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ON CONVERGENCE OF ITERATIONS FOR FIXED POINTS OF REPULSIVE TYPE

BY

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1. **Introduction.** This paper deals with the convergence of iterations of the equation

$$u = \phi(u) \tag{1.1}$$

with $u_0 \in R^n$, both as a root and a point of repulsion. Here u and ϕ are real n-vectors and ϕ is continuously differentiable with respect to u. By taking u^0 as an initial approximation to u_0 in a neighbourhood of u_0 we define the sequence of iterates $\{u^{k+1}\}$ by

$$u^{k+1} = \phi(u^k), \quad k = 1, 2, 3, \ldots. \tag{1.1'}$$

For the sake of convenience we shall, sometimes, call the equation (1.1) an iteration.

Ostrowski [2, p. 156] gave a criterion for divergence of an iteration for a fixed point of repulsive type but gave no criterion for convergence. He contended that sometimes an iteration which does not satisfy the divergence criterion may converge. He gave an example in support of his contention (see [2, p. 165]). We will give two criteria, first for convergence and the second for divergence, of an iteration. The iteration in the example given by Ostrowski satisfies the conditions of our first criterion. Thus our results sharpen, in some sense, the results of Ostrowski.

2. **Some preliminary results.** Let $x=u-u_0$. Then (1.1) is reduced to

$$x = Ax+f(x), \tag{2.1}$$

where x and f are n-vectors and A is the constant $n\times n$ matrix $(\partial\phi/\partial u)|_{u=u_0}$. It is assumed that ϕ is such that in a neighbourhood of $x=0$, $|f|\le m\,|x|^2$ for some $m>0$, where the norm, $|x|$, of $x=(x_1, x_2, \ldots, x_n)$ is defined by $|x|=\sqrt{\sum_{i=1}^n x_i^2}$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A and

$$|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_k| < 1 < |\lambda_{k+1}| \le \cdots \le |\lambda_n|.$$

There exists a real nonsingular matrix P (see [1, p. 340]) such that $PAP^{-1}=B$ where

$$B = \begin{pmatrix} D_1 & 0 & 0\ldots 0 \\ 0 & D_2 & 0\ldots 0 \\ \vdots & & \vdots \\ 0 & & D_m \end{pmatrix} \tag{2.2}$$

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the D_j are real square matrices and the other elements of B are zero. Each D_j is either of the form

$$(2.3)\qquad \begin{pmatrix} \lambda_j & 0 & 0\ldots 0 \\ \gamma & \lambda_j & 0\ldots 0 \\ 0 & \gamma & \lambda_j\ldots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \;\gamma\; \lambda_j \end{pmatrix}$$

where γ may be taken as any nonnegative number, or else D_j is of the form

$$(2.4)\qquad \begin{pmatrix} S_j & 0 & 0\ldots 0 \\ \gamma E_2 & S_j & 0\ldots 0 \\ 0 & \gamma E_2 & S_j\ldots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0\;\gamma E_2\; S_j \end{pmatrix}$$

where

$$S_j = \begin{pmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{pmatrix}$$

and E_2 is the unit 2×2 matrix. The matrix (2.3) may contain a single row and column. It is associated, of course, with the eigenvalue λ_j, while (2.4) is associated with the conjugate eigenvalues $\alpha_j \pm \beta_j$. Assume that we can write $B=B_1+B_2$, where $B_1=\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ and $B_2=\begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix}$ with all the k eigenvalues of the $k\times k$ matrix T having moduli less than unity and $n-k$ eigenvalues of W have moduli greater than unity. It can easily be seen that each diagonal element of $B_2^*B_2$ is either zero or is a real number greater than unity, and all the other elements are either zero or contain γ as a factor. Here * denotes the conjugate transpose.

Writing $y=Px$ and $Pf(P^{-1}y)=g(y)$ we get from (2.1)

$$(2.5)\qquad y = By+g(y).$$

We let $y=y_1+y_2$ where the first k components of y_1 are the same as those of y and the last $n-k$ components zero. Similarly we let $g=g_1+g_2$. Then (2.5) can be written as

$$(2.6)\qquad y_1 = B_1y_1+g_1(y_1+y_2),$$

together with

$$(2.7)\qquad y_2 = B_2y_2+g_2(y_1+y_2).$$

The function

$$h(y_1, y_2) = (B_2-E_n)y_2+g(y_1+y_2)$$

is such that $h(0,0)=0$ and $\det(\partial h/\partial y_2)$ $(=\det(B_2-E_n))\neq 0$ at $y=0$, where E_n is

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the $n\times n$ identity matrix. Therefore by the implicit function theorem (see [3, p. 147]) there exists a k-dimensional neighbourhood V_0 of the origin and a unique vector valued function ψ defined on V_0 such that

(i) ψ is continuously differentiable,
(ii) $\psi(0)=0$, and
(iii) $h(y_1, \psi(y_1))\equiv 0$ for $y_1 \in V_0$.

The function ψ is continuously differentiable because of the fact that $(\partial h/\partial y_1)$ (0, 0) exists continuously. Thus $y_2=\psi(y_1)$ defines a k-dimensional manifold M in R^n such that $B_2y_2+g_2(y_1+y_2)\equiv y_2$ for all points on M. Using the fact that $\det(\partial h/\partial y_1)|_{y_1=0}$ is 0, it is easy to check that $J_\psi=\det(\partial\psi/\partial y_1)|_{y_1=0}$ is zero.

When talking about iteration we will read (1.1) as $u^{m+1}=\phi(u^m)$, $m=0, 1, 2, \ldots$, with u^0 in a neighbourhood of u_0, and (2.6) as $y_1^{m+1}=B_1y_1^m+g_1(y_1^m+y_2^m)$. A similar remark holds for (2.7). Now we give, in the next section, the two criteria in the form of theorems.

3. Theorem 3.1. *Given any $\varepsilon>0$ satisfying $|\lambda_i|+\varepsilon<1$, $i=1, 2, \ldots, k$ there exists a $\delta>0$ such that for each $y^0 \in V_0$ with $|y^0|<\delta$, the sequence of iterates $\{y^m\}$ defined by* (2.5) *converges to* 0, *provided $\{y_2^m\}$ is obtained implicitly from* (2.7) *for each m.*

Proof. We can find a constant m^* such that $|g(y)|\le m^*|y|^2$ for all y in a neighbourhood V of the origin. Choose δ such that $m^*\delta\le\varepsilon/8$ and $|y_2|=|\psi(y_1)|\le|y_1|$ for all $y\in V$ with $|y|<\delta$. This is possible because ψ is continuously differentiable, $\psi(0)=0$ and $J_\psi=0$.

Now for $|y^0|<\delta$ and $y_2^0=\psi(y_1^0)$ we show that for each m

$$|y_1^m| \le \delta(|\lambda_k|+\varepsilon)^m.$$

This result is clearly true for $m=1$, hence we assume that $|y_1^n|\le\delta(|\lambda_k|+\varepsilon)^n$. Then

$$y_1^{n+1} = B_1y_1^n+g_1(y_1^n+y_2^n),$$

and hence

$$\begin{aligned}|y_1^{n+1}| &\le |B_1|\,|y_1^n|+|g(y_1^n+y_2^n)|\\ &\le |B_1|\,|y_1^n|+m^*(|y_1^n|+y_2^n|)^2\\ &\le |B_1|\,|y_1^n|+4m^*|y_1^n|^2\\ &\le (|\lambda_k|+\varepsilon/2)\delta(|\lambda_k|+\varepsilon)^n+4m^*\delta^2(|\lambda_k|+\varepsilon)^n\\ &\le \delta(|\lambda_k|+\varepsilon)^{n+1},\end{aligned}$$

where we have used the fact that $|B_1|\le|\lambda_k|+\varepsilon/2$ and that $(|\lambda_k+\varepsilon)^{2n}\le(|\lambda_k|+\varepsilon)^n$, where the norm $|B_1|$, of $B=(b_{ij})$ is defined by $|B_1|=\sum_{i,j=1}^n|a_{ij}|$. Thus by induction the result is true for any positive integer m. Consequently the sequence $\{y_1^m\}$ approaches zero as $m\to\infty$. Since ψ is continuous the sequences $\{\psi(y_1^m)\}$ and $\{y_2^m\}$ also approach zero as $m\to\infty$.

THEOREM 3.2. *There exists a* $\delta>0$ *and a neighbourhood of the origin* $B_\delta(0)=\{y\in R^n:|y|<\delta\}$ *such that for any initial vector* $y^0\in B_\delta(0)$, $y^0\notin M$, *the sequence* $\{y^k\}$ *of iterates defined by* (2.5) *does not converge to* 0; *in fact* $y^k\notin M$ *and* $|y_2^{k+1}|>|y_2^k|$ *for each k.*

Proof. The det $(\partial g/\partial y)|_{y=0}$ is zero and $J_\psi=0$, so given $\varepsilon>0$ we can choose a $\delta>0$ such that

$$|g(y)-g(z)| \le \varepsilon\,|y-z|$$

and

$$|\psi(y_1)-\psi(z_1)| \le \varepsilon\,|y_1-z_1|$$

whenever $|y|\le\delta$ and $|z|\le\delta$. Let $\xi^{m+1}=y_2^{m+1}-\psi(y_1^m)$. Then

$$\xi^{m+1} = B_2y_2^m+g_2(y_1^m+y_2^m)-\psi(y_1^{m+1}). \tag{2.8}$$

Now consider the identity

$$0 = B_2\psi(y_1^m)+g_2(y_1^m+\psi(y_1^m))-\psi(y_1^m) \tag{2.9}$$

(since $h(y_1^m,\psi(y_1^m))\equiv 0$).
Subtracting (2.9) from (2.8) we get

$$\begin{aligned}\xi^{m+1} &= B_2\xi^m+g_2(y_1^m+y_2^m)-g_2(y^m+\psi(y_1^m))+\psi(y_1^m)-\psi(y_1^{m+1}),\\ \xi^{m+1*}\xi^{m+1} &= |\xi^{m+1}|^2 = \xi^{m*}B_2^*B_2\xi^m+\xi^{m*}B_2^*\{g_2(y_1^m+y_2^m)\\ &\quad -g_2(y_1^m+\psi(y_2^m))+\psi(y_1^m)-\psi(y_1^{m+1})\}\\ &\quad +\{g_2^*(y_1^m+y_1^m)-g_2^*(y_1^m+\psi(y_1^m))\\ &\quad +\psi^*(y_1^m)-\psi^*(y_1^{m+1})\}\cdot\{g_2(y_1^m+y_2^m)\\ &\quad -g_2(y_1^m+\psi(y_1^m))+\psi(y_1^m)-\psi(y_1^{m+1})\}.\end{aligned}$$

We note that $|g_2(y_1^m+y_2^m)-g_2(y_1^m+\psi(y_1^m))| \le\varepsilon\,|\xi^m|$. In view of the nature of the elements of $B_2^*B_2$ we obtain the following estimate:

$$\begin{aligned}(|\xi^{m+1}|)^2 &\ge \sum_{i=k+1}^{n}|\lambda_i|^2|\xi_i|^2-\beta(\gamma,\varepsilon)\\ &> (|\xi^m|)^2-\beta(\gamma,\varepsilon),\end{aligned}$$

where ξ_i is the ith component of the vector ξ^m and β is a real number depending on γ and ε. Since γ can be made as small as we please by a suitable choice of the matrix P and ε is arbitrary, we can make β as small as we please. Consequently $\xi^{m+1}\ne 0$ whenever $\xi^m\ne 0$. Therefore if we do not choose the initial approximation on the manifold no subsequent approximation will lie on the manifold.

It is easy to verify, by considering $y_2^{m+1*}y_2^{m+1}$, that $|y_2^{m+1}|>|y_2^m|$ for all m for which $|y^m|$ remains small. The proof of the theorem is complete.

4. In the notation of Ostrowski [2] the solid angle L is given by

$$L = \{y\in R^n:|y_{k+1}|+\cdots+|y_n| > |y_1|+\cdots+|y_k|\}.$$

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This is indeed the case if $y \notin m$.

The following example was considered by Ostrowski:

$$x = (x/2)[1-(x^2+y^2)]$$
$$y = (y/2)[3+(x^2+y^2)].$$

Here u_0 is (0, 0) and

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

which has one of the eigenvalues greater than unity. The manifold m is the line $y=0$. If the initial approximation (x^0, y^0) is in a suitable neighbourhood, say $V=\{(x, y): x^2+y^2<1\}$, and is such that $y^0=0$, then implicit iteration is automatically built in. Thus the conditions of the Theorem 3.1 are satisfied and the iteration converges.

REMARK. It may be noted, as was pointed out by the referee, that Theorems 3.1 and 3.2, together with the preliminary results, go through with slight modifications under the weaker assumption on $f(x)$, namely that $|f(x)| \leq m\,|x|^{1+\eta}$, where $\eta > 0$.

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