,b),$$

where by $(\cdot\,,\cdot)_{W^{1,2}_{\mathrm{per}}(0,b)}$ we denote the inner product of the Hilbert space $W^{1,2}_{\mathrm{per}}(0,b)$. So we have $\nabla\varphi(u) = I - G_1(u)$ for all $u \in W^{1,2}_{\mathrm{per}}(0,b)$, $\nabla\varphi(u)$ being the gradient of φ at $u \in W^{1,2}_{\mathrm{per}}(0,b)$. Because of the compact embedding of $W^{1,2}_{\mathrm{per}}(0,b)$ into $C(T)$, we can easily check that G_1 is compact. Since the origin is an isolated critical point which is a local minimizer of φ, from [2, Corollary 2], we have the following.

Proposition 3.3 *If hypotheses* (H_f) *hold, then there exists* $\rho_0 > 0$ *small such that* $d_{\mathrm{LS}}(\nabla\varphi, B_\rho, 0) = 1$ *for all* $0 < \rho \le \rho_0$ *(here* d_{LS} *denotes the Leray–Schauder degree map and* $B_\rho = \{x \in W^{1,2}_{\mathrm{per}}(0,b) : \|x\| < \rho\}$*).*

In the next proposition, we produce the first nontrivial solution of problem (1.1).

Proposition 3.4 *If hypotheses* (H_f) *hold, then there exists* $x_0 \in C^1_{\mathrm{per}}(T)$*,* $x_0 \ne 0$*, solution of problem* (1.1).

Proof Since the origin is a strict local minimizer of φ, we can find $\rho > 0$ such that

$$\varphi(0) = 0 < \inf_{\partial B_\rho} \varphi.$$

Also because of hypothesis (H_f)(vi), we have $\varphi(w_0) \le 0 = \varphi(0) < \inf_{\partial B_\rho}\varphi$. These facts, combined with Proposition 3.1, permit the use of the mountain pass theorem (see Bartolo–Benci–Fortunato [3] and Gasinski–Papageorgiou [6, p. 648]), which gives $x_0 \in W^{1,2}_{\mathrm{per}}(0,b)$ such that

$$0 = \varphi(0) < \inf_{\partial B_\rho}\varphi \le \varphi(x_0) \quad \text{and} \quad \varphi'(x_0) = 0.$$

From the inequality we obtain $x_0 \neq 0$, while from the equality we have $V(x_0) = N(x_0)$, and hence $x_0 \in C^1_{\text{per}}(T)$ and solves problem (1.1). ∎

From [8, Theorems 1, 2] we have the following.

Proposition 3.5 *If hypotheses* (H_f) *hold, then there exists* $r_0 > 0$ *small such that* $d_{\mathrm{LS}}(\nabla\varphi, B_r(x_0), 0) = -1$ *for all* $0 < r \leq r_0$*, where*

$$B_r(x_0) = \{x \in W^{1,2}_{\text{per}}(0,b) : \|x - x_0\| < r\}.$$

In the next proposition we compute the Leray–Schauder degree of $\nabla\varphi$ for large balls.

Proposition 3.6 *If hypotheses* (H_f) *hold, then there exists* $R_0 > 0$ *such that*

$$d_{\mathrm{LS}}(\nabla\varphi, B_R, 0) = (-1)^k$$

for all $R \geq R_0$.

Proof For $\varepsilon > 0$, we consider the continuous, linear operator $V_\varepsilon := V + \varepsilon I$ from $W^{1,2}_{\text{per}}(0,b)$ into $W^{1,2}_{\text{per}}(0,b)^*$. Clearly V_ε is strongly monotone, hence surjective. Moreover, by Banach's theorem $V_\varepsilon^{-1} \in \mathcal{L}(W^{1,2}_{\text{per}}(0,b)^*, W^{1,2}_{\text{per}}(0,b))$. Because of the compact embedding of $L^1(T)$ into $W^{1,2}_{\text{per}}(0,b)^*$, we see that $V_\varepsilon^{-1} \colon L^1(T) \to W^{1,2}_{\text{per}}(0,b)$ is a completely continuous linear operator. Let $f_\varepsilon(t,x) = f(t,x) + \varepsilon x$ and consider the Nemitsky operator for f_ε, $N_\varepsilon \colon C(T) \to L^1(T)$ defined by

$$N_\varepsilon(x)(.) = f_\varepsilon(., x(.)) \text{ for all } x \in C(T).$$

Clearly N_ε is bounded, continuous. Then $x \to V_\varepsilon^{-1} \circ N_\varepsilon(x)$ is a compact map from $W^{1,2}_{\text{per}}(0,b)$ into itself.

Let $\theta \in (\lambda_k, \lambda_{k+1})$ and consider the compact homotopy

$$h(\beta, x) = V_\varepsilon^{-1} \circ (\beta N_\varepsilon + (1-\beta)\theta I)(x).$$

Claim: We can find $R_0 > 0$ such that $0 \neq h(\beta, x)$ for all $\beta \in [0,1]$, all $\|x\| = R$, and all $R \geq R_0$.

We proceed by contradiction. Suppose that the claim is not true. We can find $\{\beta_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0,b)$ such that

$$\beta_n \to \beta \in [0,1], \ \|x_n\| \to \infty, \text{ and } h(\beta_n, x_n) = 0 \text{ for all } n \geq 1. \tag{3.9}$$

From the equality in (3.9), we have

$$V(x_n) + \varepsilon x_n = \beta_n N_\varepsilon(x_n) + (1-\beta_n)\theta x_n. \tag{3.10}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,2}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T).$$

Dividing (3.10) by $\|x_n\|$, we obtain

$$V(y_n) + \varepsilon y_n = \beta_n \frac{N(x_n)}{\|x_n\|} + \beta_n \varepsilon y_n + (1-\beta_n)\theta y_n. \tag{3.11}$$

From the proof of Proposition 3.1, we know that (at least for a subsequence)

$$\frac{N(x_n)}{\|x_n\|} \xrightarrow{w} h = gy \text{ in } L^1(T),$$

with $g \in L^\infty(T)_+$, $\lambda_k \le g(t) \le \lambda_{k+1}$ a.e. on T, and $y_n \to y$ in $W^{1,2}_{\mathrm{per}}(0,b)$. Hence $\|y\| = 1$. Therefore, if we pass to the limit as $n \to \infty$ in (3.11), we obtain

$$V(y) = (\beta g + (1-\beta)\theta - (1-\beta)\varepsilon)y. \tag{3.12}$$

As in the proof of Proposition 3.1, we consider three distinct cases, corresponding to three different possibilities for the weight function

$$m = \beta g + (1-\beta)\theta - (1-\beta)\varepsilon \in L^\infty(T)_+.$$

Case 1: $\beta = 1$ and $g(t) = \lambda_k$ a.e. on T. From (3.12) we have $V(y) = \lambda_k y$. Hence

$$-y''(t) = \lambda_k y(t) \text{ a.e. on } T,\ y(0) = y(b),\ y'(0) = y'(b).$$

Therefore, $y \in E(\lambda_k)$, $y \ne 0$. We take duality brackets of (3.11) with $y_n^0 \in E(\lambda_k)$ (recall $y_n = y_n^0 + \widehat{y}_n$, $y_n^0 \in E(\lambda_k)$, $\widehat{y}_n \in V = E(\lambda_k)^\perp$). We obtain

$$\|(y_n^0)'\|_2^2 + \varepsilon\|y_n^0\|_2^2 = \beta_n \int_0^b \frac{N(x_n)}{\|x_n\|} y_n^0\,dt + \beta_n\varepsilon\|y_n^0\|_2^2 + (1-\beta_n)\int_0^b \theta y_n y_n^0\,dt.$$

Hence

$$\beta_n \int_0^b \Big(\frac{N(x_n)}{\|x_n\|} - \lambda_k y_n^0\Big) y_n^0\,dt + (1-\beta_n)\int_0^b (\theta y_n - (\lambda_k+\varepsilon)y_n^0)y_n^0\,dt = 0. \tag{3.13}$$

Note that

$$\int_0^b (\theta y_n - (\lambda_k+\varepsilon)y_n^0)y_n^0\,dt \to \int_0^b (\theta - (\lambda_k+\varepsilon))y^2\,dt$$

(recall $y \in E(\lambda_k)$). Choosing $\varepsilon < \theta - \lambda_k$, we have

$$\int_0^b (\theta - (\lambda_k+\varepsilon))y^2\,dt = (\theta - \lambda_k - \varepsilon)\|y\|_2^2.$$

We may assume that $\beta_n \ne 1$ for all $n \ge 1$ or otherwise we have a sequence of nontrivial solutions and we are done (see (3.10). Hence, we can find $n_0 \ge n_1$ such that

$$(1-\beta_n)\int_0^b (\theta y_n - (\lambda_k+\varepsilon)y_n^0)y_n^0\,dt > 0 \text{ for all } n \ge n_0.$$

From (3.13) it follows that

$$\beta_n \int_0^b \Big(\frac{N(x_n)}{\|x_n\|} - \lambda_k y_n^0 \Big) y_n^0 \, dt < 0 \text{ for all } n \geq n_0. \tag{3.14}$$

Since $y \in E(\lambda_k)$, we have

$$\frac{\|x_n^0\|}{\|x_n\|} \to 1 \text{ as } n \to \infty.$$

So, by virtue of hypothesis $(\mathrm{H}_f)(\mathrm{v})$, we have that

$$0 < \beta_n \gamma_1 \leq \beta_n \int_0^b (f(t, x_n(t)) - \lambda_k x_n(t)) x_n^0(t) \, dt \text{ for all } n \geq n_0.$$

Therefore,

$$0 < \frac{\beta_n \gamma_1}{\|x_n\|^2} \leq \beta_n \int_0^b \Big(\frac{N(x_n)}{\|x_n\|} - \lambda_k y_n^0 \Big) y_n^0 \, dt \text{ for all } n \geq n_0. \tag{3.15}$$

Comparing (3.14) and (3.15), we reach a contradiction.

Case 2 $\beta = 1$ and $g(t) = \lambda_{k+1}$ a.e. on T. We treat this case similarly to Case 1, using this time the second part of hypothesis $(\mathrm{H}_f)(\mathrm{v})$.

Case 3: $\beta \in [0, 1)$ or ($g \neq \lambda_k$ and $g \neq \lambda_{k+1}$). In this case we have

$$\lambda_k < \beta g(t) + (1 - \beta)\theta < \lambda_{k+1} \text{ a.e. on } T.$$

From (3.12) we have

$$\begin{cases} -y''(t) = (\beta g(t) + (1 - \beta)\theta - (1 - \beta)\varepsilon) y(t) \text{ a.e. on } T, \\ y(0) = y(b), y'(0) = y'(b). \end{cases}$$

If we chose $\varepsilon > 0$ small, from [1, Proposition 2] we deduce that $y = 0$, a contradiction to the fact that $\|y\| = 1$.

So in all three cases we have reached a contradiction. This means that the claim is true. Because of the claim and the homotopy invariance of the Leray–Schauder degree, we have

$$d_{\mathrm{LS}}(\nabla\varphi, B_R, 0) = d_{\mathrm{LS}}(I - V_\varepsilon^{-1} \circ (\theta I), B_R, 0) \text{ for all } R \geq R_0. \tag{3.16}$$

But since $\theta \in (\lambda_k, \lambda_{k+1})$, from the Leray–Schauder index formula (see [14, p. 619]) we have

$$d_{\mathrm{LS}}(I - V_\varepsilon^{-1} \circ (\theta I), B_R, 0) = (-1)^k \text{ for all } R > 0. \tag{3.17}$$

From (3.16) and (3.17), we conclude that

$$d_{\mathrm{LS}}(\nabla\varphi, B_R, 0) = (-1)^k \text{ for all } R \geq R_0. \qquad \blacksquare$$

Now we are ready for the multiplicity result concerning problem (1.1).

Theorem 3.7 *If hypotheses* (H_f) *hold, then problem* (1.1) *has at least two nontrivial solutions* x_0, $u_0 \in C^1(T)$.

Proof We choose $0 < \rho \le \rho_0$, $0 < r \le r_0$, and $R \ge R_0$ such that

$$B_\rho \cap B_r(x_0) = \varnothing \quad \text{and} \quad \overline{B}_\rho,\ \overline{B}_r(x_0) \subseteq B_R.$$

Then from the additivity and excision properties of the Leray–Schauder degree map, we have

$$d_{\mathrm{LS}}(\nabla\varphi, B_R, 0) = d_{\mathrm{LS}}(\nabla\varphi, B_\rho, 0) + d_{\mathrm{LS}}(\nabla\varphi, B_r(x_0), 0) + d_{\mathrm{LS}}\big(\nabla\varphi, B_R \setminus \big(\overline{B_\rho \cup B_r(x_0)}\big), 0\big).$$

Using Propositions 3.3, 3.5, and 3.6, we have

$$(-1)^k = 1 + (-1) + d_{\mathrm{LS}}\big(\nabla\varphi, B_R \backslash \big(\overline{B_\rho \cup B_r(x_0)}\big), 0\big).$$

Hence

$$d_{\mathrm{LS}}\big(\nabla\varphi, B_R \backslash \big(\overline{B_\rho \cup B_r(x_0)}\big), 0\big) = (-1)^k.$$

Therefore, from the solution property of the Leray–Schauder degree, we infer that there exists $u_0 \in B_R \backslash (\overline{B_\rho \cup B_r(x_0)})$. Hence $u_0 \ne 0$, $u_0 \ne x_0$ such that $V(u_0) = N(u_0)$. Hence

$$-u_0''(t) = f(t, u_0(t)) \text{ a.e. on } T, \quad u_0(0) = u_0(b), \quad u_0'(0) = u_0'(b), \quad u_0 \in C^1(T).$$

Therefore we have shown that (1.1) has at least two nontrivial solutions $x_0, u_0 \in C^1(T)$. ■

References

[1] S. Aizicovici, N. S. Papageorgiou, and V. Staicu, *Periodic solutions for second order differential inclusions with the scalar p-Laplacian.* J. Math. Anal. Appl. **322**(2006), no. 2, 913–929. doi:10.1016/j.jmaa.2005.09.077

[2] H. Amann, *A note on degree theory for gradient mappings.* Proc. Amer. Math. Soc. **85**(1982), no. 4, 591–595. doi:10.2307/2044072

[3] P. Bartolo, V. Benci, and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" nonresonance at infinity.* Nonlinear Anal. **7**(1983), no. 9, 981–1012. doi:10.1016/0362-546X(83)90115-3

[4] C. Fabry and A. Fonda, *Periodic solutions of nonlinear differential equations with double resonance.* Ann. Mat. Pura Appl. **157**(1990), 99–116. doi:10.1007/BF01765314

[5] A. Fonda and J. Mawhin, *Quadratic forms, weighted eigenfunctions and boundary value problems for nonlinear second order ordinary differential equations.* Proc. Royal Soc. Edinburgh Sect. A **112**(1989), no. 1-2, 145–153.

[6] L. Gasiński and N. S. Papageorgiou, *Nonlinear Analysis*. Series in Mathematical Analysis and Applications 9. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[7] P. Habets and G. Metze, *Existence of periodic solutions of Duffing equations.* J. Differential Equations **78**(1989), no. 1, 1–32. doi:10.1016/0022-0396(89)90073-9

[8] H. Hofer, *A note on the topological degree at a critical point of mountain pass-type.* Proc. Amer. Math. Soc. **90**(1984), no. 2, 309–315. doi:10.2307/2045362

[9] R. Iannacci and M. N. Nkashama, *Unbounded perturbations of forced second order ordinary differential equations at resonance.* J. Differential Equations **69**(1987), no. 3, 289–309. doi:10.1016/0022-0396(87)90121-5

[10] S. Kyritsi and N. S. Papageorgiou, *Solutions for doubly resonant nonlinear nonsmooth periodic problems.* Proc. Edinburgh Math. Soc. **48**(2005), no. 1, 199–211. doi:10.1017/S0013091504000264

[11] E. Landesman and A. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance.* J. Math. Mech. **19**(1969/1970), 609–625,

[12] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*. Applied Mathematical Sciences 74. Springer-Verlag, New York, 1989.

[13] P. Omari and F. Zanolin, *Nonresonance conditions on the potential for a second order periodic boundary value problem.* Proc. Amer. Math. Soc. **117**(1993), no. 1, 125–135. doi:10.2307/2159707

[14] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. III*. Springer-Verlag, New York, 1985.

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