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KOMLOS LIMITS AND FATOU'S LEMMA IN SEVERAL DIMENSIONS

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ABSTRACT. Using Komlos' Theorem, a sequence decomposition result due to Gaposhkin, and two results due to Artstein, we prove a result concerning the properties of Komlos limits. We then show that a stronger version of Fatou's Lemma in several dimensions can be deduced from Artstein's version of the Lemma. The version of Fatou's Lemma proved here subsumes the most recent version of the Lemma in several dimensions given by Balder.

1. **Introduction.** Using Komlos' Theorem [7], a sequence decomposition result due to Gaposhkin [4], and two results due to Artstein [1], we prove a result concerning the properties of Komlos limits. We then show that a stronger version of Fatou's Lemma in several dimensions can be deduced from Artstein's version of the Lemma [1]. The version of Fatou's Lemma proved here subsumes the most recent version of the Lemma given by Balder [2], as well as the versions given by Schmeidler [8], Hildenbrand and Mertens [6], and Hildenbrand [5].

2. **Preliminaries.** Let (Ω, Σ, μ) denote a finite measure space, $L^1(\Omega, \Sigma, \mu) \equiv L^1$ the space of all equivalence classes of real-valued integrable functions defined on Ω, and $L^1_m(\Omega, \Sigma, \mu) \equiv L^1_m$ the m-fold product of L^1. Thus, if $f \in L^1_m$, then $f = (f^1, \ldots, f^m)$, where $f^i \in L^1$ for $i = 1, \ldots, m$, and $\|f^i\|_1 = \int_\Omega |f^i|\, d\mu$. For $\{f_n\}_n$ and f in L^1_m, $\lim_n \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu$ means $\lim_n \int_\Omega f^i_n\, d\mu = \int_\Omega f^i\, d\mu$, for each i. Inequalities between vectors should be understood as componentwise inequalities.

We will denote by $Ls\big(f_n(\omega)\big)$ the set of all limit points of $\{f_n(\omega)\}_n$, and by $\operatorname{co} Ls\big(f_n(\omega)\big)$ the convex hull of $Ls\big(f_n(\omega)\big)$.

In proving the stronger version of Fatou's Lemma we will use the following generalization of Komlos' Theorem [7].

KOMLOS' THEOREM IN R^m. If $\{f_n\}_n \subset L^1_m$, with $\sup_n \|f^i_n\|_1 < \infty$ for all i, then there is a subsequence $\{f_{nk}\}_k$ and an $f^\wedge \in L^1_m$ such that,

(1) $\sum_{1 \leq j \leq k}(1/k) f_{nj}(\omega) \longrightarrow f^\wedge(\omega)$ a.e. $[\mu]$ as $k \longrightarrow \infty$ (i.e., $\{f_{nk}\}_k$ converges a.e. $[\mu]$ Cesàro to $f^\wedge$),

(2) any further subsequence extracted from $\{f_{nk}\}_k$ also converges a.e. $[\mu]$ Cesàro to $f^\wedge$.

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We will refer to any subsequence satisfying conclusions (1) and (2) of Komlos' Theorem as a K sequence and we will refer to the corresponding Cesàro limit $f^\wedge \in L_m^1$ as the K limit.

3. **Results.** Our first result concerns the properties of K limits.

PROPOSITION 1. *Let $\{f_n\}_n \subset L_m^1$ be a K-sequence with K-limit $f^\wedge \in L_m^1$. Then*
(1) $f^\wedge(\omega) \in \operatorname{co} Ls\{f_n(\omega)\}$ a.e. $[\mu]$, and
(2) there exists an $f^* \in L_m^1$ such that $f^*(\omega) \in Ls\big(f_n(\omega)\big)$ a.e. $[\mu]$ and

$$\int_\Omega f^*\,d\mu = \int_\Omega f^\wedge\,d\mu.$$

Our second result is a stronger version of Fatou's Lemma in several dimensions.

PROPOSITION 2. *Let $\{f_n\}_n \subset L_m^1$ be such that $\sup_n \|f_n^i\|_1 < \infty$ for all i and $\lim_n \int_\Omega f_n\,d\mu$ exists. Without loss of generality, assume that $\{f_n\}_n$ is a K sequence.*

If for any subsequence $\{f_{nk}\}_k$ of $\{f_n\}_n$, $\{\max(0, -\sum_{1\le j\le k}(1/k)f_{nj}\}_k$ is uniformly integrable, then there exists an $f^* \in L_m^1$ such that
(1) $f^*(\omega) \in Ls\big(f_n(\omega)\big)$ a.e. $[\mu]$, and
(2) $\int_\Omega f^*\,d\mu \le \lim_n \int_\Omega f_n\,d\mu$.
If $\{\sum_{1\le j\le k}(1/k)f_{nj}\}_k$ is uniformly integrable, then (2) holds with equality.

REMARK. The most recent version of Fatou's Lemma in several dimensions is due to Balder [2].

BALDER'S LEMMA. *Let $\{f_n\}_n \subset L_m^1$ be such that $\{\max(0, -f_n)\}_n$ is uniformly integrable and $\lim_n \int_\Omega f_n\,d\mu$ exists. Then there exists $f^* \in L_m^1$ such that*
(1) $f^*(\omega) \in Ls\big(f_n(\omega)\big)$ a.e. $[\mu]$, and
(2) $\int_\Omega f^*\,d\mu \le \lim_n \int_\Omega f_n\,d\mu$.
If $\{f_n\}_n$ is uniformly integrable, then (2) holds with equality.

Recall that if $\{f_n\}_n$ is uniformly integrable, then $\sup_n \|f_n^i\|_1 < \infty$ for all i, but the converse is not true in general. Moreover, if $\{\max(0, -f_n)\}_n$ is uniformly integrable, then $\{\max(0, -\sum_{1\le k\le n}(1/n)f_k)\}_n$ is uniformly integrable. However, the converse is not true in general. A similar statement can be made concerning $\{f_n\}_n$ and $\{\sum_{1\le k\le n}(1/n)f_k\}_n$.

4. **Proofs.** First, we will need the following generalization, to m dimensions, of Gaposhkin's [4] sequence decomposition lemma (Lemma C.I).

GAPOSHKIN'S LEMMA. *If $\{f_n\}_n \subset L_m^1$, with $\sup_n \|f_n^i\|_1 < \infty$ for all i, then there is a subsequence $\{f_{nk}\}_k$ such that, for each k, $f_{nk} = g_k + h_k$, where $\{g_k\}_k$ converges weakly [i.e., in $\sigma(L_m^1, L_m^\infty)$] to some $g^0 \in L_m^1$, and $\lim_k h_k(\omega) = 0$ a.e. $[\mu]$.*

We will also need the following results due to Artstein [1]:

ARTSTEIN'S RESULTS. *(i) (Theorem A, Fatou's Lemma). Let* $\{f_n\}_n \subset L^1_m$ *be such that* $\{f_n\}_n$ *is uniformly integrable and* $\lim_n \int_\Omega f_n \, d\mu$ *exists. Then there exists* $f^* \in L^1_m$ *such that*

(1) $f^*(\omega) \in Ls\big(f_n(\omega)\big)$ *a.e.* $[\mu]$*, and*

(2) $\int_\Omega f^* \, d\mu = \lim_n \int_\Omega f_n \, d\mu$.

(ii) (Proposition C). Let $\{g_n\}_n \subset L^1_m$ *be such that* $\{g_n\}_n$ *is uniformly integrable. If* $\{g_n\}_n$ *converges weakly to some* $g^0 \in L^1_m$*, then* $g^0(\omega) \in \operatorname{co} Ls\{g_n(\omega)\}$ *a.e.* $[\mu]$.

PROOF OF PROPOSITION 1. By Gaposhkin's Lemma, we can assume, without loss of generality, that $\{f_n\}_n$ is such that for each $n, f_n = g_n + h_n$, where $\{g_n\}_n$ converges weakly to some $g^0 \in L^1_m$ and $\lim_n h_n(\omega)$ a.e. $[\mu]$. Since $\{f_n\}_n$ is a K sequence with K limit $f^\wedge$, $\{g_n\}_n$ is also a K sequence with K limit $f^\wedge$. Also, $Ls\big(f_n(\omega)\big) = Ls\big(g_n(\omega)\big)$ a.e. $[\mu]$. Applying Artstein's Theorem A to $\{g_n\}_n$, it follows that there exists an $f^* \in L^1_m$ such that $f^*(\omega) \in Ls\big(g_n(\omega)\big)$ a.e. $[\mu]$, and such that

$$\int_\Omega f^* \, d\mu = \lim_n \int_\Omega g_n \, d\mu = \int_\Omega g^0 \, d\mu.$$

Since $\{g_n\}$ converges weakly to g^0, $\{\sum_{1 \le k \le n} (1/n) g_k\}_n$ must also converge weakly to g^0. Thus, since $\{\sum_{1 \le k \le n} (1/n) g_k\}_n$ converges to $f^\wedge$ a.e. $[\mu]$, $g^0 = f^\wedge$ a.e. $[\mu]$. So f^* satisfies (2).

By Artstein's Proposition C, $g^0(\omega) \in \operatorname{co} Ls\{g_n(\omega)\}$ a.e. $[\mu]$. Thus, since $\operatorname{co} Ls\{g_n(\omega)\} = \operatorname{co} Ls\{f_n(\omega)\}$ a.e.$[\mu]$ and $f^\wedge = g^0$ a.e. $[\mu]$, $f^\wedge(\omega) \in \operatorname{co} Ls\{f_n(\omega)\}$ a.e. $[\mu]$, proving (1). ∎

PROOF OF PROPOSITION 2. Let $f^\wedge \in L^1_m$ be the K limit corresponding to the K sequence $\{f_n\}_n$. By part (2) of Proposition 1, there exists an $f^* \in L^1_m$ such that $f^*(\omega) \in Ls\big(f_n(\omega)\big)$ a.e. $[\mu]$ and $\int_\Omega f^* \, d\mu = \int_\Omega f^\wedge \, d\mu$. Now let $\{f_{nk}\}_k$ be any subsequence of $\{f_n\}_n$ such that $\{\max(0, -\sum_{1 \le j \le k} (1/k) f_{nj})\}_k$ is uniformly integrable. Since $\max\big(0, -(1/k)\sum_{1 \le j \le k} f_{nj}(\omega)\big) \longrightarrow \max\big(0, -f^\wedge(\omega)\big)$ a.e. $[\mu]$, as $k \longrightarrow \infty$ we have

$$\int_\Omega \max(0, -f^\wedge) \, d\mu = \lim_k \int_\Omega \max\Big(0, -(1/k) \sum_{1 \le j \le k} f_{nj}\Big) \, d\mu.$$

Moreover,

$$\lim_n \int_\Omega f_n \, d\mu = \lim_k \int_\Omega f_{nk} \, d\mu = \lim_k (1/k) \sum_{1 \le j \le k} \int_\Omega f_{nj} \, d\mu.$$

We have then

$$\begin{aligned}
\lim_n \int_\Omega f_n \, d\mu &= \lim_k \int_\Omega f_{nk} \, d\mu = \lim_k (1/k) \sum_{1 \le j \le k} \int_\Omega f_{nj} \, d\mu \\
&= \liminf_k \int_\Omega \Big[\max\Big(0, (1/k) \sum_{1 \le j \le k} f_{nj}\Big) - \max\Big(0, -(1/k) \sum_{1 \le j \le k} f_{nj}\Big)\Big] \, d\mu \\
&\ge \liminf_k \int_\Omega \Big[\max\Big(0, (1/k) \sum_{1 \le j \le k} f_{nj}\Big)\Big] \, d\mu \\
&\qquad - \limsup_k \int_\Omega \max\Big(0, -(1/k) \sum_{1 \le j \le k} f_{nj}\Big) \, d\mu \\
&\ge \int_\Omega \max(0, f^\wedge) \, d\mu - \int_\Omega \max(0, -f^\wedge) \, d\mu = \int_\Omega f^\wedge \, d\mu = \int_\Omega f^* \, d\mu.
\end{aligned}$$

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Finally, since $(1/k)\sum_{1\le j\le k} f_{nj}(\omega) \longrightarrow f^{\wedge}(\omega)$ a.e. $[\mu]$ as $k \longrightarrow \infty$, it follows that if $\{\sum_{1\le j\le k}(1/k)f_{nj})\}_k$ is uniformly integrable, then

$$\lim_n \int_\Omega f_n\,d\mu = \lim_k \int_\Omega f_{nk}\,d\mu = \lim_k (1/k) \sum_{1\le j\le k} \int_\Omega f_{nj}\,d\mu = \int_\Omega f^{\wedge}\,d\mu = \int_\Omega f^*\,d\mu.$$

■

REMARKS. In Balder [3], using Chacon's Biting Lemma (a result equivalent to Gaposhkin's sequence splitting result), Balder showed that his version of Fatou's Lemma (i.e., Balder [2]) could be deduced from Artstein's version of the Lemma (i.e., Artstein [1]). Here, using Gaposhkin's sequence splitting result [4] and Komlos' Theorem [7], we have shown, in a very direct way, that an even stronger version of Fatou's Lemma can be deduced from Artstein's Lemma.

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