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CHARACTERIZATIONS OF THE BOREL σ-FIELDS OF THE FUZZY NUMBER SPACE

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Abstract

In this paper, we characterize Borel σ-fields of the set of all fuzzy numbers endowed with different metrics. The main result is that the Borel σ-fields with respect to all known separable metrics are identical. This Borel field is the Borel σ-field making all level cut functions of fuzzy mappings from any measurable space to the fuzzy number space measurable with respect to the Hausdorff metric on the cut sets. The relation between the Borel σ-field with respect to the supremum metric d_∞ is also demonstrated. We prove that the Borel field is induced by a separable and complete metric. A global characterization of measurability of fuzzy-valued functions is given via the main result. Applications to fuzzy-valued integrals are given, and an approximation method is presented for integrals of fuzzy-valued functions. Finally, an example is given to illustrate the applications of these results in economics. This example shows that the results in this paper are basic to the theory of fuzzy-valued functions, such as the fuzzy version of Lebesgue-like integrals of fuzzy-valued functions, and are useful in applied fields.

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1. Introduction and preliminaries

The measurability of fuzzy-valued functions is a basic concept for fuzzy set theory and applications. As in the classical case, measurability of these functions is also basic in fuzzy analysis. It is closely related to the establishment of fuzzy Lebesgue-like integrals, which is a key step for applications of fuzzy analysis.

Throughout this paper, (X, Ω) denotes a measurable space. A *multifunction* $F : X \to R$ is a function from X to the set $\mathcal{P}(R)$ of all nonempty subsets of the real

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line R. The function F is called *measurable* if, for every closed subset C of R, $F^{-1}(C) = \{x \in X \mid F(x) \cap C \neq \emptyset\} \in \Omega$. Recall also that a mapping F from X to a metric space Y is called measurable if for each open set $U \subset Y$, $F^{-1}(U) \in \Omega$. It is well-known that if F is compact valued, the measurability of a multifunction F is equivalent to the measurability of a function $F : X \to K(R)$, where $K(R)$ is the metric space of all compact subsets of R endowed with the Hausdorff metric H [2, Theorem 3.2]. Fuzzy mappings are natural generalizations of multifunctions. First, we recall the definition of measurability of a fuzzy mapping.

Definition 1.1. A fuzzy mapping $\tilde{F} : X \to R$ is a function from X to the set $\tilde{\mathcal{P}}(R)$ of all normal fuzzy subsets of R. If $\tilde{F} : X \to R$ is a fuzzy mapping and C is a subset of R, then $\tilde{F}^{-1}(C)$ denotes the function from X to $[0, 1]$ defined by $\tilde{F}^{-1}(C)(x) = \sup_{r \in C} \tilde{F}(x)(r)$ for each $x \in X$. The fuzzy mapping $\tilde{F}$ is called measurable if, for every closed subset C of R, $\tilde{F}^{-1}(C)$ is measurable as a function from X to $[0, 1]$.

Definition 1.2. A fuzzy set u on R is called a fuzzy number if it satisfies the following conditions:

(1) u is normal, that is, there exists $x \in R$ such that $u(x) = 1$;
(2) u is upper semicontinuous;
(3) u is fuzzy convex, that is, $u(rx + (1 - r)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and $r \in I$;
(4) $\operatorname{supp} u = \overline{\{x \in R \mid u(x) > 0\}}$ is compact, where "$^-$" denotes the closure of the set under consideration.

Let E^1 denote the set of all fuzzy numbers. Following Kim's [8] notation, the set of all fuzzy sets on R^n which satisfies only (1), (2) and (4) is denoted by $\mathcal{F}(R^n)$. Zhang [10] discussed the measurability of fuzzy-valued functions by using supporting functions of fuzzy numbers, and gave characterizations for measurability of these functions in terms of graphs and supporting functions. Kim [8] showed that the measurability of a fuzzy mapping from X with values in $\mathcal{F}(R^n)$ is equivalent to the measurability of the function regarded as a unary function from X to $\mathcal{F}(R^n)$ endowed with the Skorokhod metric.

In this paper, we consider the case of all metrics defined on the set of fuzzy numbers, and discuss the measurability of a fuzzy function from X to E^1 endowed with all known metrics defined on the set of all fuzzy numbers. Let $K(R)$ and $I(R)$ denote the families of nonempty compact subsets and nonempty closed intervals of the real line R, respectively. Note that $K(R)$ is metrized by the Hausdorff metric

$$H(A, B) = \max\Big\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\Big\}$$

and it is both complete and separable with respect to the Hausdorff metric H, and $I(R)$ is closed in $K(R)$.

For a fuzzy set u in R, we denote the cut sets of u by

$$L_\alpha u = \begin{cases} \{x \mid u(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1, \\ \operatorname{supp} u & \text{if } x = 0, \end{cases}$$

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where $\operatorname{supp} u = \{x \mid u(x) > 0\}$ is the support of u. It is easy to see that $u \in E^1$ if and only if $L_\alpha u \in I(R)$ for each $\alpha \in [0, 1]$. For $u \in E^1$, let $L_\alpha u = [L_\alpha^- u, L_\alpha^+ u]$.

Let $u, v \in E^1$; for each $\alpha \in [0, 1]$, define

$$H(L_\alpha u, L_\alpha v) = \max\{|L_\alpha^- u - L_\alpha^- v|, |L_\alpha^+ u - L_\alpha^+ v|\}.$$

For $u \in E^1$, the endograph of u is defined by

$$\operatorname{end}(u) = \{(x, \alpha) \in R \times [0, 1] \mid u(x) \geq \alpha\},$$

while the endograph metric between u and v is defined as follows:

$$D'(u, v) = H(\operatorname{end}(u), \operatorname{end}(v)).$$

It is well-known that (E^1, D') is a separable but not a complete metric space.

Theorem 1.3 [1]. *A fuzzy mapping F is measurable if and only if the composition $L_\alpha \circ F$ is measurable for all $\alpha \in [0, 1]$.*

Lemma 1.4 [2]. *Let $I(R)$ be a metric space endowed with the Hausdorff metric H. For $u \in E^1$, we define $F_u : [0, 1] \to I(R)$ by $F_u(\alpha) = L_\alpha u$. Then F_u has right-limit on $[0, 1)$ and is right-continuous at 0.*

Let $\overline{\bigcup_{\beta>\alpha} L_\beta u} = L_{\alpha^+} u$. The right-limit of F_u in $K(R)$ at $\alpha \in [0, 1)$ is just $L_{\alpha^+} u = [L_{\alpha^+}^-, L_{\alpha^+}^+]$. Let $j_u(\alpha) = H(L_\alpha u, L_{\alpha^+} u)$; then F_u is continuous at α if and only if $j_u(\alpha) = 0$, that is, $L_\alpha u = L_{\alpha^+} u$. Let $J(u) = \{\alpha \mid j_u(\alpha) > 0\}$; then $J(u)$ is at most countable.

Remark 1.5. Since F_u is continuous at 0, it follows that $J(u) \subset (0, 1)$.

2. Main results

In this section, we prove that if a fuzzy mapping $\tilde{F}$ is E^1-valued, then the measurability of $\tilde{F}$ is equivalent to the measurability of it regarded as a function $\tilde{F} : X \to E^1$, where E^1 is considered as the metric space endowed with any known separable metrics defined on it existing in the literature. We first consider the endograph metric D'. To this end, we need to characterize the Borel σ-field of E^1. Unless otherwise stated, we assume that the spaces $I(R)$ and E^1 are considered as metric spaces endowed with the Hausdorff metric H and the endograph metric D', respectively.

Lemma 2.1. *For $\alpha \in I$, the cut set application L_α can be considered as a function from E^1 to $K(R)$, defined by $u \mapsto L_\alpha u$. Then, for $\alpha \in (0, 1)$, L_α is continuous at $u \in E^1$ if and only if $\alpha \notin J(u)$.*

Proof. We prove by contradiction that the condition is necessary, that is, for $u \in E^1$, if $\alpha_0 \in J(u)$, we prove that L_α is not continuous at u. By definition, $L_{\alpha_0} u \neq L_{\alpha_0^+} u$, which yields

$$L_{\alpha_0}^- u < L_{\alpha_0^+}^- u \quad \text{or} \quad L_{\alpha_0}^+ u > L_{\alpha_0^+}^+ u.$$

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Without loss of generality, let us suppose that $L^-_{\alpha_0}u < L^-_{\alpha_0^+}u$. For $n \in N$ such that $\alpha_0 > 1/n$, define

$$u_n(x) = \begin{cases} \alpha_0 - \dfrac{1}{n} + \dfrac{1}{n} \times \dfrac{x - L^-_{\alpha_0-1/n}u}{L^-_{\alpha_0^+}u - L^-_{\alpha_0-1/n}u}, & x \in [L^-_{\alpha_0-1/n}u, L^-_{\alpha_0^+}u], \\ u(x), & \text{otherwise.} \end{cases}$$

Then $u_n \in E^1$ and $u_n \to u(D')$, but $L^-_{\alpha_0}u_n = L^-_{\alpha_0^+}u \neq L^-_{\alpha_0}u$. So, $L_{\alpha_0}u_n \nrightarrow L_{\alpha_0}u$, that is, L_{α_0} is not continuous at u.

To prove the sufficiency of the condition, let $\alpha_0 \notin J(u)$. If L_{α_0} is not continuous at u, then there exists a sequence $\{u_n\}$ in E^1 such that $u_n \to u(D')$, but there is an $\epsilon_0 > 0$ such that $H(L_{\alpha_0}u_n, L_{\alpha_0}u) \geq 2\epsilon_0$. Without loss of generality, let us assume that $|L^-_{\alpha_0}u_n - L^-_{\alpha_0}u| \geq 2\epsilon_0, n = 1, 2, \ldots$. Since $\alpha_0 \notin J(u)$, for the above ϵ_0, there exists $\delta > 0$ such that

$$H(L_\alpha u, L_{\alpha_0}u) \leq \epsilon_0 \quad \text{for } |\alpha - \alpha_0| \leq \delta. \tag{2.1}$$

We now have two cases.

Case (i). There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$L^-_{\alpha_0}u_{n_k} - L^-_{\alpha_0}u \geq 2\epsilon_0.$$

To simplify the notation, the subsequence $\{u_{n_k}\}$ is denoted by $\{u_n\}$.

Let $P = (L^-_{\alpha_0+\delta}u, \alpha_0 + \delta)$, so $P \in \text{end}(u)$. The two lines $y = \alpha_0$ and $x = L^-_{\alpha_0}u + 2\epsilon_0$ separate the plane into four parts: the left-upper part, the right-upper part, the left-lower part and the right-lower part of the plane. Hence, $\text{end}(u_n)$ disjoints from the interior of the left-upper part of the plane. However, the distances from P to the two lines $y = \alpha_0$ and $x = L^-_{\alpha_0}u + 2\epsilon_0$ are δ and $L^-_{\alpha_0}u + 2\epsilon_0 - L^-_{\alpha_0+\delta}u$, respectively.

By inequality (2.1), $L^-_{\alpha_0+\delta}u - L^-_{\alpha_0}u \leq \epsilon_0$, so $L^-_{\alpha_0}u + 2\epsilon_0 - L^-_{\alpha_0+\delta}u \geq \epsilon_0$; thus, $d(P, \text{end}(u_n)) \geq \min\{\delta, \epsilon_0\}$ and, therefore,

$$D'(u_n, u) = H(\text{end}(u_n), \text{end}(u)) \geq d(P, \text{end}(u_n)) \geq \min\{\delta, \epsilon_0\}.$$

This contradicts $u_n \to u(D')$.

Case (ii). There exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that

$$L^-_{\alpha_0}u - L^-_{\alpha_0}u_{n_l} \geq 2\epsilon_0.$$

As in case (i), we also use $\{u_n\}$ to denote its subsequence $\{u_{n_l}\}$.

Let $Q = (L^-_{\alpha_0}u_n, \alpha_0)$, so $Q \in \text{end}(u_n)$. As in case (i), the two lines $y = \alpha_0 - \delta$ and $x = L^-_{\alpha_0}u - \epsilon_0$ separate the plane into four parts. Then $\text{end}(u)$ disjoints from the interior of the left-upper part of the plane. However, the distances from Q to the two lines $y = \alpha_0 - \delta$ and $x = L^-_{\alpha_0}u - \epsilon_0$ are δ and $L^-_{\alpha_0}u - \epsilon_0 - L^-_{\alpha_0}u_n$, respectively. So, $L^-_{\alpha_0}u - \epsilon_0 - L^-_{\alpha_0}u_n \geq \epsilon_0$ and $d(Q, \text{end}(u)) \geq \min\{\epsilon_0, \delta\}$; therefore,

$$D'(u_n, u) = H(\text{end}(u_n), \text{end}(u)) \geq d(Q, \text{end}(u)) \geq \min\{\delta, \epsilon_0\}.$$

This also contradicts that $u_n \to u(D')$, which completes the proof. □

LEMMA 2.2. *Let $\mathcal{B}_{D'}$ denote the Borel σ-field of E^1 with respect to the endograph metric D', that is, $\mathcal{B}_{D'}$ is the Borel σ-field generated by the set of all open sets with respect to the endograph metric. Then the class $\mathcal{B}_{D'}$ coincides with the smallest σ-field of subsets of E^1 for which the maps $L_\alpha : u \to L_\alpha u$ are measurable for all $\alpha \in [0, 1]$.*

PROOF. Let $\mathcal{B}$ be the smallest σ-field of subsets of E^1 for which the maps $L_\alpha : u \to L_\alpha u$ are measurable for all $\alpha \in [0, 1]$. Now we prove that for each $\alpha_0 \in (0, 1)$, L_{α_0} is measurable with respect to $\mathcal{B}_{D'}$. It is well-known in analysis that the Borel σ-field of $K(R)$ is generated by all mappings f, continuous and bounded on $K(R)$, so it is enough to prove that $f(L_{\alpha_0})$ is a pointwise limit of a sequence of continuous maps on E^1 with respect to the endograph metric.

First, we show that for each $\epsilon > 0$, $g_\epsilon(u) = \int_{\alpha_0-\epsilon}^{\alpha_0} f(L_\alpha u)\, d\alpha$ is continuous on E^1. If $u_n \to u(D')$, then, from Lemma 2.1, $L_\alpha u_n \to L_\alpha u$ for $\alpha \notin J(u)$. Since $J(u)$ is at most countable, we obtain from the Lebesgue bounded convergence theorem that $g_\epsilon(u_n) \to g_\epsilon(u)$. Thus, g_ϵ is continuous on E^1. It follows from left-continuity of $L_\alpha u$ as a function of α that $g_\epsilon(u_n)/\epsilon \to f(L_{\alpha_0} u)$ for each $\alpha_0 \in (0, 1)$, as $\epsilon \to 0$. Hence, $f(L_{\alpha_0})$ is a pointwise limit of a sequence of continuous maps on E^1. In other words, L_{α_0} is measurable with respect to $\mathcal{B}_{D'}$; therefore, $\mathcal{B} \subset \mathcal{B}_{D'}$.

Now we prove the reverse inclusion $\mathcal{B}_{D'} \subset \mathcal{B}$. Since the endograph metric is weaker than the Skorokhod metric, it follows that $\mathcal{B}_{D'} \subset \mathcal{B}_s$, where $\mathcal{B}_s$ is the Borel σ-field of E^1 with respect to the Skorokhod metric. By a result of Kim [8], we know that $\mathcal{B}_s \subset \mathcal{B}$. Therefore, $\mathcal{B}_{D'} \subset \mathcal{B}$, which completes the proof. □

LEMMA 2.3. *Let a fuzzy mapping $\tilde{F} : X \to R$ be E^1-valued; then $\tilde{F}$ is measurable if and only if it is measurable when considered as a function from X to the metric space E^1 endowed with the endograph metric D'.*

PROOF. By Theorem 1.3, a fuzzy mapping $\tilde{F}$ is measurable if and only if for each $\alpha \in [0, 1]$, $L_\alpha \tilde{F}$ is a measurable multifunction or, equivalently, measurable when considered as a function from X to the metric space $K(R)$ endowed with the Hausdorff metric H. Thus, the result follows immediately from Lemma 2.2. □

Now we consider the relation between the Borel σ-field $\mathcal{B}$ and the supremum metric d_∞. Note that the topology on E^1 generated by the metric d_∞ is not separable. The following example shows that the Borel σ-field with respect to the metric d_∞ is different from that generated by the set of all open (closed) sets with respect to the metric d_∞.

EXAMPLE 2.4. Define $\tilde{F} : I \to E^1$ as follows:

$$\tilde{F}(t)(x) = \begin{cases} 1, & x = 0, \\ t, & x \in (0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

where $I = [0, 1]$. For each $t_0 \in I$, clearly, $d_\infty(\tilde{F}(t), \tilde{F}(t_0)) = 1$ for $t \neq t_0$. Thus, $\tilde{F}$ is not measurable with respect to the supremum metric d_∞. The reason is as follows.

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Since the subspace topology on $\tilde{F}(I)$ with respect to the supremum metric is the discrete topology, the generated Borel field is the power set of $\tilde{F}(I)$. Also, since $\tilde{F}$ is injective, each subset of I is the reverse image of a Borel set of $\tilde{F}(I)$ under $\tilde{F}$. Since there exists no Borel set (in fact, there exists no Lebesgue measurable set) in I, $\tilde{F}$ is not measurable.

On the contrary, note that for each fixed $\alpha \in I$,

$$L_\alpha \circ \tilde{F}(t) = \begin{cases} [0,1], & \alpha \in [0,t], \\ \{1\}, & \text{otherwise.} \end{cases}$$

Thus, $L_\alpha \circ \tilde{F} : I \to I(R)$ is measurable, since

$$(L_\alpha \circ \tilde{F})^{-1}(\{[0,1]\}) = [0,\alpha], (L_\alpha \circ \tilde{F})^{-1}(\{1\}) = (\alpha, 1]$$

are measurable sets. Thus, each $L_\alpha \circ \tilde{F}$ is measurable and hence $\tilde{F}$ is measurable.

In contrast with Example 2.4, we have the following result.

Theorem 2.5. *Let $\mathcal{B}^*$ be the Borel σ-field on E^1 generated by the set of all open balls with respect to the metric d_∞. Then we have $\mathcal{B} = \mathcal{B}^*$.*

Proof. First, for $u \in E^1$ and $\epsilon > 0$, we consider

$$\overline{B_{d_\infty}(u,\epsilon)} = \{v \in E^1 \mid d_\infty(u,v) \le \epsilon\}.$$

Let $\Gamma = [0,1] \cap Q$, the set of all rational numbers in the unit interval. For each $\alpha \in \Gamma$, let

$$\overline{B_\alpha} = \{v \in E^1 \mid H(L_\alpha(u), L_\alpha(v)) \le \epsilon\};$$

then $\overline{B}_\alpha \in \mathcal{B}$, since $\overline{B}_\alpha = L_\alpha^{-1}(\overline{B_H(L_\alpha(u),\epsilon)})$. Thus,

$$\overline{B_{d_\infty}(u,\epsilon)} = \bigcap_{\alpha\in\Gamma} \overline{B_\alpha)} \in \mathcal{B}.$$

Hence, $\mathcal{B}^* \subseteq \mathcal{B}$.

Conversely, consider

$$F_\alpha : E^1 \to I(R).$$

Note that $\mathcal{B}$ is the smallest Borel σ-field on E^1 making all F_α measurable. For $\alpha \in \Gamma$ and $\epsilon > 0$,

$$F^{-1}(\overline{B(L_\alpha(u),\epsilon)} = \{v \in E^1 \mid H(L_\alpha u, L_\alpha v) \le \epsilon\};$$

thus,

$$\bigcap_{\alpha\in\Gamma} F^{-1}(\overline{B(L_\alpha(u),\epsilon)}) = \overline{B_{d_\infty}(u,\epsilon)} \in \mathcal{B}^*.$$

This shows that $\mathcal{B} = \mathcal{B}^*$ and the proof is complete. □

Example 2.4 and Theorem 2.5 show that $\mathcal{B}^* \neq \mathcal{B}_{d_\infty}$. This phenomenon of course results from the fact that the metric d_∞ is not separable.

From Lemma 2.3 and the corresponding result in Kim [8], we have the following corollary.

Corollary 2.6. *If ρ is a metric on E^1 such that ρ is topologically finer than D' but coarser than d_s, then $\mathcal{B}_\rho = \mathcal{B}$, where d_s is the Skorokhod metric defined by Kim* [8].

Remark 2.7. By a result of Fan [4], we have the following topological inclusions on the topologies on E^1 generated by various known metrics:

$$\mathcal{T}_{D'} \subset \mathcal{T}_{d_p} \subset \mathcal{T}_D \subset \mathcal{T}_{d_s},$$

where D is the sendograph metric and d_p is the d_p metric for $p \geq 1$. For the definition of the sendograph metric and the L_p metric, the reader is suggested to refer to the work of Diamond and Kloeden [3]. For the definition of the Skorokhod metric and related properties, please refer to the work of Joo and Kim [6, 8]. Here $\mathcal{T}_\rho$ is the topology generated by the metric ρ for any metric ρ.

Hence, we have the following equalities on Borel σ-fields on E^1, since all topologies considered are separable:

$$\mathcal{B}_{D'} = \mathcal{B}_{d_p} = \mathcal{B}_D = \mathcal{B}_{d_s}.$$

Combining the above results, we have the following theorem, which is the main result of this paper.

Theorem 2.8. *On the fuzzy number space E^1, we have the following equalities on the Borel σ-fields generated by all known metrics defined on it and the measurability of E^1-valued functions:*

$$\mathcal{B} = \mathcal{B}_{D'} = \mathcal{B}_{d_p} = \mathcal{B}_D = \mathcal{B}_{d_s} = \mathcal{B}^* \subset \mathcal{B}_{d_\infty}.$$

Joo and Kim [6] proved that the Skorokhod metric d_s is topologically equivalent to the modified Skorokhod metric d_s^*; the latter is a complete metric, so it is both complete and separable. Thus, we have the following corollary.

Corollary 2.9. *A fuzzy mapping F defined on a measurable space (X, Ω) is measurable if and only if F is a single-valued measurable function with respect to the separable and complete metric d_s^*, so the Borel σ-field $\mathcal{B}$ is actually the Borel σ-field on E^1 generated by the complete and separable metric d_s^*.*

3. Applications

Let $F : X \to R$ be a fuzzy mapping, that is, F is a mapping from a measurable space (X, Ω) to E^1. From the definition of measurability, to check the measurability of F, one needs to check the measurability of all real functions $L_\alpha^- \circ F$ and $L_\alpha^+ \circ F$ for all $\alpha \in [0, 1]$; this is actually infeasible in practice.

By Theorem 2.8, to check the measurability of F, it suffices to check the measurability of a single-valued function $F : X \to E^1$ with respect to any separable metric mentioned in Theorem 2.8 or to check the measurability of the inverse images of balls with respect to the complete but nonseparable metric d_∞; this may simplify the checking of measurability considerably. Since the fuzzy number set E^1 with

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respect to different metrics can be embedded into a Banach space or topological vector space [3], Theorem 2.8 may enable us to incorporate the study of measurability or even integrals of fuzzy-valued functions into the more general setting of abstract vector-valued functions. The following theorem is a result along this line of thought.

THEOREM 3.1. *Let $F : X \to R$ be a fuzzy mapping, where (X, Ω) is a measurable space. Then F is measurable if and only if for each $u \in E^1$, the function $g_u : X \to [0, \infty)$ defined by $g_u(x) = d(u, F(x))$ is measurable with respect to any of the separable metrics mentioned in Theorem 2.8 (or with respect to $\mathcal{B}^*$ in the case of the metric d_∞).*

PROOF. For each $u \in E^1$, consider $h : E^1 \to [0, \infty)$ defined by $h(v) = d(u, v)$; it is well-known from topology that h is a continuous function; thus, it is also measurable. Clearly, $g_u = h \circ F$ is the composition of two measurable functions, so g_u is measurable.

Conversely, if all g_u are measurable, for each open set $V \subset E^1$, by separability of (E^1, d), we can assume that $V = \bigcup_{n=1}^{\infty} B_d(u_n, r_i)$ is a countable union of open balls in E^1 with respect to the metric d (note that this step of the proof can also be replaced by an open set which is a union of countable open balls with respect to the metric d_∞); thus,

$$f^{-1}(V) = \bigcup_{n=1}^{\infty} g_{u_n}^{-1}([0, r_n)) \in \Omega,$$

which completes the proof. □

Let $f : (X, \Omega, \mu) \to E^1$ be a function. Consider the integral of f. By Theorem 2.8, if f is measurable, then, for each α, $L_\alpha \circ f$ is measurable; this is equivalent to the fact that $L_\alpha \circ f$ is measurable for $\alpha \in C$, for some countable dense subset of [0,1]. The integrability of $L_\alpha \circ f$ for $\alpha \in C$ ensures the integrability of f.

Let

$$A_\alpha = \int_\Omega L_\alpha \circ f \, d\mu. \tag{3.1}$$

Note that I_α is a closed interval on the real line and it is anti-monotone with respect to α, so the integral of f is

$$A = \bigvee_{\alpha \in C} \alpha \chi_{A_\alpha}. \tag{3.2}$$

Since C is infinite, to compute the integral of f by (3.2), one needs to compute A_α by using (3.1) for infinitely many α, which is impossible in practice. To compute the integral effectively, for example, we can use the endograph metric D' and Theorem 2.8, since D' has the following finite approximation property.

THEOREM 3.2 [5]. *For $u, v \in E^1$, suppose that for a fixed $\varepsilon > 0$, $C = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite subset of I such that $H(C, I) \le \varepsilon$ and, for $\alpha \in C$, $H(u_\alpha, v_\alpha) \le \varepsilon$. Then*

$$D'(u, v) \le \sqrt{2}\varepsilon. \tag{3.3}$$

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By using Theorem 3.2, we present the following simple procedure to numerically calculate the integral of a fuzzy-valued function f to any designated accuracy.

Algorithm 1 Algorithm of the integral for a fuzzy-valued function

For a measurable fuzzy-valued function $f(X, \Omega, \mu) \to E^1$ and $\varepsilon > 0$.
STEP 1. Take finite levels $C \subset I$ such that $H(C, I) \leq \varepsilon$.
STEP 2. For each $\alpha \in C$, calculate the numerical value A'_α of the integral $L_\alpha \circ f$ such that $H(A_\alpha, A'_\alpha) \leq \varepsilon$; this step can be done by using methods in numerical analysis.
STEP 3. Let $A' = \bigvee_{\alpha \in C} a \cdot A'_\alpha$.
Then A' is an approximation of the integral A of f. By (3.3), the accuracy is shown as

$$D'(A, A') \leq \sqrt{2}\varepsilon.$$

The above illustration shows that fuzzy-valued integrals can be quite easily computed by using Theorem 2.8 and Algorithm 1. Like integrals in analysis, fuzzy-valued integrals are widely used in various applications related to fuzzy sets. The following is a simple example of an economic application given by Luo and Han [9]. Note that in the reference, no computation method is given as to the final result of the integral, which makes the model difficult to use. By using Algorithm 1, the outcome integral can be calculated efficiently, thus making the economic model practical.

EXAMPLE 3.3. Suppose that n firms have demands for some resource S. Let $\lambda \in I$ be the degree of constraint for the firms, which may be taken as a function of the resource. Note that $\lambda = 0$ means free constraint, $\lambda = 1$ means the most strict constraint, that is, the values of the resource amounts are all 0 and $\lambda \in (0, 1)$ means something in between.

If λ is fixed, then the λ constraint is a hard constraint; for moving $\lambda \in [0, 1]$, it means a soft constraint. If $\lambda \in [0, 1]$, $u_\lambda \subset R^+$ means demand for resource S. Based on practical meaning, it is reasonable to assume that $u \in E^1$, where u is called the membership function of the firm for the resource.

If an economic system is composed of n firms, the demand of the ith firm is $u^i, i = 1, 2, \ldots, n$. The total soft demand of the economic system is $\sum_{i=1}^{n} u^i$ (where the sum is the addition of the fuzzy numbers).

For a big economic system, there might be many firms. Thus, we cannot compute the total demand by considering only the individual firms. Also, a big economic system has the characteristic that the survival of some firms may be of negligible effect, whereas the survival of some other firms may be fatal for the system. Also, some firms may not be significant individually, but they may be very important as an entity; hence, it is ideal to describe the economic system by a measure space (X, Ω, μ), where X is the set of all firms, Ω is a σ-algebra on X, representing the possible combinations of the individual firms to be considered and μ is a finite measure on (X, Ω). In general, $\mu(X) = 1$, that is, μ is a probability measure on (X, Ω).

To study the asymptotic behaviour of the economic system when the number of firms is very big, we may consider the case when X is an infinite set. For $x \in X$, let $F(x)$ be a soft demand for S; then $F(x)$ is a fuzzy-valued mapping. The total soft demand of the economic system for S is

$$A = \int_X F(x)\,d\mu,$$

which can be calculated by using Algorithm 1.

4. Concluding remarks

Butnariu [1] studied the measurability of a function $f : (S, \Omega) \to R^n$, which is an upper semicontinuous and compact support-valued fuzzy function. Let $\mathcal{F}(R^n)$ denote the set of all fuzzy sets on R^n which are normal, upper semicontinuous and compact supported. Butnariu raised the following questions.

(i) Kaleva [7] has proved that $\mathcal{B}^*\mathcal{F}(R^n) \subset \mathcal{BF}(R^n)$ in the setting of all fuzzy sets. Is the converse inclusion true?

(ii) Under what conditions is the measurability of a function $F : (S, \Omega) \to R^n$, which is $\mathcal{F}(R^n)$ valued, equivalent to the measurability of F with respect to the metric d_p?

Our results in this paper can be regarded as answers to the above two questions in the case of fuzzy-valued mappings. Theorem 2.8 and Corollary 2.9 show that the measurabilities of fuzzy number valued functions are the same with respect to all known separable metrics on the fuzzy number sets E^1. Thus, we can freely use any separable metric to establish a fuzzy version of Lebesgue-like integrals for fuzzy-valued functions, since their measurabilities are all equivalent.

As illustrated by Algorithm 1 and Example 3.3, and considering the importance of the measurability concept in real analysis, we believe that the results in this paper can be applied to the study of fuzzy functions not only confined to integrals but also to other problems like differentiation. For related work in these directions, we refer to the classical works of Diamond and Kloeden [3] and Kaleva [7]. As shown in Section 3, the results of this paper can make fuzzy-valued functions much more useful in applied sciences.

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