Canad. Math. Bull. Vol. **53** (2), 2010 pp. 286–294
doi:10.4153/CMB-2010-036-5

Orders of π-Bases

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Abstract. We extend the scope of B. Shapirovskii's results on the order of π-bases in compact spaces and answer some questions of V. Tkachuk.

Introduction

The notion of π-base is an essential tool for studying the internal structure of a topological space as well as its external properties (embeddings, functions and the like); this was established primarily in the work of Boris Shapirovskii [6, 7] containing major discoveries. In this paper we attempt to show the full natural scope of his ideas regarding the order of π-bases.

In Section 1, we decipher and refine the method of induction used by Shapirovskii [7, Section 3].

In Section 2, using the results of Section 1, we describe a canonical form for π-bases in regular spaces and prove that canonical π-bases always exist. In Lemma 2.4 we give a characterization of free sequences and with its help we derive a series of new results, starting with the central Theorem 2.6.

Section 3 deals with the natural question as to whether or not the assumptions in our theorems could be further relaxed. We give some examples to the contrary which also solves three problems of V. Tkachuk [8].

The idea for this paper originated from the observation that our Lemma 2.4 could be used in place of final compactness (that is, small $L(X)$), even in the original Shapirovskii argument made for compact spaces.

We have used [2, 3] as general references for definitions and notation. $\mathcal{ON}$ is the class of ordinal numbers. Additions and multiplications are ordinal operations. We write $[\gamma, \delta)$, or $\delta \setminus \gamma$, for $\{\alpha : \gamma \leq \alpha < \delta\}$. We denote by $\mathcal{T}_X$ the family of all non-empty open subsets of a topological space X.

1 Canonical κ-Functions

Definition 1.1 For an infinite cardinal κ, a *canonical κ-function* is a class function $\phi = \phi_\kappa \colon \mathcal{ON} \to [\mathcal{ON} \times \kappa]^{<\omega}$ satisfying the following two conditions:

(i) for every ordinal α, $\phi(\alpha) \subseteq \alpha \times \kappa$,

(ii) for every ordinal δ of the form $\delta = \kappa \cdot \epsilon$ there is $\gamma(\delta) < \delta$ such that

$$\big[[\gamma(\delta), \delta) \times \kappa \big]^{<\omega} \subseteq \phi'' \delta.$$

Received by the editors May 7, 2007.
Published electronically April 6, 2010.
The author's research was supported by NSERC
AMS subject classification: **54A25**, 03E10, 03E75, 54A35.
Keywords: Shapirovskii π-base, point-countable π-base, free sequences, canonical form for ordinals.

Definition 1.2 A *τ-strong canonical* (κ, λ)*-function* is a function $\psi \colon \lambda \to [\lambda \times \kappa]^\tau$ satisfying the following two conditions:

(i) $(\forall \alpha \in \mathrm{dom}(\psi))\ \psi(\alpha) \subseteq \alpha \times \kappa$.
(ii) for every ordinal $\delta \leq \lambda$ with $\mathrm{cf}(\delta) = \kappa^+$ there is $\gamma(\delta) < \delta$ such that

$$\big[\, [\gamma(\delta), \delta) \times \kappa \big]^{\tau} \subseteq \psi''\delta.$$

Definition 1.3 Let κ be an infinite cardinal. Define a class-function $\sigma = \sigma_\kappa \colon \mathcal{ON} \to \mathcal{ON}$ by the following rule:

- $\sigma(0) = 0$,
- $\sigma(1) = \kappa$,
- $\sigma(\alpha + 1) = \sigma(\alpha) + |\sigma(\alpha)|$, for $\alpha > 0$,
- $\sigma(\beta) = \sup\{\sigma(\alpha) : \alpha < \beta\}$, for β limit.

Lemma 1.4 *Every ordinal δ has the following unique σ_κ-normal form:*

$$\delta = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1}) + \Delta,$$

where $n \in \omega$,

$$|\sigma(\alpha_0)| > |\sigma(\alpha_1)| > |\sigma(\alpha_2)| > \cdots > |\sigma(\alpha_{n-2})| > |\sigma(\alpha_{n-1})|, \alpha_{n-1} > 0,$$

and $\Delta < \kappa$.

Proof To visualize, we partition $\mathcal{ON}$ into intervals $[0], [1, \kappa^+), \dots, [\mu, \mu^+), \dots$. This is the finest partition of $\mathcal{ON}$ into intervals that are closed under σ. Then we choose descending α_i from different intervals, excluding the first.

Existence. Since σ is increasing continuous, and $\sigma(1) = \kappa$, if $\delta \geq \kappa$, then $\exists! \alpha_0 > 0$ such that $\sigma(\alpha_0) \leq \delta < \sigma(\alpha_0 + 1)$. Similarly, if $\mathrm{type}(\delta \setminus \sigma(\alpha_0)) \geq \kappa$, then $\exists! \alpha_1 > 0$ such that $\sigma(\alpha_1) \leq \mathrm{type}(\delta \setminus \sigma(\alpha_0)) < \sigma(\alpha_1 + 1)$. Eventually, we get to $\alpha_{n-1} > 0$ (if any, otherwise set $n = 0$) such that

$$\sigma(\alpha_{n-1}) \leq \mathrm{type}(\delta \setminus (\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-2})) < \sigma(\alpha_{n-1} + 1),$$

but now $\mathrm{type}(\delta \setminus (\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1})) < \kappa$. Put

$$\Delta = \mathrm{type}(\delta \setminus (\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1})).$$

Uniqueness is now easily proved by induction on the length of the normal form. It follows that the lexicographic ordering of the σ-normal forms (that is, the ordinal sequences $\langle \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \Delta \rangle$) coincides with the natural order of their values in $\mathcal{ON}$, but we will not need this explicitly. ∎

Definition 1.5 Define a total pressing-down (save for $\gamma(0) = 0$) class-function $\gamma = \gamma_\kappa \colon \mathcal{ON} \to \mathcal{ON}$ as follows. For every ordinal δ with the σ_κ-normal form $\delta = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-2}) + \sigma(\alpha_{n-1}) + \Delta$, set

- $\gamma(\delta) = 0$, if $n = 0$,
- $\gamma(\delta) = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-2})$, otherwise.

Theorem 1.6 *For every infinite cardinal κ there is a canonical κ-function $\phi = \phi_\kappa$.*

Proof *Step 1:* For every ordinal δ with $\Delta = 0$ in its normal form, let

$$\delta' = \gamma(\delta) + \sigma(\alpha_{n-1} + 1)$$

(for $\delta > 0$ this is also $\delta' = \delta + |\sigma(\alpha_{n-1})|$). Then fix a function

$$f_\delta \colon [\delta, \delta') \longrightarrow \left[[\gamma(\delta), \delta') \times \kappa \right]^{<\omega}$$

such that

(i) f_δ is onto, and
(ii) $\forall \xi \in [\delta, \delta')\ f_\delta(\xi) \subseteq \xi \times \kappa$.

This is very easy to arrange, because for every α, $|\sigma(\alpha+1)| = |[\sigma(\alpha), \sigma(\alpha+1))| \geq \kappa$. We may start with an arbitrary surjection mapping $|\sigma(\alpha_{n-1} + 1))|$-many times to every member of the range.

Step 2: Now consider ordinals δ of the form $\delta = \kappa \cdot \epsilon$. These are the same as the just considered ordinals with $\Delta = 0$ in their normal form.

Suppose that we have finitely many functions $h_0, \dots, h_{n-1}$ such that $(\forall i < n)$ $h_i \colon [\delta, \delta + \kappa) \to [(\delta + \kappa) \times \kappa]^{<\omega}$ and $(\forall \xi \in \operatorname{dom}(h_i))\ h_i(\xi) \subseteq \xi \times \kappa$. Then set $H = H[h_0, \dots, h_{n-1}]$ and fix a function with the same domain and co-domain such that $(\forall i)(\forall \xi \in [\delta, \delta + \kappa))\ (\exists \eta \geq \xi)\ H(\eta) = h_i(\xi)$ (and so $H(\eta) = h_i(\xi) \subseteq \xi \times \kappa \subseteq \eta \times \kappa$). In other words, H is a combination of $h_0, \dots, h_{n-1}$ mapping onto the union of their ranges.

Step 3: Finally, define $\phi = \phi_\kappa$ on the ordinal intervals of the form $[\delta, \delta + \kappa)$ with $\delta = \kappa \cdot \epsilon$, simultaneously for all such δ, by the following explicit rule. Find the normal form $\kappa \cdot \epsilon = \delta = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1}) + 0$. Then set $\phi \restriction [\delta, \delta + \kappa) = H[h_0, \dots, h_{n-1}]$, where $h_i = f_{\sigma(\alpha_0)+\sigma(\alpha_1)+\cdots+\sigma(\alpha_i)} \restriction [\delta, \delta + \kappa)$.

Step 4: We are left to check that the function ϕ just defined satisfies Definition 1.1. It is transparent that the first condition is satisfied, and the second is in the following assertion.

Claim 1.7 Suppose that for every δ with $\Delta = 0$ (and $n \geq 0$) in its normal form,

$$\operatorname{ran}(f_\delta) \subseteq \operatorname{ran}(\phi \restriction [\delta, \delta')).$$

Then for every such δ with $n \geq 1$,

$$\left[[\gamma(\delta), \delta) \times \kappa \right]^{<\omega} \subseteq \operatorname{ran}(\phi \restriction [\gamma(\delta), \delta)).$$

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Proof This is straightforward, by induction on n and then by a subinduction on α_{n-1} in the normal form for δ.

The case $\alpha_{n-1} = \beta + 1$ is explicit, and for α_{n-1} a limit ordinal, use

$$\big[[\gamma(\delta), \gamma(\delta) + \sigma(\alpha_{n-1})) \times \kappa\big]^{<\omega} = \bigcup_{\beta<\alpha_{n-1}} \big[[\gamma(\delta), \gamma(\delta) + \sigma(\beta)) \times \kappa\big]^{<\omega}.$$

The equation is true, because $\{\beta < \alpha_{n-1} : \gamma(\gamma(\delta) + \sigma(\beta)) = \gamma(\delta)\}$ is cofinal in α_{n-1}. ■

Theorem 1.8 *If $(\kappa^+)^\kappa = \kappa^+$ and for every cardinal μ with $\kappa^+ \le \mu < \lambda$ we have $\mu^\kappa = \mu$, then there is a κ-strong (κ, λ)-function. Under CH, there is an ω-strong $(\omega, \aleph_\omega)$-function.*

Proof This time consider σ_{κ^+}-normal forms for ordinals $\delta \in \mathcal{ON}$ and the regressive function γ_{κ^+}. Otherwise do, *mutatis mutandis*, as in the proof of Theorem 1.6. It will still be possible to define $f_\delta \colon [\delta, \delta') \twoheadrightarrow [[\gamma(\delta), \delta') \times \kappa]^\kappa$ because for every δ of the form $\delta = \kappa^+ \cdot \epsilon$ we have $|[\delta, \delta')|^\kappa = |[\delta, \delta')| \ge \kappa^+$. ■

2 Shapirovskii π-Bases in Regular Spaces

Recall [2] that $\mathcal{R} \subseteq \mathcal{T}_X$ is a π-base for the topology $\mathcal{T}_X$ of X if and only if $\forall U \in \mathcal{T}_X$ $\exists R \in \mathcal{R}$ with $R \subseteq U$. A family $\mathcal{R} \subseteq \mathcal{T}_X$ is a local π-base for $p \in X$ if and only if $\forall U \in \mathcal{T}_X$ with $p \in U$ $\exists R \in \mathcal{R}$ with $R \subseteq U$. The π-character of a point p in X is the cardinal $\pi\chi(p, X) = \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathcal{T}_X$ is a local π-base for $p\}$ and the π-character of X is $\pi\chi(X) = \sup\{\pi\chi(p, X) : p \in X\}$.

For $P = \{p_\alpha : \alpha < \mu\}$ and $\delta < \mu$, we write $P_\delta = \{p_\alpha : \alpha < \delta\}$ and $P^\delta = \{p_\alpha : \delta \le \alpha < \mu\}$. Then $P = \{p_\alpha : \alpha < \mu\} \subseteq X$ is called *left-separated* if $\overline{P_\delta} \cap P^\delta = \varnothing$, for every $\delta < \mu$ (this means that all initial segments of P are relatively closed in P). It is well known and easy to see that every space X has a dense subspace left-separated in the order-type $d(X)$.

Definition 2.1 Suppose X is a regular topological space with $\pi\chi(X) = \kappa$. Suppose the density of X, $d(X) = \lambda$, is an infinite cardinal and $P = \{p_\alpha : \alpha < \lambda\}$, is a left-separated dense subspace, left-separated as written. Then $\mathcal{S} = \{S_{\alpha,i} : \langle\alpha, i\rangle \in \lambda\times\kappa\} \subseteq \mathcal{T}_X$, or $\mathcal{S}$ together with P, is called a *Shapirovskii π-base for X* if the following conditions are satisfied:

(i) $\{S_{\alpha,i} : i < \kappa\}$ a local π-base for p_α in X,
(ii) $\overline{P_\alpha} \cap \bigcup\{\overline{S_{\beta,i}} : \alpha \le \beta, i < \kappa\} = \varnothing$,
(iii) $(\forall\delta = \kappa\cdot\epsilon)(\forall A \in [[\gamma(\delta), \delta) \times \kappa]^{<\omega})$ if $\bigcap_{a\in A} \overline{S_a} \ne \varnothing$, then $\bigcap_{a\in A} \overline{S_a} \cap \bigcup\{\overline{P_\alpha} : \alpha < \delta\} \ne \varnothing$, and therefore $\bigcap_{a\in A} \overline{S_a} \cap \overline{P_\delta} \ne \varnothing$.

We will also say that $\mathcal{S}$ as above is a *κ-strong Shapirovskii π-base*, if the condition (iii) is replaced by the following:

(iii′) $(\forall\delta = \kappa^+\cdot\epsilon)\big(\forall A \in [[\gamma(\delta), \delta)\times\kappa]^\kappa\big)$ if $\bigcap_{a\in A} \overline{S_a} \ne \varnothing$, then $\bigcap_{a\in A} \overline{S_a} \cap \overline{P_\delta} \ne \varnothing$.

Theorem 2.2 *Every regular space has a Shapirovskii π-base.*

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Proof Let $Q = \{q_\alpha : \alpha < \lambda\}$ be a left-separated dense subspace of X. Define $P = \{p_\alpha : \alpha < \lambda\}$ and $\mathcal{S} = \{S_{\alpha,i} : \langle \alpha, i\rangle \in \lambda \times \kappa\}$ by induction on $\delta < \lambda$.

Suppose at the stage $\delta \geq 0$ we have $P_\delta = \{p_\alpha : \alpha < \delta\}$ and $\mathcal{S} = \{S_{\alpha,i} : \langle \alpha, i\rangle \in \delta \times \kappa\}$. If $\bigcap\{\overline{S_a} : a \in \phi_\kappa(\delta)\} \cap \overline{P_\delta} = \varnothing$ and $\bigcap\{\overline{S_a} : a \in \phi_\kappa(\delta)\} \neq \varnothing$, pick p_δ in $\bigcap\{\overline{S_a} : a \in \phi_\kappa(\delta)\}$. Otherwise, put $p_\delta = q_\xi$, where ξ is the least index of a member of $Q \setminus \overline{P_\delta}$.

Next, p_δ being thus defined, pick a π-base $\mathcal{B}$ for p_δ of size $|\mathcal{B}| \leq \kappa$ and with $\overline{P_\delta} \cap \overline{B} = \varnothing$ for each member B of $\mathcal{B}$, index it as $\{S_{\delta,i} : i \in \kappa\}$. This completes the induction. ■

Theorem 2.3 *Under the cardinal assumptions of Theorem* 1.8, *every regular space with* $\pi\chi(X) \leq \kappa$ *and* $d(X) \leq \lambda$ *has a* κ*-strong Shapirovskii* π*-base.*

Recall [2] that $P = \{p_\alpha : \alpha < \mu\} \subseteq X$ is a *free sequence* in the space X if $\overline{P_\delta} \cap \overline{P^\delta} = \varnothing$, for every $\delta < \mu$. Let $\mathcal{F}(X) = \sup\{|P| : P \text{ is a free sequence in } X\}$.

The following characterization of free sequences parallels Shapirovskii's characterization of discrete sets in [5].[1] It says that small $\mathcal{F}(X)$ can be viewed as a compactness-like reflection property of the space X, and this is precisely what we will need in the sequel.

Lemma 2.4 *Let X be any topological space and κ any infinite cardinal. Then* (i) $\mathcal{F}(X) \leq \kappa$ *if and only if* (ii) *for every* $Y \subseteq X$, *every family* $\mathcal{U} \subseteq T_X$ *such that* $(\forall A \in [Y]^{\leq\kappa})$ $(\exists U \in \mathcal{U})$ $\overline{A} \subseteq U$ *has a subfamily* $\mathcal{V} \subseteq \mathcal{U}$ *of size* $|\mathcal{V}| \leq \kappa$ *covering* Y.

Proof *Sufficiency.* Suppose Y and $\mathcal{U}$ are as in (ii), but (ii) fails, so there is no $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ covering Y. We will pick up a free sequence $P = \{p_\alpha : \alpha < \kappa^+\}$ by induction on $\delta < \kappa^+$. Suppose that at the stage $\delta \geq 0$ we have $P_\delta = \{p_\alpha : \alpha < \delta\}$ and $\{U_\alpha : \alpha < \delta\} \subseteq \mathcal{U}$. Then pick $U_\delta \in \mathcal{U}$ with $\overline{P_\delta} \subseteq U_\delta$ and $p_\delta \in Y \setminus \bigcup_{\alpha \leq \delta} U_\alpha$.

We claim that P is a free sequence. Indeed, for $\delta < \kappa^+$, $\overline{P_\delta} \subseteq U_\delta$ and $P^\delta \cap U_\delta = \varnothing$, whereupon $\overline{P_\delta} \cap \overline{P^\delta} = \varnothing$.

Necessity. Now assume that (i) fails, and there is a free sequence $P = \{p_\alpha : \alpha < \kappa^+\}$. Let $\mathcal{U} = \{X \setminus \overline{P^\delta} : \delta < \kappa^+\}$. Because κ^+ is a regular cardinal, $(\forall A \in [P]^{\leq\kappa})$ $(\exists \delta < \kappa^+)$ $\overline{A} \subseteq \overline{P_\delta} \subseteq X \setminus \overline{P^\delta}$. Therefore, $\mathcal{U}$ is as in (ii) with $Y = P$. Let $\mathcal{V}$ be a subfamily of $\mathcal{U}$ of size $|\mathcal{V}| \leq \kappa$. Then, again by regularity of κ^+, $(\exists \delta < \kappa^+)$ such that $\bigcup \mathcal{V} \subseteq \bigcup_{\gamma \leq \delta}(X \setminus \overline{P^\gamma}) = X \setminus \overline{P^\delta}$. Therefore $\bigcup \mathcal{V}$ is disjoint from P^δ, and thus does not cover P. ■

Recall (see [1]) that a space X is *initially* κ*-compact*, if every cover of cardinality at most κ has a finite subcover, and X is $(\kappa, \kappa^+]$*-compact*, if every cover of X of cardinality κ^+ has a subcover of cardinality κ.

The fact that $t(X) = \mathcal{F}(X)$ in compact spaces is well known, but we will need these weaker covering properties of X as a factor in the relationship between $t(X)$ and $\mathcal{F}(X)$. The following is folklore.

[1] For an important different approach to free sequences, see [10, 11].

Lemma 2.5 *Suppose that X is a regular space. Then*

(i) *X is initially κ-compact and $\mathcal{F}(X) \le \kappa \Rightarrow t(X) \le \kappa$.*
(ii) *X is $(\kappa, \kappa^+]$-compact and $t(X) \le \kappa \Rightarrow \mathcal{F}(X) \le \kappa$.*

Proof (i) Let $Y \subseteq X$ with $Y = \bigcup\{\overline{A} : A \in [Y]^{\le\kappa}\}$ and observe that Y is initially κ-compact. It is sufficient to show that Y is closed. So fix $p \notin Y$. We will find a neighbourhood of p disjoint from Y. For every $A \in [Y]^{\le\kappa}$ find U_A, a neighbourhood of p with $\overline{U_A} \cap \overline{A} = \varnothing$. Then $\mathcal{U} = \{X \setminus \overline{U_A} : A \in [Y]^{\le\kappa}\}$ is a cover of Y as in of Lemma 2.4(ii). Therefore, there is a $\mathcal{V} \subseteq \mathcal{U}$, $|\mathcal{V}| \le \kappa$, $\mathcal{V}$ covers Y, and since Y is initially κ-compact, a finite $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$ which also covers Y. But then $\bigcap\{U_A : X \setminus \overline{U_A} \in \mathcal{W}\}$ is a neighbourhood of p disjoint from Y, as wanted.

(ii) Suppose $P = \{p_\alpha : \alpha < \kappa^+\} \subseteq X$. By tightness, $\overline{P} = \bigcup\{\overline{P_\alpha} : \alpha < \kappa^+\}$. If P were a free sequence in X, then $\mathcal{U} = \{X \setminus \overline{P^\delta} : \delta < \kappa^+\}$ would be an increasing κ^+-cover of X, contradicting $(\kappa, \kappa^+]$-compactness of X. ∎

For a family $\mathcal{R}$ of subsets of X and a point $p \in X$, the *order of p in $\mathcal{R}$* is the cardinal $\operatorname{ord}(p, \mathcal{R}) = |\{R \in \mathcal{R} : p \in R\}|$. The *order of* $\mathcal{R}$ is $\operatorname{ord}(\mathcal{R}) = \sup\{\operatorname{ord}(p, \mathcal{R}) : p \in X\}$. Finally, a family $\mathcal{R}$ is *point-κ* if $\operatorname{ord}(\mathcal{R}) \le \kappa$, *i.e.,* if every point belongs to at most κ members of $\mathcal{R}$.

Theorem 2.6 *Suppose X is a regular initially κ-compact space with $\pi\chi(X) = \kappa$ and no free sequences of length κ^+ (that is, $\mathcal{F}(X) \le \kappa$). Then any Shapirovskii π-base is point-κ.*

Proof Let $\mathcal{S}$ be a Shapirovskii π-base for X, as displayed in the definition, and suppose that $\mathcal{R} \subseteq \mathcal{S}$, $|\mathcal{R}| = \kappa^+$. We must show that $\bigcap \mathcal{R} = \varnothing$.

For some $I \in [\lambda \times \kappa]^{\kappa^+}$, $\mathcal{R} = \{S_{\alpha,i} : \langle \alpha, i\rangle \in I\}$, and so $|\pi_0'' I| = \kappa^+$, where π_0 denotes the projection from the square to the first coordinate, $\pi_0(\langle a, b\rangle) = a$. Pick $\delta \in \lambda$, the least ordinal such that $|(\pi_0'' I) \cap \delta| = \kappa^+$. Then $\operatorname{cf}(\delta) = \kappa^+$. Let $J = I \cap ([\gamma(\delta), \delta) \times \kappa)$. Then $|\pi_0'' J| = \kappa^+$, because $\gamma(\delta) < \delta$.

Let $\mathcal{Q} = \{S_{\alpha,i} : \langle \alpha, i\rangle \in J\} \subseteq \mathcal{R}$. By Lemma 2.5(i), $t(X) \le \kappa$. Since $t(X) \le \kappa < \kappa^+ = \operatorname{cf}(\delta)$,

$$\overline{P_\delta} = \bigcup\{\overline{P_\alpha} : \alpha < \delta\}. \tag{1}$$

Since by the choice of δ, $\pi_0'' J$ is cofinal in δ, $\bigcap\{\overline{Q} : Q \in \mathcal{Q}\} \cap \overline{P_\delta} = \varnothing$. This uses (1) above and Definition 2.1(ii). Therefore, $\mathcal{U} = \{X \setminus \overline{Q} : Q \in \mathcal{Q}\}$ is an open cover of $\overline{P_\delta}$, and $(\forall A \in [\overline{P_\delta}]^{\le\kappa})\ (\exists Q \in \mathcal{Q})$ with $\overline{A} \subseteq X \setminus \overline{Q}$.

Since $\mathcal{F}(X) \le \kappa$, Lemma 2.4 applies (with $Y = \overline{P_\delta}$), and $\exists \mathcal{V} \subseteq \mathcal{U}$, $|\mathcal{V}| \le \kappa$, such that $\mathcal{V}$ is also a cover of $\overline{P_\delta}$. Since $\overline{P_\delta}$ is initially κ-compact, there if a finite $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$ which is a cover of $\overline{P_\delta}$. Say, $\mathcal{W} = \{X \setminus \overline{S_a} : a \in A\}$, for some finite $A \subseteq J$, and so we have $(\bigcap_{a \in A} \overline{S_a}) \cap \overline{P_\delta} = \varnothing$. Since δ with $\operatorname{cf}(\delta) = \kappa^+$ is *a fortiori* of the form $\delta = \kappa \cdot \epsilon$ for some ϵ, by Definition 2.1(iii) this implies that $\bigcap_{a \in A} \overline{S_a} = \varnothing$, and therefore $\bigcap_{a \in A} S_a = \varnothing$. But $\{S_a : a \in A\} \subseteq \mathcal{Q} \subseteq \mathcal{R}$, and so $\bigcap \mathcal{R} = \varnothing$. ∎

Corollary 2.7 *Every regular countably compact space with countable π-character and no uncountable free sequences has a point-countable π-base.*

This corollary gives a partial positive answer to [8, Problem 4.4]. The core of our argument, in fact, also proves the following variations.

Corollary 2.8 *Every regular initially $t(X)^+$-compact space with*

$$\kappa = \max\{\pi\chi(X), t(X)\}$$

has a point-κ π-base.

Corollary 2.9 *Every first-countable initially ω_1-compact regular space has a point-countable π-base.*

Corollary 2.10 *Let $\kappa = \max\{\pi\chi(X), t(X)\}$. If $d(X) \leq \kappa^+$, then X has a point-κ π-base.*

This is, in essence, Tkachuk's Theorem 3.2 [8].

Corollary 2.11 *Suppose X is a regular space which is initially $\mathcal{F}(X)$-compact. Let $\kappa = \max\{\mathcal{F}(X), \pi\chi(X)\}$. Then X has a point-κ π-base. In fact, any Shapirovskii π-base is point-κ.*

Corollary 2.12 *Every regular countably compact space with no uncountable free sequences has a point-$\pi\chi(X)$ π-base.*

In the presence of a nice cardinal arithmetic, the covering restrictions can be altogether omitted, but only when the density of the space is not too large.

Theorem 2.13 *Suppose that κ and λ are cardinals such that $(\kappa^+)^\kappa = \kappa^+$ and for every μ with $\kappa^+ \leq \mu < \lambda$, $\mu^\kappa = \mu$. Then every regular space X with $\pi\chi(X) \leq \kappa$, $\mathcal{F}(X) \leq \kappa$ and $d(X) \leq \lambda$ has a point-κ π-base.*

Corollary 2.14 *Under CH, every regular space with $\mathcal{F}(X) = \omega$ and $d(X) \leq \aleph_\omega$ has a point-$\pi\chi(X)$ π-base.*

Corollary 2.15 *Under CH, every regular first-countable space with $d(X) \leq \aleph_\omega$ and no uncountable free sequences has a point-countable π-base.*

3 Counterexamples to Weaker Assumptions

Shapirovskii [7, Theorem 3.2][2] provided the main tool for proving that a space does not have a point-κ π-base, which we will need in the following weak form:

($\star$) If $\max\{\kappa^+, s(X)\} < d(X)$, then X does not have a point-κ π-base.

This criterion is used in every example below.

Example 3.1 There is in, *ZFC*, a first-countable zero-dimensional left-separated space X such that $d(X) = |X| \geq (\beth_\omega)^+$, $hL(X) = \beth_\omega$, and hence $s(X) = \beth_\omega$.

[2] If we define $m(X) = \min\{\sup\{(\mathrm{ord}(p, \mathcal{R}))^+ : p \in X\} : \mathcal{R} \text{ is a } \pi\text{-base for } X\}$, it states that $d(X) \leq m(X) \cdot s(X)$.

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By ($\star$), X cannot have a point-countable π-base. This is one of the celebrated generalized L-spaces of Stevo Todorčević (see [9, Theorem 16]). This example gives a negative answer to [8, Problem 4.1].

Example 3.2 There is in *ZFC* a zero-dimensional first-countable space left-separated in the order-type $\mathfrak{b}$ with no discrete subspace of size $\mathfrak{b}$. It has a point-countable π-base if and only if $\mathfrak{b} = \omega_1$.

This is another L-space of Stevo Todorčević [9]. In the case of $\mathfrak{b} = \omega_1$, whatever the value of $\mathcal{F}(X)$ is, the space has a point-countable π-base by Corollary 2.10.

Example 3.3 Consistently, relative to the existence of a supercompact cardinal, there is a first-countable hereditarily Lindelöf (hence with $\mathcal{F}(Y) = s(Y) = \omega$) space Y left-separated in the order-type $\omega_2 = 2^\omega$ without a point-countable π-base.

There is in *ZFC* a zero-dimensional space X left-separated in the order-type $d(X) = |X| = 2^{\beth_{\omega_1}}$ with $\chi(X) = \omega_1$ and $s(X) = hL(X) = \beth_{\omega_1}$. This is still another generalized L-space of Todorčević [9]. By ($\star$), it does not have a point-ω_1 π-base.

By a result of Magidor [4, Corollary 3] $V \models 2^{\beth_{\omega_1}} = (\beth_{\omega_1})^{++}$ is consistent, relative to the existence of a supercompact cardinal.[3] Force with $Fn(\omega, \beth_{\omega_1})$ from V as a ground model. This will preserve $(\beth_{\omega_1})^+$ (in the form of $(\aleph_1)^{V[G]}$) and all cardinals above it, while collapsing all cardinals below it to $\aleph_0$. By a routine computation (counting names and using the generic collapsing function), $2^{\aleph_0} = \aleph_2$ in $V[G]$. We claim that the space X from the ground model V will possess in the generic extension $V[G]$ all the properties of Y stated above. As usual, the topology of Y is understood to be generated in $V[G]$ from $\mathcal{T}_X$ as a base.

All topological base properties (*i.e.,* those which can be formulated in terms of a base and are invariant under choosing a base) of X will be inherited by Y. These include regularity, left-separated structure, and "p is a complete accumulation point of A," provided A is in V (even if the cardinal $|A|^V$ is collapsed). The only property that needs an argument is the hereditary Lindelöfness of Y. It is sufficient to check that a set of size $\aleph_1$ in the extension contains a point of complete accumulation of itself. Now, a set A of cardinality $\aleph_1$ in $V[G]$ has a name $\mathring{A}$ in V indexed by the ordinals in $(\beth_{\omega_1})^+$. Since our forcing poset has size $\beth_{\omega_1}$, there is a single condition in G which evaluates in V $(\beth_{\omega_1})^+$-many points of $\mathring{A}$, say a set B. Now a complete accumulation point $b \in B$ of B in X (which exists by $hL(X) < |B|$ in V), as we remarked, is the same for $B \subseteq A$ in $V[G]$.

This space does not have a point-countable π-base for the cardinal arithmetic reason alone (since it has $\chi(Y) = \mathcal{F}(Y) = \aleph_0$ and $|Y| = \aleph_2 \leq \aleph_\omega$). This shows that the cardinal assumption in Corollary 2.15 (and *a fortiori* in Theorem 2.13) is necessary. This example also gives a negative answer to [8, Problems 4.3, 4.6].

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[3]The author would like to thank Moti Gitik for the following comment: "It is possible to replace the supercompact cardinal with the strong cardinal, as was done by Segal and Merimovitch".

https://doi.org/10.4153/CMB-2010-036-5 Published online by Cambridge University Press

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