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THE NORMALITY IN PRODUCTS WITH A COUNTABLY COMPACT FACTOR

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ABSTRACT. It is known that the product $\omega_1 \times X$ of ω_1 with an M_1-space may be non-normal. In this paper we prove that the product $\kappa \times X$ of an uncountable regular cardinal κ with a paracompact semi-stratifiable space is normal iff it is countably paracompact. We also give a sufficient condition under which the product of a normal space with a paracompact space is normal, from which many theorems involving such a product with a countably compact factor can be derived.

1. **Introduction.** As is well known, the product of a normal countably compact space with a metric space is normal, see [5], [12] and [17]. Kombarov [11] later generalized this by proving that the product of a normal countably compact space with a sequential paracompact space is normal. Since then the normality of products with a countably compact factor or more specially, with a cardinal factor, was investigated in [3], [7], [10], [14], [15] and [18]. Observe that the product of a normal countably compact space with a Lašnev space is normal. However, there exists an M_1-space X such that $\omega_1 \times X$ is not normal [3]. On the other hand, the equivalence of normality and countable paracompactness was established for many cases in the theory of product spaces, see [8], [9], [13], [15], [16], [19] and [20], in particular, it is well known by [16] that the product of a normal countably paracompact space with a metric space is normal iff it is countably paracompact. In section 2 of this paper, we prove that the product $\kappa \times X$ of an uncountable regular cardinal κ with a paracompact semi-stratifiable space is normal iff it is countably paracompact. In section 3, in place of a semi-stratifiable space, we consider general paracompact spaces. A sufficient condition under which the product of a normal space with a paracompact space is normal is given, so that many theorems involving such a product can be derived.

All spaces considered here are regular T_1. By N we denote the set of positive integers, and by κ a cardinal with the usual order to topology when consider it as a space. For a set Γ, $|\Gamma|$ is the cardinality of Γ and $\Gamma^{<\omega}$ is the set of all finite subsets of Γ.

2. **Products with a semi-stratifiable factor.** A space X is said to be *semi-stratifiable* [4] if there exists a function g of $X \times N$ into the topology of X, satisfying

(i) $\bigcap_{n\in N} g(x,n) = \{x\}$ for each $x \in X$,

(ii) if $\{x_n\}$ is a sequence of points in X with $x \in \bigcap_{n\in N} g(x_n,n)$ for some $x \in X$, then $\{x_n\}$ converges to x.

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It is well known that semi-stratifiable spaces are perfect and subparacompact and that the class of semi-stratifiable spaces contains σ-spaces.

The following lemma, due to [8], is very useful in proving our main theorem in this section.

LEMMA 2.1. *Let X be a countably paracompact space, and let E and F be a pair of disjoint subsets. Suppose that F is closed and there exist open sets U_n, $n \in N$ such that $E \subset \bigcap_{n\in N} U_n$ and $\bigcap_{n\in N} \overline{U}_n \cap F = \emptyset$. Then E and F are separated by open sets.*

LEMMA 2.2([15]). *A space X is normal iff for any disjoint closed sets F and K of X there exists a σ-locally finite open cover $\mathcal{U}$ of X such that $\overline{U}$ is disjoint from F or K for each $U \in \mathcal{U}$.*

LEMMA 2.3. *Let X be a countably compact space and Y a semi-stratifiable space. Then $X \times Y$ is countably metacompact.*

PROOF. Let g be a function of $Y \times N$ into the topology of Y satisfying (i) and (ii) above. Let $\{G_n : n \in N\}$ be an increasing open cover of $X \times Y$. We shall now construct for each $n \in N$ inductively two σ-locally finite collections $\mathcal{G}_n$ and $\mathcal{F}_n$ of closed subsets of Y satisfying the following conditions.

(1) $\mathcal{G}_n = \{F \setminus \bigcup_{x\notin G_n} g(p(x), n) : F \in \mathcal{F}_{n-1}\}$

(2) $\mathcal{F}_n = \bigcup\{\mathcal{F}_F : F \in \mathcal{F}_{n-1}\}$ such that $\mathcal{F}_F$ refines $\{F \cap g\big(p(x), n\big) : x \notin G_n\}$

where, $\mathcal{F}_0 = \{Y\}$ and $p: X \times Y \longrightarrow Y$ is the projection.

Assume that the above construction has been already performed for values no greater than n. Then let

$$\mathcal{G}_{n+1} = \{F \setminus \bigcup_{x\notin G_{n+1}} g\big(p(x), n+1\big) : F \in \mathcal{F}_n\}$$

Since $\mathcal{F}_n$ is σ-locally finite collection of closed sets of Y, so is $\mathcal{G}_{n+1}$.

To define $\mathcal{F}_{n+1}$, fix an $F \in \mathcal{F}_n$. Since Y is perfect and subparacompact, the collection

$$\{F \cap g\big(p(x), n+1\big) : x \notin G_{n+1}\}$$

has a σ-discrete closed refinement $\mathcal{F}_F$. Since $\mathcal{F}_n$ is σ-locally finite and $\bigcup \mathcal{F}_F \subset F$, we see that the collection $\mathcal{F}_{n+1} = \bigcup\{\mathcal{F}_F : F \in \mathcal{F}_n\}$ is σ-locally finite. Thus we have inductively accomplished the desired construction.

Now set $\mathcal{G} = \bigcup_{n\in N} \mathcal{G}_n$. We assert that $Y = \bigcup \mathcal{G}$.

Assume the contrary and pick $y \in Y \setminus \bigcup \mathcal{G}$. Then there exist $(x_1, y_1) \notin G_1$ and $F_1 \in \mathcal{F}_1$ such that $y \in F_1 \subset g(y_1, 1)$. Proceeding by induction, it follows from $y \in F_n \setminus \bigcup \mathcal{G}_n$ that there exist $(x_{n+1}, y_{n+1}) \notin G_{n+1}$ and $F_{n+1} \in \mathcal{F}_{n+1}$ such that $y \in F_{n+1} \subset g(y_{n+1}, n+1)$. Thus we have obtained a sequence $\{(x_n, y_n) \notin G_n : n \in N\}$ with $y \in \bigcap_{n\in N} g(y_n, n)$. It follows from the above definition of semi-stratifiable spaces that the sequence $\{y_n\}$ converges to y. Since X is countably compact, the sequence $\{x_n : n \in N\}$ has a cluster point x in X so that the point (x, y) in $X \times Y$, being a cluster point of $\{(x_n, y_n) : n \in N\}$, is not in any G_n. This contradicts with the assumption of $\{G_n : n \in N\}$ being a cover of $X \times Y$. Thus $Y = \bigcup \mathcal{G}$.

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For each $n \in N$ write $\mathcal{G}_n = \bigcup_{m \in N} \mathcal{G}_{nm}$, where $\mathcal{G}_{nm}$ is locally finite. Let $G_{nm} = \bigcup \mathcal{G}_{nm}$; then G_{nm} is closed and $X \times G_{nm} \subset G_n$ for each $n, m \in N$. Moreover, let $C_n = X \times \bigcup_{i,j \leq n} G_{ij}$. It follows that C_n is closed and $C_n \subset G_n$. It is easy to see that $\{C_n : n \in N\}$ covers $X \times Y$. Thus $X \times Y$ is countably metacompact.

THEOREM 2.4. *Let X be a paracompact semi-stratifiable space and $cf(\kappa) \geq \omega_1$. Then $\kappa \times X$ is normal iff it is countably paracompact.*

PROOF. Since normal countably metacompact spaces are countably paracompact, the necessity follows from Lemma 2.3. To prove the sufficiency, let A and B be any disjoint closed sets in $\kappa \times X$. In [18, Theorem 4.1], Yajima proved that there exists a σ-locally finite closed cover $\mathcal{F} = \bigcup_{n \in N} \mathcal{F}_n$ of X satisfying that for each $F \in \mathcal{F}$, there exists a $\lambda(F) \in \kappa$ such that

$$\Big(\big(\lambda(F), \kappa\big) \times F\Big) \cap A = \emptyset \text{ or } \Big(\big(\lambda(F), \kappa\big) \times F\Big) \cap B = \emptyset.$$

Take a locally finite open expansion $\mathcal{G}_n = \{G_F : F \in \mathcal{F}_n\}$ of $\mathcal{F}_n$ for each $n \in N$ given by the paracompactness of X so that $\{\kappa \times G_F : F \in \mathcal{F}_n\}$ is locally finite in $\kappa \times X$.

Now for each $F \in \mathcal{F}$, if $\Big(\big(\lambda(F), \kappa\big) \times F\Big) \cap A = \emptyset$, by the perfect normality of X, there exist open sets V_n, $n \in N$, such that

$$\big(\lambda(F), \kappa\big) \times F = \bigcap_{n \in N} \big(\lambda(F), \kappa\big) \times V_n = \bigcap_{n \in N} \big(\lambda(F), \kappa\big) \times \overline{V}_n \subset \kappa \times X \setminus A.$$

It follows from Lemma 2.1 that there exists an open set $U(F, 0) \subset \kappa \times G_F$ such that

$$\big(\lambda(F), \kappa\big) \times F \subset U(F, 0) \subset \overline{U(F, 0)} \subset \kappa \times X \setminus A.$$

Moreover for each $F \in \mathcal{F}$, since $[0, \lambda(F)] \times X$ is paracompact, there exist open sets $U(F, 1)$ and $U(F, 2)$ of $\kappa \times X$ such that $[0, \lambda(F)] \times F \subset U(F, 1) \cup U(F, 2) \subset \kappa \times G_F$ and $\overline{U(F, i)}$, $i = 1, 2$, is disjoint from A or B.

For each $n \in N$ put

$$\mathcal{U}_n = \{U(F, i) : F \in \mathcal{F}_n \text{ and } i = 0, 1, 2\}.$$

Then $\mathcal{U}_n$ is locally finite such that $\mathcal{U} = \bigcup_{n \in N} \mathcal{U}_n$ covers $\kappa \times X$ and for each $U \in \mathcal{U}_n$, $\overline{U}$ is disjoint from A or B. It follows from Lemma 2.2 that $\kappa \times X$ is normal. The theorem is proved.

Since normal countably metacompact spaces are countably paracompact, Lemma 2.3 actually says that the normal product of a countably compact space with a semi-stratifiable space is countably paracompact. But we do not know if the inversion is true or not. We pose here the following

PROBLEM 2.5. Let X be a normal countably compact space and Y a paracompact semi-stratifiable space. Is then the normality of $X \times Y$ equivalent to its countable paracompactness?

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3. **Products with a paracompact factor.** A space X is said to be *shrinking* if for every open cover $\mathcal{U}$ of X, there exists an open cover $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ of X such that $\overline{V(U)} \subset U$ for each $U \in \mathcal{U}$. The open cover $\mathcal{V}$ is called a *shrinking* of $\mathcal{U}$. It is well known that paracompact spaces are shrinking and shrinking spaces are normal.

The following lemma from Bešlagić [2] is often used for the proof of the shrinking property of spaces.

LEMMA 3.1. *If for each open cover $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ of a space X, there exists an open cover $\{V_{\alpha,n} : \alpha \in \kappa$ and $n \in N\}$ such that $\overline{V}_{\alpha,n} \subset U_\alpha$ for each $\alpha \in \kappa$ and $n \in N$. Then X is shrinking.*

Let κ be a regular uncountable cardinal. In [19], the author proved that the normal product $\kappa \times X$ of κ and a semi-stratifiable space X with $\chi(X) < \kappa$ is shrinking. In a letter to the author, Yajima kindly pointed out that this actually is true for any subparacompact space X with $\chi(X) < \kappa$. However the idea used there is very useful. In fact, using the idea we shall establish Theorem 3.2 below from which many theorems involving products with a κ-compact factor can be derived. But we first give some terminology.

Let X and Y be spaces, and $\mathcal{U}$ a cover of $X \times Y$. A subset S of Y is called *stable* to $\mathcal{U}$, if every $x \in X$ has a neighborhood O_x such that $O_x \times S \subset U$ for some $U \in \mathcal{U}$.

THEOREM 3.2. *Let X be a normal (shrinking) space and Y a paracompact space. If for every binary (any) open cover $\mathcal{U}$ of $X \times Y$, each point in Y has a stable neighborhood to $\mathcal{U}$. Then $X \times Y$ is normal (shrinking).*

PROOF. Let $\mathcal{U}$ be any open cover of $X \times Y$ and suppose that each point y in Y has a stable neighborhood V_y to $\mathcal{U}$. It suffices to show that $\mathcal{U}$ has a shrinking. Put for each $U \in \mathcal{U}$

$$G(U, y) = \bigcup\{P : P \text{ is open in } X \text{ such that } P \times V_y \subset U\}.$$

Then $X = \bigcup\{G(U, y) : U \in \mathcal{U}\}$. It follows from the normality (resp. shrinking property) of X that there exists an open cover $\{H(U, y) : U \in \mathcal{U}\}$ of X such that $\overline{H(U, y)} \subset G(U, y)$ for each $U \in \mathcal{U}$. Take a locally finite open cover $\{W_y : y \in Y\}$ of Y with $\overline{W}_y \subset V_y$ given by the paracompactness of Y. Now for each $U \in \mathcal{U}$ define

$$W_U = \bigcup\{H(U, y) \times W_y : y \in Y \text{ such that } \overline{H(U, y) \times W_y} \subset U\}.$$

One sees easily that $\{W_U : U \in \mathcal{U}\}$ is a shrinking of $\mathcal{U}$. The proof of the theorem is complete.

It is known by [6] that the product of a normal countably compact space with a first countable paracompact space is normal. We now have

COROLLARY 3.3. *Let X be a shrinking countably compact space and Y a first countable paracompact space. Then $X \times Y$ is shrinking.*

PROOF. Let $\mathcal{U}$ be any open cover of $X \times Y$. Let $y \in Y$ and take a neighborhood base $\{V_n : n \in N\}$ of y. For each $n \in N$, put

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$$G_n = \bigcup\{P : P \text{ is open in } X \text{ such that } P \times V_n \subset U \text{ for some } U \in \mathcal{U}\}.$$

It follows that $X = \bigcup_{n\in N} G_n$. Since X is countably compact, $X = \bigcup_{n=1}^{m} G_n$ for some $m \in N$. One sees that $V_y = \bigcap_{n=1}^{m} V_n$ is a stable neighborhood of y to $\mathcal{U}$. By Theorem 3.2, $X \times Y$ is shrinking.

A space is said to be *κ-paracompact* if its every open cover of cardinality $\leq \kappa$ admits a locally finite open refinement, and a space is *κ-collectionwise normal* if for every discrete collection $\{F_\gamma : \gamma \in \Gamma\}$ of closed sets of the space with $|\Gamma| \leq \kappa$ there exists a collection $\{U_\gamma : \gamma \in \Gamma\}$ of mutually disjoint open sets such that $F_\gamma \subset U_\gamma$ for each $\gamma \in \Gamma$.

Let I^κ be the product space of κ copies of $I = [0, 1]$, and $A(\kappa)$ the one-point compactification of the discrete space of cardinality κ. Then it is known from Morita [12] and Alas [1] respectively that a space X is κ-paracompact and normal iff $X \times I^\kappa$ is normal, and a space X is κ-collectionwise normal and countably paracompact iff $X \times A(\kappa)$ is normal. Thus the same proof as in Corollary 3.3 also shows the following

COROLLARY 3.4 ([12, THEOREM 4.1]). *Let X be a normal κ-compact space and Y a paracompact space with $\chi(Y) \leq \kappa$. Then $X \times Y$ is normal and thus collectionwise normal and κ-paracompact.*

COROLLARY 3.5 ([11, THEOREM 1.1]). *The product of a normal countably compact space with a sequential paracompact space is normal and thus collectionwise normal and countably paracompact.*

PROOF. Let X be a normal countably compact space and Y a sequential paracompact space. Let $\{V_1, V_2\}$ be a binary open cover of $X \times Y$. Let $\overline{y} \in Y$. Put

$$G_i = \{x \in X : (x, \overline{y}) \in V_i\}, i = 1, 2.$$

Then G_i, $i = 1, 2$, is open and $X = G_1 \cup G_2$. Since X is normal, there exist open sets H_1, H_2 such that $X = H_1 \cup H_2$ and $\overline{H}_i \subset G_i$ for $i = 1, 2$. Put

$$S_i = \{y \in Y : \overline{H}_i \times \{y\} \subset V_i\}, i = 1, 2.$$

Then S_i is open. Otherwise, for example, we can find a sequence $\{y_n \in Y \setminus S_1 : n \in N\}$ which converges to a point $y_o \in S_1$. For each $n \in N$, choose $x_n \in \overline{H}_1$ so that $(x_n, y_n) \notin V_1$. Let $x_0 \in \overline{H}_1$ be a cluster point of the sequence $\{x_n : n \in N\}$ so that (x_0, y_0) is a cluster point of the sequence $\{(x_n, y_n) \notin V_1 : n \in N\}$. It follows that $(x_0, y_0) \notin V_1$, this is impossible. S_1 thus is open. It is easy to see that $S_1 \cap S_2$ is a stable neighborhood of $\overline{y}$ to $\{V_1, V_2\}$. Theorem 3.2 then implies that $X \times Y$ is normal.

A space is called *strongly κ-compact* if the closure of any subset of cardinality $\leq \kappa$ is compact [11]. It is easy to see that strongly κ-compact spaces are κ-compact. In order to show that normal products with a strong κ-compact factor are collectionwise normal and κ-paracompact with the aid of Alas' result and Morita's result mentioned above, we need the following lemma.

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LEMMA 3.6. *Let X be a strongly κ-compact space and Y a compact space. Then $X \times Y$ is strongly κ-compact.*

PROOF. Let F be any subset of $X \times Y$ of cardinality κ and $\mathcal{G} = \{G_\gamma : \gamma \in \Gamma\}$ a collection of open sets of $X \times Y$ with $\overline{F} \subset \bigcup \mathcal{G}$. We have to find a finite subcollection of $\mathcal{G}$ which covers $\overline{F}$.

Index F by κ as $F = \{(x_\alpha, y_\alpha) : \alpha \in \kappa\}$. Let $p: X \times Y \longrightarrow X$ be the projection. Note that p is closed. Now for each $\varphi \in \Gamma^{<\omega}$ let

$$V_\varphi = \{x \in X : (\{x\} \times Y) \cap \overline{F} \subset \bigcup_{\gamma \in \varphi} G_\gamma\}.$$

Then V_φ is open since Y is compact. On the other hand,

$$p(\overline{F}) \subset \bigcup \{V_\varphi : \varphi \in \Gamma^{<\omega}\}.$$

However, since p is closed, $p(\overline{F}) = \overline{\{x_\alpha : \alpha \in \kappa\}}$. It follows from the strong κ-compactness of X that there exist finitely many $\varphi_1, \ldots, \varphi_n \in \Gamma^{<\omega}$ such that

$$\overline{\{x_\alpha : \alpha \in \kappa\}} \subset \bigcup_{i=1}^{n} V_{\varphi_i}.$$

Let us put for each $\gamma \in \varphi_i$, $i = 1, \ldots, n$,

$$H_{\varphi_i,\gamma} = (V_{\varphi_i} \times Y) \cap G_\gamma.$$

Then $\{H_{\varphi_i,\gamma} : \gamma \in \varphi_i \text{ and } i = 1, \ldots, n\}$ covers $\overline{F}$.

COROLLARY 3.7 ([11, THEOREM 1.4]). *Let X be a normal strongly κ-compact space and Y a paracompact space with $t(Y) \leq \kappa$. Then $X \times Y$ is normal, and thus collectionwise normal and κ-paracompact.*

PROOF. Let V_i, G_i, H_i and S_i, for $i = 1, 2$, be as in the proof of Corollary 3.5. It remains to prove that $S_i, i = 1, 2$, is open. This is essentially done in Kombarov [11]. Indeed, for example, let $y \in S_1$ such that $y \in \overline{F}$ for some $F \subset Y \setminus S_1$ with $|F| \leq \kappa$. Index F as $F = \{y_\alpha : \alpha \in \kappa\}$. For each $\alpha \in \kappa$ choose $x_\alpha \in \overline{H}_1$ so that $(x_\alpha, y_\alpha) \notin V_1$. Take a neighborhood base $\{V_\gamma : \gamma \in \Gamma\}$ of y. For each $\gamma \in \Gamma$ define $R_\gamma = \{x_\alpha : \alpha \in \kappa$ and $y_\alpha \in V_\gamma\}$. It follows that the collection $\{\overline{R}_\gamma : \gamma \in \Gamma\}$ of compact sets has the finite intersection property. And thus we may pick an $x \in \bigcap\{\overline{R}_\gamma : \gamma \in \Gamma\} \subset \overline{H}_1$. Since $(x, y) \in V_1$, there exists a neighborhood H of x such that $H \times V_{\gamma_0} \subset V_1$ for some $\gamma_0 \in \Gamma$. We can find some $x_\alpha \in H \cap R_{\gamma_0}$. One sees easily that $(x_\alpha, y_\alpha) \in V_1$, a contradiction proving that $S_i, i = 1, 2$, is open.

COROLLARY 3.8. *Let X be a paracompact space with $t(X) \leq \kappa$. Then $X \times \kappa^+$ is collectionwise normal and κ-paracompact.*

COROLLARY 3.9 ([10, THEOREM 2.7]). *Let X be a paracompact space. If $X \times \kappa$ is orthocompact, then it is shrinking, where $cf(\kappa) \leq \omega_1$.*

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PROOF. If $X \times \kappa$ is orthocompact, by [10, Lemma 1.1], X has orthocaliber κ, *i.e.*, if $x \in X$ and $\mathcal{U}$ is a collection of neighborhoods of x with $|\mathcal{U}| = \kappa$, then there exists $\mathcal{V} \in \mathcal{U}$ such that $x \in \mathrm{Int}(\bigcap \mathcal{V})$ with $|\mathcal{V}| = \kappa$. Now let $\mathcal{G} = \{G_\gamma : \gamma \in \Gamma\}$ be any open cover of $X \times \kappa$ and fix $x \in X$. Then for each $\alpha \in \kappa$, there exist $f(\alpha) < \alpha$ and a neighborhood V_α of x such that $V_\alpha \times \big(f(\alpha), \alpha\big] \subset G_\gamma$ for some $\gamma \in \Gamma$. Take $\theta \subset \kappa$ such that $|\theta| = \kappa$ and $V_x = \bigcap\{V_\alpha : \alpha \in \theta\}$ is a neighborhood of x. Then, although $\theta \neq \kappa$, using the Pressing Down Lemma, we can regard V_x as a stable neighborhood of x to $\mathcal{G}$. Hence, the corollary is proved.

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