COMPOSITIO MATHEMATICA

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Compositio Math. **154** (2018), 1593–1632.

doi:10.1112/S0010437X18007170

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

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Abstract

Let Y be a compact Kähler normal space and let $\alpha \in H^{1,1}_{BC}(Y)$ be a Kähler class. We study metric properties of the space $\mathcal{H}_\alpha$ of Kähler metrics in α using Mabuchi geodesics. We extend several results of Calabi, Chen, and Darvas, previously established when the underlying space is smooth. As an application, we analytically characterize the existence of Kähler–Einstein metrics on $\mathbb{Q}$-Fano varieties, generalizing a result of Tian, and illustrate these concepts in the case of toric varieties.

Introduction

Let Y be a compact Kähler manifold and $\alpha_Y \in H^{1,1}(Y,\mathbb{R})$ a Kähler class. The space $\mathcal{H}_{\alpha_Y}$ of Kähler metrics ω_Y in α_Y can be seen as an infinite dimensional Riemannian manifold, whose tangent spaces $T_{\omega_Y}\mathcal{H}_{\alpha_Y}$ can all be identified with $\mathcal{C}^\infty(Y,\mathbb{R})$. Mabuchi has introduced in [Mab87] an L^2-metric on $\mathcal{H}_{\alpha_Y}$, by setting

$$\langle f,g\rangle_{\omega_Y} := \int_Y f\, g\, \frac{{\omega_Y}^n}{V_{\alpha_Y}},$$

where $n = \dim_{\mathbb{C}} Y$ and $V_{\alpha_Y} = \int_Y {\omega_Y}^n = \alpha_Y^n$ denotes the volume of α_Y.

Mabuchi studied the corresponding geometry of $\mathcal{H}_{\alpha_Y}$, showing, in particular, that it can formally be seen as a locally symmetric space of non-positive curvature. Semmes [Sem92] re-interpreted the geodesic equation as a complex homogeneous equation, while Donaldson [Don99] strongly motivated the search for smooth geodesics through its connection with the uniqueness of constant scalar curvature Kähler metrics.

In a series of remarkable works [Che00, CC02, CT08, Che09, CS12], Chen and his collaborators have studied the metric and geometric properties of the space $\mathcal{H}_{\alpha_Y}$, showing in particular that it is a path metric space (a non-trivial assertion in this infinite-dimensional setting). A key step from [Che00] has been the production of $\mathcal{C}^{1,\bar{1}}$-geodesics, which turn out to minimize the intrinsic distance d. Very recently, such a regularity result was improved by Chu *et al.* [CTW17]: they showed that geodesics are $C^{1,1}$. It follows from the work of Lempert and Vivas [LV13], Darvas and Lempert [DL12], and Ross and Witt-Nyström [RW15] that one cannot expect better regularity, but for the toric setting.

The metric study of the space $(\mathcal{H}_{\alpha_Y}, d)$ has been recently pushed further by Darvas [Dar17b, Dar17c, Dar15]. He characterized there the metric completion of $(\mathcal{H}_{\alpha_Y}, d)$ and showed that such a completion is non-positively curved in the sense of Alexandrov. He also introduced several

Received 21 August 2016, accepted in final form 19 December 2017, published online 19 July 2018.
2010 Mathematics Subject Classification 53C55 (primary), 32W20, 53C25 (secondary).
Keywords: Kähler metrics, Monge–Ampère equation, Mabuchi distance.

Finsler-type metrics on $\mathcal{H}_{\alpha_Y}$, which turn out to be quite useful (see [DR17, BBJ15]). For each $p \geqslant 1$, we set

$$d_p(\phi_0, \phi_1) := \inf\{\ell_p(\phi) \mid \phi \text{ is a path joining } \phi_0 \text{ to } \phi_1\}, \quad \forall \phi_0, \phi_1 \in \mathcal{H}_{\omega_Y}, \tag{1}$$

where

$$\ell_p(\phi) := \int_0^1 |\dot{\phi}_t|_p \, dt = \int_0^1 \left(\int_Y |\dot{\phi}_t|^p \, \mathrm{MA}(\phi_t) \right)^{1/p} dt,$$

and $\mathrm{MA}(\phi_t) := (\omega_Y + dd^c \phi_t)^n / V_{\alpha_Y}$. The goal of this article is to extend these studies to the case when the underlying space has singularities.

From now on, let Y be a compact Kähler normal space and $\alpha_Y \in H^{1,1}_{BC}(Y)$ a Kähler class, where $H^{1,1}_{BC}(Y)$ denotes the Bott–Chern cohomology space. We fix a base point ω_Y representing α_Y and work with the space of Kähler potentials

$$\mathcal{H}_{\omega_Y} := \{\phi \in C^\infty(Y, \mathbb{R}) : \omega_Y + dd^c \phi \text{ is a Kähler form}\}.$$

Our first main result extends the main results of [Che00] and [Dar15, Theorem 1], as follows.

Theorem A.

(i) $(\mathcal{H}_{\omega_Y}, d_p)$ *is a metric space.*

(ii) $d_p(\phi_0, \phi_1) = (\int_Y |\dot{\phi}_0|^p \, \mathrm{MA}(\phi_0))^{1/p} = (\int_Y |\dot{\phi}_1|^p \, \mathrm{MA}(\phi_1))^{1/p}$, $\forall \phi_0, \phi_1 \in \mathcal{H}_{\omega_Y}$.

As we are going to discuss, in Remark 1.11, the singularities of Y prevent us from defining the distance d_p as in (1). We instead work on a resolution of Y and there define d_p as a limit of path length metrics. We refer to Definition 1.10 and Remark 1.14 for the precise definition of d_p.

Following [Dar17c, Dar15] we then study the metric completion of the space $(\mathcal{H}_{\alpha_Y}, d_p)$ and establish the following generalization of [Dar15, Theorem 2].

Theorem B. *Let Y be a projective normal variety and assume that ω_Y is a Hodge form. The metric completion of $(\mathcal{H}_{\omega_Y}, d_p)$ is a geodesic metric space, which can be identified with the finite-energy class $(\mathcal{E}^p(Y, \omega_Y), I_p)$.*

Finite-energy classes have been introduced in [GZ07] and further studied in [BEGZ10, BBGZ13]; we recall their definition in § 2. The Mabuchi geodesics can be extended to finite-energy geodesics, which are still metric geodesics. A key technical tool here is Theorem 3.6, which compares d_p and I_p, where

$$I_p(\phi_0, \phi_1) := \left(\int_Y |\phi_0 - \phi_1|^p \left[\frac{\mathrm{MA}(\phi_0) + \mathrm{MA}(\phi_1)}{2} \right] \right)^{1/p}.$$

This is a natural quantity which allows one to define the 'strong topology' on $\mathcal{E}^p(Y, \omega_Y)$.

The metric completion of $(\mathcal{H}_{\alpha_Y}, d)$ has been considered by Streets in his study of the Calabi flow [Str16] and also plays an important role in recent works by Berman *et al.* [BBJ15] and Berman *et al.* [BDL16]. There is no doubt that the extension to the singular setting will play a leading role in subsequent applications. We illustrate this here by generalizing Tian's analytic criterion [Tia97, PSSW08], using results in [BBEGZ] and an idea in [DR17].

Theorem C. *Let (Y, D) be a log Fano pair. It admits a unique Kähler–Einstein metric if and only if there exists $\varepsilon, M > 0$, such that, for all $\phi \in \mathcal{H}_{\mathrm{norm}}$,*

$$\mathcal{F}(\phi) \leqslant -\varepsilon d_1(0, \phi) + M.$$

https://doi.org/10.1112/S0010437X18007170 Published online by Cambridge University Press

Here, $\mathcal{F}$ is a functional whose critical points are Kähler–Einstein potentials (§ 5) and $\mathcal{H}_{\mathrm{norm}}$ is the set of potentials in $\mathcal{H}_{\omega_Y}$ normalized such that the supremum is 0. This result has been independently obtained by Darvas [Dar17a] using a different approach.

Our results should also be useful in analyzing more generally constant scalar curvature Kähler (cscK) metrics on mildly singular varieties (see, for example, the recent construction by Arezzo and Spotti of cscK metrics on crepant resolutions of Calabi–Yau varieties with non-orbifold singularities [AS16]).

A way to establish these results is to consider a resolution of singularities $\pi : X \to Y$ and to work with the space $\mathcal{H}_\omega$ of potentials associated with the form $\omega = \pi^*\omega_Y$. All these results actually hold in the more general setting when ω is merely a semi-positive and big form (i.e. $\int_X \omega^n > 0$). We approximate $\mathcal{H}_\omega$ by spaces of Kähler potentials $\mathcal{H}_{\omega+\varepsilon\omega_X}$ and show that the most important metric properties of $(\mathcal{H}_{\omega+\varepsilon\omega_X}, d_\varepsilon)$ pass to the limit.

The organization of the paper is as follows. Section 1 starts with a recap of Mabuchi geodesics and metrics. Theorem A is proved in § 1.2, where we develop a low-regularity approach for understanding geodesics by approximation. We introduce in § 2 classes of finite-energy currents and compare their natural topologies with the one induced by the Mabuchi distances in § 3. We study finite-energy geodesics in § 4 and prove Theorem B. We finally prove Theorem C in § 5.

1. The space of Kähler currents

Let (Y, ω_Y) be a compact Kähler normal space of dimension n. It follows from the definition of $H^{1,1}_{BC}(Y)$ (see, for example, [BEG13, Definition 4.6.2]) that any other Kähler metric on Y in the same Bott–Chern cohomology class of ω_Y can be written as

$$\omega_\phi = \omega_Y + dd^c\phi,$$

where $d = \partial + \overline{\partial}$ and $d^c = (1/2i\pi)(\partial - \overline{\partial})$. Let $\mathcal{H}_{\omega_Y}$ be the space of *Kähler potentials*

$$\mathcal{H}_{\omega_Y} = \{\phi \in C^\infty(Y, \mathbb{R}); \omega_\phi = \omega + dd^c\phi > 0\}.$$

This is a convex open subset of the Fréchet vector space $C^\infty(Y) := C^\infty(Y, \mathbb{R})$ and thus itself a Fréchet manifold, which is, moreover, parallelizable:

$$T\mathcal{H}_{\omega_Y} = \mathcal{H}_{\omega_Y} \times C^\infty(Y).$$

For any $\phi \in \mathcal{H}_{\omega_Y}$, each tangent space $T_\phi\mathcal{H}_{\omega_Y}$ is identified with $C^\infty(Y)$.

As two Kähler potentials define the same metric when (and only when) they differ by an additive constant, we set

$$\mathcal{H}_{\alpha_Y} = \mathcal{H}_{\omega_Y}/\mathbb{R},$$

where $\mathbb{R}$ acts on $\mathcal{H}_{\omega_Y}$ by addition. The set $\mathcal{H}_{\alpha_Y}$ is therefore the space of Kähler metrics on Y in the cohomology class $\alpha_Y := \{\omega_Y\} \in H^{1,1}_{BC}(Y)$.

In the whole article we fix $\pi : X \to Y$ a resolution of singularities and set $\omega = \pi^*\omega_Y$, $\alpha = \pi^*\alpha_Y$. Since α is no longer Kähler, we fix ω_X a Kähler form on X and set

$$\omega_\varepsilon := \omega + \varepsilon\omega_X$$

for $\varepsilon > 0$. We will study the geometry and the topology of the spaces

$$\mathcal{H}_\alpha = \pi^*\mathcal{H}_{\alpha_Y} \quad \text{and} \quad \mathcal{H}_\omega = \pi^*\mathcal{H}_{\omega_Y}$$

by approximating them by the spaces $\mathcal{H}_{\alpha_\varepsilon}, \mathcal{H}_{\omega_\varepsilon}$, where

$$\mathcal{H}_{\omega_\varepsilon} := \{\varphi \in C^\infty(X,\mathbb{R}); \omega_\varepsilon + dd^c\varphi > 0\} \quad \text{and} \quad \alpha_\varepsilon := \{\omega_\varepsilon\}.$$

All the properties that we are going to establish actually hold for cohomology classes α that are merely *semi-positive* and *big* (not necessarily the pull-back of a Kähler class under a desingularization).

Our analysis will focus on the ample locus of α.

Definition 1.1. The ample locus $\mathrm{Amp}(\alpha)$ of α is the Zariski open set of those points $x \in X$, such that α can be represented by a positive closed $(1,1)$-current that is a smooth positive form near x.

We then let $\mathcal{H}_\omega$ denote the space of potentials $\varphi \in C^\infty(X,\mathbb{R})$ such that ω_φ is a Kähler form in $\mathrm{Amp}(\alpha)$. In our main case of interest, i.e. when $\alpha = \pi^*\alpha_Y$ for some Kähler class α_Y on a normal space Y, the ample locus

$$\mathrm{Amp}(\alpha) = \pi^{-1}(Y^{\mathrm{reg}})$$

is the preimage of the set of regular points of Y.

1.1 The Riemannian structure

1.1.1 *Mabuchi geodesics.*

Definition 1.2 [Mab87]. The *Mabuchi metric* is the L^2 Riemannian metric on $\mathcal{H}_\omega$. It is defined by

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_X \psi_1 \psi_2 \frac{(\omega + dd^c\varphi)^n}{V_\alpha},$$

where $\varphi \in \mathcal{H}_\omega$, $\psi_1, \psi_2 \in C^\infty(X)$, and $(\omega + dd^c\varphi)^n / V_\alpha$ is the volume element, normalized so that it is a probability measure. Here, $V_\alpha := \alpha^n = \int_X \omega^n$.

In the following, we shall also use the notation $\omega_\varphi := \omega + dd^c\varphi$ and

$$\mathrm{MA}(\varphi) := V_\alpha^{-1} \omega_\varphi^n.$$

Geodesics between two points φ_0, φ_1 in $\mathcal{H}_\omega$ correspond to the extremals of the energy functional

$$\varphi \mapsto H(\varphi) = \frac{1}{2} \int_0^1 \int_X (\dot\varphi_t)^2 \, \mathrm{MA}(\varphi_t)\, dt,$$

where $\varphi = \varphi_t$ is a smooth path in $\mathcal{H}_\omega$ joining φ_0 and φ_1. The geodesic equation is formally obtained by computing the Euler–Lagrange equation for this energy functional (with fixed end points). It is given by

$$\ddot\varphi \, \mathrm{MA}(\varphi) = \frac{n}{V_\alpha} \, d\dot\varphi \wedge d^c\dot\varphi \wedge \omega_\varphi^{n-1}. \tag{2}$$

We are interested in the boundary value problem for the geodesic equation: given φ_0, φ_1, two distinct points in $\mathcal{H}_\omega$, can one find a path $(\varphi(t))_{0 \leqslant t \leqslant 1}$ in $\mathcal{H}_\omega$ which is a solution of (2) with end points $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$?

For each path $(\varphi_t)_{t \in [0,1]}$ in $\mathcal{H}_\omega$, we set

$$\varphi(x, t + is) = \varphi_t(x), \quad x \in X, \quad t + is \in S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\};$$

i.e. we associate with each path (φ_t) a function φ on the complex manifold $M = X \times S$, which only depends on the real part of the strip coordinate: we consider S as a Riemann surface with boundary and use the complex coordinate $z = t + is$ to parametrize the strip S. Set $\omega(x, z) := \omega(x)$.

Semmes observed [Sem92] that the path φ_t is a geodesic in $\mathcal{H}_\omega$ if and only if the associated function φ on $X \times S$ is a ω-psh solution of the homogeneous complex Monge–Ampère equation

$$(\omega + dd^c_{x,z}\varphi)^{n+1} = 0. \tag{3}$$

This motivates the following.

Definition 1.3. The function

$$\varphi = \sup\{u\,;\, u \in \mathrm{PSH}(M, \omega) \text{ and } u \leqslant \varphi_{0,1} \text{ on } \partial M\}$$

is the Mabuchi geodesic joining φ_0 to φ_1.

Here $\mathrm{PSH}(M, \omega)$ denotes the set of ω-psh functions on M: these are functions $u : M \to \mathbb{R} \cap \{-\infty\}$ that are locally the sum of a plurisubharmonic and a smooth function, such that $\omega + dd^c_{x,z}u \geqslant 0$ in the sense of currents (see § 2.1.1 for more details).

Proposition 1.4. *Let $(\varphi_t)_{0 \leqslant t \leqslant 1}$ be the Mabuchi geodesic joining φ_0 to φ_1. Then:*

(i) $\varphi \in \mathrm{PSH}(M, \omega)$ *is uniformly bounded on M and continuous on* $\mathrm{Amp}(\{\omega\}) \times \bar{S}$;

(ii) $|\varphi(x, z) - \varphi(x, z')| \leqslant A|\Re(z) - \Re(z')|$ *with* $A = \|\varphi_0 - \varphi_1\|_{L^\infty(X)}$;

(iii) $\varphi_{|\{\Re(z)=0\}} = \varphi_0$, $\varphi_{|\{\Re(z)=1\}} = \varphi_1$ *and* $(\omega + dd^c_{x,z}\varphi)^{n+1} = 0$.

It is, moreover, the unique bounded ω-psh solution to this Dirichlet problem.

We thank Hoang Chinh Lu for sharing his ideas on the continuity of φ.

Proof. The proof follows from a classical balayage technique together with a barrier argument, as noted by Berndtsson [Ber15]. Set $A = \|\varphi_1 - \varphi_0\|_{L^\infty(X)}$.

Observe that the function $\varphi_0 - At$, with $t = \Re(z)$, is ω-psh on M and $\varphi_0 - At|_{\partial M} \leqslant \varphi_{0,1}$. Hence, it belongs to the family $\mathcal{F}$ defining the upper envelope φ, so $\varphi_0 - At \leqslant \varphi_t$.

Similarly, $\varphi_0 + At$ is a ω-psh function on M and $\varphi_0 + At|_{\partial M} \geqslant \varphi_{0,1}$. Since $(\omega + dd^c_{x,z}(\varphi_0 + At))^{n+1} = 0$, it follows from the maximum principle that $u \leqslant \varphi_0 + At$, for any $u \in \mathcal{F}$ in the family. Therefore,

$$\varphi_0 - At \leqslant \varphi_t \leqslant \varphi_0 + At.$$

Similar arguments show that

$$\varphi_1 + A(t-1) \leqslant \varphi_t \leqslant \varphi_1 - A(t-1).$$

The upper semi-continuous regularization φ^* of φ satisfies the same estimates, showing, in particular, that $\varphi^*|_{\partial M} = \varphi_{0,1}$. Since φ^* is ω-psh, we infer $\varphi^* \in \mathcal{F}$; hence, $\varphi^* = \varphi$. Thus φ is ω-psh and uniformly bounded, proving the first statement in part (i). Classical balayage arguments show that $(\omega + dd^c_{x,z}\varphi)^{n+1} = 0$, proving part (iii).

We now prove part (ii). Consider the function

$$\chi_t(x) = \max\{\varphi_0(x) - A\log|z|, \varphi_1(x) + A(\log|z| - 1)\}$$

and note that it belongs to $\mathcal{F}$ and has the right boundary values.

Since $\chi_- = \varphi_0(x) - At \leqslant \varphi$ with equality at $t = 0$, we infer, for all x,

$$-A = \frac{\partial \chi_-}{\partial t}_{|t=0} \leqslant \dot{\varphi}_0(x).$$

Similarly $\chi_+ = \varphi_1(x) + A(t-1) \leqslant \varphi$ with equality at $t = 1$ yields, for all x, $\dot{\varphi}_1(x) \leqslant +A = (\partial\chi_+/\partial t)_{|t=1}$. Since $t \mapsto \varphi_t(x)$ is convex (by subharmonicity in z), we infer that for a.e. t, x, $-A \leqslant \dot{\varphi}_0(x) \leqslant \dot{\varphi}_t(x) \leqslant \dot{\varphi}_1(x) \leqslant +A$.

It remains to show that φ is continuous on $\mathrm{Amp}(\{\omega\}) \times \bar{S}$. We can assume, without loss of generality, that $\varphi_0 < \varphi_1$. Indeed, given any $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$, there exists $C > 0$, such that $\varphi_0 < \varphi_1 + C$. By Lemma 1.8, the Mabuchi geodesic joining φ_0 and $\varphi_1 + C$ is $\psi_t = \varphi_t + Ct$, $t \in [0, 1]$. The continuity of $(x, t) \to \psi_t(x)$ will then imply the continuity of $(x, t) \to \varphi_t(x)$.

We change notation slightly, replacing the strip S by the annulus $D := \{z = e^{t+is} \in \mathbb{C} : 1 \leqslant |w| \leqslant e\}$. We are going to express the function φ as a global Θ-psh envelope on the compact manifold $X \times \mathbb{P}^1$, where we view the annulus D as a subset of the Riemann sphere, $\mathbb{C} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. The form $\Theta(x, z) = \omega(x) + A\omega_{\mathrm{FS}}(z)$ is a semi-positive and big form on the compact Kähler manifold $\widetilde{M} := X \times \mathbb{P}^1$, so the viscosity approach of [EGZ17] can be applied, showing that the envelope φ is continuous on $\mathrm{Amp}(\{\omega\}) \times \bar{S}$. Here, ω_{FS} denotes the Fubini–Study metric on $\mathbb{P}^1$ and $A > 0$ is a constant to be chosen next.

Consider $U = \max(U_0, U_1)$, where $U_0(x, z) := \varphi_0(x)$ and

$$U_1(x, z) := \varphi_1(x) + A(\log|z|^2 - \log(|z|^2 + 1) + \log(e^2 + 1) - 2).$$

We choose $A > 0$ so large that $U(x, 1) \equiv \varphi_0(x)$. Note that $U(x, e) \equiv \varphi_1(x)$ since $\varphi_0 < \varphi_1$. Both U_0 and U_1 are Θ-psh on $\widetilde{M}$, hence so is U.

Fix ρ a local potential of $A\omega_{\mathrm{FS}}$ in D, such that $\rho|_{\partial D} = 0$ and let F be a continuous S^1-invariant function on $\widetilde{M}$, such that:

(a) $F = \varphi_{0,1}$ on $X \times \partial D$;
(b) $F(x, z) \geqslant U(x, z) \geqslant \varphi_0(x)$;
(c) $F(x, z) + \rho(z) > \varphi_t(x)$ in $X \times D$, with $t = \log|z|$.

We let the reader check that the function $F = U$ in $\widetilde{M} \backslash X \times D$ and

$$F(x, z) := (1 - \log|z|)\varphi_0(x) + (\log|z|)\varphi_1(x) - \rho(z) + (\log|z|)(1 - \log|z|),$$

for $(x, z) \in X \times D$, does the job.

We claim that for all $(x, z) \in X \times D$,

$$P_\Theta(F)(x, z) + \rho(z) = \varphi_{\log|z|}(x)\,,$$

where

$$P_\Theta(F) := \sup\{v : v \in \mathrm{PSH}(\widetilde{M}, \Theta) \text{ and } v \leqslant F\}.$$

Indeed $P_\Theta(F) + \rho$ is ω-psh in $X \times D$ and has boundary values $\leqslant \varphi_{0,1}$. It follows from the definition of the geodesic that $P_\Theta(F) + \rho \leqslant \varphi_t$. Conversely, $F + \rho \geqslant U + \rho \in \mathrm{PSH}(X \times D, \omega)$ and $U = \varphi_{0,1}$ on ∂M, thus $P_\Theta(F) + \rho = \varphi_{0,1}$ on ∂M. Condition (c) ensures that $M = X \times D$ does not meet the contact set $\{P_\Theta(F) = F\}$, since $F + \rho > \varphi_t \geqslant P_\Theta(F) + \rho$. It thus follows from a balayage argument [BT82] that $(\Theta + dd^c P_\Theta(F))^{n+1} = 0$ in M, and the maximum principle yields

$$P_\Theta(F) + \rho = \varphi_t.$$

The continuity of φ on $\mathrm{Amp}(\{\omega\}) \times \bar{S}$ now follows from [EGZ17], together with the following easy observation: the arguments in [EGZ17, § 2.2] ensure that if F is a smooth function on $\widetilde{M}$, $P_\Theta(F)$ is a Θ-psh function, continuous on $\mathrm{Amp}(\{\Theta\})$. The same result holds if F is merely continuous. Indeed, let F_j be a sequence of smooth functions on $\widetilde{M}$ converging uniformly to F. Taking the envelope at both sides of the inequality $F_j \leqslant F + \|F_j - F\|_{L^\infty(X)}$, we get $P_\Theta(F_j) \leqslant P_\Theta(F) + \|F_j - F\|_{L^\infty(X)}$. Hence, $\|P_\Theta(F_j) - P_\Theta(F)\|_{L^\infty(X)} \leqslant \|F_j - F\|_{L^\infty(X)}$. Thus, $P_\Theta(F_j)$ converges uniformly to $P_\Theta(F)$, and so $P_\Theta(F)$ is a Θ-psh function that is continuous on $\mathrm{Amp}(\{\Theta\}) = \mathrm{Amp}(\{\omega\}) \times \bar{S}$. □

Remark 1.5. If one could choose F smooth in this proof, it would follow from [BD12] (or [Ber13, Theorem 1.2]) that $\varphi \in \mathcal{C}^{1,\bar{1}}(\mathrm{Amp}(\alpha) \times S)$. This would also provide a compact proof of Chen's regularity result.

We now observe that geodesics in $\mathcal{H}_\omega$ are projections of those in $\mathcal{H}_{\omega_\varepsilon}$,

PROPOSITION 1.6. *Let φ denote the geodesic joining φ_0 to φ_1 in $\mathcal{H}_\omega$ and let φ^ε denote the corresponding geodesic in the space $\mathcal{H}_{\omega_\varepsilon}$. The map $\varepsilon \mapsto \varphi^\varepsilon$ is increasing and φ^ε decreases to φ as ε decreases to zero. Moreover,*

$$\varphi = P(\varphi^\varepsilon),$$

where P denotes the projection operator onto the space $\mathrm{PSH}(M, \omega)$.

Recall that, for an upper semi-continuous function $u : M \to \mathbb{R}$, its projection $P(u)$ is defined by

$$P(u) := \sup\{v \in \mathrm{PSH}(M, \omega); v \leqslant u\}.$$

The function $P(u)$ is either identical to $-\infty$ or belongs to $\mathrm{PSH}(M, \omega)$. It is the greatest ω-psh function on M that lies below u.

Proof. Set $\psi := P(\varphi^\varepsilon)$. Since $\omega \leqslant \omega_\varepsilon$, it follows from the envelope point of view that $\varphi \leqslant \varphi^\varepsilon$. Thus, $\varphi = P(\varphi) \leqslant P(\varphi^\varepsilon) = \psi$ and $\psi \in \mathrm{PSH}(M, \omega)$. Now $\psi \leqslant \varphi$, since $\psi \leqslant \varphi^\varepsilon = \varphi_0, \varphi_1$ on ∂M and $\psi \in \mathrm{PSH}(M, \omega)$. Thus, $\psi = P(\varphi^\varepsilon) = \varphi$.

Fix $\varepsilon' \leqslant \varepsilon$. The inclusion $\mathrm{PSH}(M, \omega_{\varepsilon'}) \subset \mathrm{PSH}(M, \omega_\varepsilon)$ implies similarly that $\varphi \leqslant \varphi^{\varepsilon'} \leqslant \varphi^\varepsilon$. The decreasing limit v of φ^ε, as ε decreases to zero, satisfies both $\varphi \leqslant v$ and $v \in \mathrm{PSH}(M, \omega)$ with boundary values φ_0, φ_1, thus $v = \varphi$. □

It will also be interesting to consider *subgeodesics*.

DEFINITION 1.7. A subgeodesic is a path (φ_t) of functions in $\mathcal{H}_\omega$ (or in larger classes of ω-psh functions) such that the associated function is a ω-psh function on $X \times S$.

We shall soon need the following simple observation.

LEMMA 1.8. *Fix $c \in \mathbb{R}$, $\varphi, \psi \in \mathcal{H}_\omega$ and let $(\varphi_t)_{0 \leqslant t \leqslant 1}$ denote the Mabuchi geodesic joining $\varphi = \varphi_0$ to $\varphi_1 = \psi$. Then $\psi_t(x) := \varphi_t(x) - ct, 0 \leqslant t \leqslant 1, x \in X$, is the Mabuchi geodesic joining φ to $\psi - c$.*

Proof. The proof follows from Definition 1.3 and the definition of envelopes, since $\sup\{v; v \in \mathrm{PSH}(M, \omega)$ and $v \leqslant \varphi, v \leqslant \psi - c$ on $\partial M\} = \varphi_t - ct$. □

1.1.2 *Mabuchi and other Finsler distances.* When ω is Kähler, the length of a smooth path $(\varphi_t)_{t\in[0,1]}$ in $\mathcal{H}_\omega$ is defined in a standard way,

$$\ell(\varphi) := \int_0^1 |\dot{\varphi}_t|\,dt = \int_0^1 \sqrt{\int_X \dot{\varphi}_t^2\,\mathrm{MA}(\varphi_t)}\,dt.$$

The distance between two points in $\mathcal{H}_\omega$ is then

$$d(\varphi_0,\varphi_1) := \inf\{\ell(\varphi)\mid \varphi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\}.$$

It is easy to verify that d defines a semi-distance (i.e. non-negative, symmetric, and satisfying the triangle inequality). It is, however, non-trivial to check that d is non-degenerate (see [MM05] for a striking example).

Observe that d induces a distance on $\mathcal{H}_\alpha$ (that we abusively still denote d) compatible with the Riemannian splitting $\mathcal{H}_\omega = \mathcal{H}_\alpha \times \mathbb{R}$, by setting

$$d(\omega_\varphi,\omega_\psi) := d(\varphi,\psi)$$

whenever the potentials φ,ψ of $\omega_\varphi,\omega_\psi$ are normalized by $E(\varphi)=E(\psi)=0$ (see § 2.2.1 for the definition of the functional E).

It is rather easy to check that $(\mathcal{H}_\alpha, d)$ is not a complete metric space. We shall describe the metric completion $(\overline{\mathcal{H}}_\alpha, d)$ in § 4. Following Darvas [Dar15], we introduce a family of distances that generalize d:

Definition 1.9. For $p \geqslant 1$ and ω Kähler, we set

$$d_p(\varphi_0,\varphi_1) := \inf\{\ell_p(\varphi)\mid \varphi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\},$$

where $\ell_p(\varphi) := \int_0^1 |\dot{\varphi}_t|_p\,dt = \int_0^1 (\int_X |\dot{\varphi}_t|^p\,\mathrm{MA}(\varphi_t))^{1/p}\,dt$.

Note that $d_2 = d$ is the Mabuchi distance. Mabuchi geodesics have constant speed with respect to all the Finsler structures ℓ_p, as was observed by Berndtsson [Ber09, Lemma 2.1]: for any $\mathcal{C}^1$-function χ,

$$t \mapsto \int_X \chi(\dot{\varphi}_t)\,\mathrm{MA}(\varphi_t)$$

is constant along a geodesic. Indeed

$$\begin{aligned}\frac{d}{dt}\int_X \chi(\dot{\varphi}_t)\,\mathrm{MA}(\varphi_t) &= \int_X \chi'(\dot{\varphi}_t)\ddot{\varphi}_t\,\mathrm{MA}(\varphi_t) + \frac{n}{V_\alpha}\int_X \chi(\dot{\varphi}_t)\,dd^c\dot{\varphi}_t\wedge\omega_{\varphi_t}^{n-1}\\ &= \int_X \chi'(\dot{\varphi}_t)\bigg\{\ddot{\varphi}_t\,\mathrm{MA}(\varphi_t) - \frac{n}{V_\alpha}d\dot{\varphi}_t\wedge d^c\dot{\varphi}_t\wedge\omega_{\varphi_t}^{n-1}\bigg\} = 0\end{aligned}$$

since $\ddot{\varphi}_t\,\mathrm{MA}(\varphi_t) - (n/V_\alpha)\,d\dot{\varphi}_t\wedge d^c\dot{\varphi}_t\wedge\omega_{\varphi_t}^{n-1} = 0$. Applying this observation to $\chi(t)=t^p$ shows that Mabuchi geodesics have constant ℓ_p-speed.

When ω is merely semi-positive, there are fewer smooth paths within $\mathcal{H}_\omega$. It is natural to consider smooth paths in $\mathcal{H}_{\omega_\varepsilon}$ and pass to the limit in the previous definitions.

Definition 1.10. Assume ω is semi-positive and big. Let $\varphi_0,\varphi_1 \in \mathcal{H}_\omega$. We define the Mabuchi distance between φ_0 and φ_1 as

$$d_p(\varphi_0,\varphi_1) := \liminf_{\varepsilon\to 0} d_{p,\varepsilon}(\varphi_0,\varphi_1),$$

where $d_{p,\varepsilon}$ is the distance with respect to the Kähler form $\omega_\varepsilon := \omega + \varepsilon\omega_X$.

We will show in Theorem 1.13 that it is a distance, which moreover does not depend on the way we approximate ω by Kähler classes.

Remark 1.11. For any smooth path $\psi : [0,1] \to \mathcal{H}_\omega$, we can still define

$$\ell_p(\psi) := \int_0^1 \left(\frac{1}{V} \int_X |\dot{\psi}_t|^p (\omega + dd^c \psi_t)^n \right)^{1/p} dt$$

when ω is merely semi-positive. Since $\mathrm{PSH}(M,\omega) \subset \mathrm{PSH}(M,\omega_\varepsilon)$, ψ_t is both in $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega_\varepsilon}$. Observe that

$$\begin{aligned} V_\varepsilon^{-1} \int_X |\dot{\psi}_t|^p (\omega_\varepsilon + dd^c \psi_t)^n &= V_\varepsilon^{-1} \int_X |\dot{\psi}_t|^p (\omega + dd^c \psi_t + \varepsilon \omega_X)^n \\ &\leqslant V^{-1} \int_X |\dot{\psi}_t|^p (\omega + dd^c \psi_t)^n + A\varepsilon, \end{aligned}$$

hence

$$\ell_{p,\varepsilon}(\psi) \leqslant \ell_p(\psi) + A'\varepsilon\,,$$

where $\ell_{p,\varepsilon}$ denotes the length in $\mathcal{H}_{\omega_\varepsilon}$. We infer

$$d_p(\varphi_0,\varphi_1) \leqslant \inf\{\ell_p(\psi)\ \psi \text{ smooth path joining } \varphi_0 \text{ and } \varphi_1 \text{ in } \mathcal{H}_\omega\}.$$

The converse inequality is, however, unclear, owing to the lack of positivity of ω: it is difficult to smooth out ω-psh functions if ω is not Kähler. This partially explains Definition 1.10.

1.2 Approximation by Kähler classes

Fix $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$. We let $(\varphi_t)_{0\leqslant t\leqslant 1}$ denote the Mabuchi geodesic in $\mathcal{H}_\omega$ joining φ_0 to φ_1.

DEFINITION 1.12. For $t = 0, 1$ we set

$$I(t) := \int_X |\dot{\varphi}_t|^p \,\mathrm{MA}(\varphi_t).$$

THEOREM 1.13. *Set $\omega_\varepsilon = \omega + \varepsilon\omega_X$, $\varepsilon > 0$. Then $\lim_{\varepsilon\to 0} d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1)$ exists and is independent of ω_X. More precisely,*

$$d^p_{p,\varepsilon}(\varphi_0,\varphi_1) \to I(0) = I(1).$$

In particular, $d_p(\varphi_0,\varphi_1) = I(0)^{1/p} = I(1)^{1/p}$ defines a distance on $\mathcal{H}_\omega$.

In the definition of $I(0), I(1)$, the time derivatives $\dot{\varphi}_0 = \dot{\varphi}_0^+$, $\dot{\varphi}_1 = \dot{\varphi}_1^-$ denote the right and left derivatives, respectively.

Remark 1.14. When $\omega = \pi^*\omega_Y$, for some Kähler form ω_Y on a compact normal space Y, for each $p \geqslant 1$ and $\forall \phi_0, \phi_1 \in \mathcal{H}_{\omega_Y}$, we define

$$d_p(\phi_0,\phi_1) := d_p(\varphi_0,\varphi_1) \quad \text{where } \varphi_0 = \pi^*\phi_0, \varphi_1 = \pi^*\phi_1.$$

This definition does not depend on the choice of resolution. Indeed, let $\pi' : X' \to Y$ be another resolution of Y that dominates X, i.e. there exists a holomorphic and bimeromorphic map $f : X' \to X$, such that $\pi' = \pi \circ f$. Set $\omega' := {\pi'}^*\omega_Y = f^*\omega$. We need to show that

$$d_{p,\omega}(\varphi_0,\varphi_1) = d_{p,f^*\omega}(f^*\varphi_0, f^*\varphi_1).$$

Denote by ψ_t the $f^*\omega$-geodesic joining $f^*\varphi_0$ and $f^*\varphi_1$. We claim that $\psi_t = f^*\varphi_t$. We first observe that, since ψ_t is a $f^*\omega$-psh function for each fixed t, $\psi_t = f^*\gamma_t$, where γ_t is a ω-psh function on X. Set $M' := X' \times S$, $\psi(x',t) := \psi_t(x')$, and $\gamma(x',t) := \gamma_t(x')$ for each $(x',t) \in M'$. By construction, we have that

$$f^*(\omega + dd^c\gamma)^{n+1} = (f^*\omega + dd^c\psi)^{n+1} = 0 \quad \text{on } M' := X' \times S, \quad \psi|_{\partial M'} = f^*\varphi_{0,1}.$$

The claim follows from the uniqueness of the solution of the Dirichlet problem in Proposition 1.4. The invariance of the non-pluripolar Monge–Ampère measure under bimeromorphic maps [DiN15], together with the fact that $V := \int_X \omega^n = \int_{X'} f^*\omega$, give

$$\int_X |\dot{\varphi}_0|^p \frac{(\omega + dd^c\varphi_0)^n}{V} = \int_{X'} |\dot{f^*\varphi_0}|^p \frac{(f^*\omega + dd^c f^*\varphi_0)^n}{V} = \int_{X'} |\dot{\psi}_0|^p \frac{(\omega' + dd^c\varphi_0)^n}{V}.$$

The conclusion then follows from Theorem 1.13.

Proof. Observe that $\varphi_0, \varphi_1 \in \mathcal{H}_{\omega_\varepsilon}$ and let φ_t^ε be the corresponding geodesic. It follows from [Dar15, Theorem 3.5] that

$$d_{p,\varepsilon}^p(\varphi_0,\varphi_1) = V_\varepsilon^{-1} \int_X |\dot{\varphi}_0^\varepsilon|^p (\omega_\varepsilon + dd^c\varphi_0)^n.$$

Now observe that

$$\dot{\varphi}_0^+ \leqslant \dot{\varphi}_0^\varepsilon \leqslant \frac{\varphi_t^\varepsilon - \varphi_0}{t} \quad \forall t \in (0,1),$$

where the first inequality follows from the fact that $\varepsilon \to \varphi_t^\varepsilon$ is decreasing (Proposition 1.6), while the second uses the convexity of $t \mapsto \varphi_t^\varepsilon$. Thus,

$$|\dot{\varphi}_0^\varepsilon - \dot{\varphi}_0^+| \leqslant \left| \frac{\varphi_t^\varepsilon - \varphi_0}{t} - \dot{\varphi}_0^+ \right|.$$

Letting $\varepsilon \searrow 0$ and then $t \to 0$ shows that $|\dot{\varphi}_0^\varepsilon - \dot{\varphi}_0^+|$ converges pointwise to zero. Moreover, $(\omega_\varepsilon + dd^c\varphi_0)^n = f_\varepsilon \, dV$ where dV is the Lebesgue measure and $f_\varepsilon > 0$ are smooth densities, which converge locally uniformly to $f \geqslant 0$ with $(\omega + dd^c\varphi_0)^n = f \, dV$. The dominated convergence theorem thus yields

$$\lim_{\varepsilon \to 0} d_{p,\varepsilon}^p(\varphi_0,\varphi_1) = V^{-1} \int_X |\dot{\varphi}_0^+|^p (\omega + dd^c\varphi_0)^n = I(0).$$

The argument for $I(1)$ is similar.

This shows, in particular, that d_p is a distance on $\mathcal{H}_\omega$: if $d_p(\varphi_0,\varphi_1) = 0$, then $I(0) = I(1) = 0$, hence, $\dot{\varphi}_0(x) = \dot{\varphi}_1(x) = 0$ for a.e. $x \in X$, which implies $\dot{\varphi}_t(x) = 0$ for a.e. $x \in X$, by convexity of $t \mapsto \varphi_t(x)$. Thus, $\varphi_0(x) = \varphi_1(x)$ for a.e. $x \in X$. □

We now extend the definition of the distance d_p for bounded ω-psh potentials.

Definition 1.15. Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$; then

$$d_p(\varphi_0,\varphi_1) := \liminf_{\varepsilon \to 0} \liminf_{j,k \to +\infty} d_{p,\varepsilon}(\varphi_0^j, \varphi_1^k) = \liminf_{\varepsilon \to 0} d_{p,\varepsilon}(\varphi_0,\varphi_1),$$

where φ_0^j, φ_1^k are smooth sequences of ω_ε-psh functions decreasing to φ_0 and φ_1, respectively.

Observe that $d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1)$ is well defined for potentials in $\mathcal{E}^p(X,\omega_\varepsilon)$ [Dar15], and so, in particular, for bounded ω_ε-psh functions.

PROPOSITION 1.16. *Let $\varphi_0,\varphi_1 \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$. The limit of $d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1)$ as ε goes to zero exists and does not depend on the choice of ω_X.*

Proof. First, observe that since φ_0,φ_1 are bounded, they belong to $\mathcal{E}^p(X,\omega_\varepsilon)$ for any $0 \leqslant \varepsilon \leqslant 1$. By [Dar15, Corollary 4.14] we know that the Pythagorean formula holds true, i.e.

$$d^p_{p,\varepsilon}(\varphi_0,\varphi_1) = d^p_{p,\varepsilon}(\varphi_0,\varphi_0 \vee_\varepsilon \varphi_1) + d^p_{p,\varepsilon}(\varphi_0 \vee_\varepsilon \varphi_1,\varphi_1),$$

where $\psi := \varphi_0 \vee_\varepsilon \varphi_1$ is the greatest ω_ε-psh function that lies below $\min(\varphi_0,\varphi_1)$. Fix $\varepsilon \leqslant \varepsilon'$. We claim that

$$V_\varepsilon d^p_{p,\varepsilon}(\varphi_0,\psi) \leqslant V_{\varepsilon'} d^p_{p,\varepsilon'}(\varphi_0,\psi) \quad \text{and} \quad V_\varepsilon d^p_{p,\varepsilon}(\psi,\varphi_1) \leqslant V_{\varepsilon'} d^p_{p,\varepsilon'}(\psi,\varphi_1).$$

Let ψ^ε_t, $\psi^{\varepsilon'}_t$ denote the ε-geodesic and the ε'-geodesic, both joining ψ and φ_0. Since $\varepsilon \to \psi^\varepsilon_t$ is increasing (Proposition 1.6), we have that, for any $t \in (0,1)$

$$\frac{\psi^\varepsilon_t - \psi}{t} \leqslant \frac{\psi^{\varepsilon'}_t - \psi}{t},$$

which implies $\dot\psi^\varepsilon_0 \leqslant \dot\psi^{\varepsilon'}_0$. Moreover, observe that, since $\varphi_0(x) \geqslant \psi(x)$ for all $x \in X$, Lemma 3.3 yields $\dot\psi^\varepsilon_0(x) \geqslant 0$ for all $x \in X$. It then follows that

$$\int_X |\dot\psi^\varepsilon_0|^p(\omega_\varepsilon + dd^c\psi)^n \leqslant \int_X |\dot\psi^{\varepsilon'}_0|^p(\omega_{\varepsilon'} + dd^c\psi)^n,$$

hence the claim. The same type of arguments give $V_\varepsilon d^p_{p,\varepsilon}(\psi,\varphi_1) \leqslant V_{\varepsilon'} d^p_{p,\varepsilon'}(\psi,\varphi_1)$. Hence,

$$V_\varepsilon V^{-1}_{\varepsilon'} d^p_{p,\varepsilon}(\varphi_0,\varphi_1) \leqslant d^p_{p,\varepsilon'}(\varphi_0,\varphi_0 \vee_\varepsilon \varphi_1) + d^p_{p,\varepsilon'}(\varphi_0 \vee_\varepsilon \varphi_1,\varphi_1).$$

Using again [Dar15, Corollary 4.14] and the triangle inequality we get

$$V_\varepsilon V^{-1}_{\varepsilon'} d^p_{p,\varepsilon}(\varphi_0,\varphi_1) \leqslant d^p_{p,\varepsilon'}(\varphi_0,\varphi_1) + 2d^p_{p,\varepsilon'}(\varphi_0 \vee_\varepsilon \varphi_1,\varphi_0 \vee_{\varepsilon'} \varphi_1).$$

Moreover, since $\varphi_0 \vee_{\varepsilon'} \varphi_1 \geqslant \varphi_0 \vee_\varepsilon \varphi_1$, [Dar15, Lemma 5.1] yields

$$\begin{aligned} d^p_{p,\varepsilon'}(\varphi_0 \vee_\varepsilon \varphi_1,\varphi_0 \vee_{\varepsilon'} \varphi_1) &\leqslant \frac{1}{V_{\varepsilon'}} \int_X (\varphi_0 \vee_{\varepsilon'} \varphi_1 - \varphi_0 \vee_\varepsilon \varphi_1)^p \,(\omega_{\varepsilon'} + dd^c(\varphi_0 \vee_\varepsilon \varphi_1))^n \\ &\leqslant \frac{1}{V_{\varepsilon'}} \int_X (\varphi_0 \vee_{\varepsilon'} \varphi_1 - \varphi_0 \vee_\varepsilon \varphi_1)^p \,(\omega + \omega_X + dd^c(\varphi_0 \vee_\varepsilon \varphi_1))^n \\ &:= V^{-1}_{\varepsilon'} \eta(\varepsilon,\varepsilon'). \end{aligned}$$

Observe that $\eta(\varepsilon,\varepsilon')$ converges to 0 as ε' goes to 0. From above, we have

$$V_\varepsilon d^p_{p,\varepsilon}(\varphi_0,\varphi_1) \leqslant V_{\varepsilon'} d^p_{p,\varepsilon'}(\varphi_0,\varphi_1) + \eta(\varepsilon,\varepsilon').$$

Hence, the limit exists.

Now, let $\omega_X, \widetilde\omega_X$ be two Kähler metrics on X, such that

$$\omega_X \leqslant \widetilde\omega_X \leqslant C\omega_X$$

https://doi.org/10.1112/S0010437X18007170 Published online by Cambridge University Press

for some $C>0$. Assume first $\varphi_0 \leqslant \varphi_1$. Set $\widetilde{\omega}_\varepsilon := \omega + \varepsilon\widetilde{\omega}_X$ and observe that $\omega_\varepsilon \leqslant \widetilde{\omega}_\varepsilon \leqslant \omega_{\varepsilon'}$, where $\varepsilon' = \varepsilon C$. Let $\varphi_t^\varepsilon, \tilde{\varphi}_t^\varepsilon$ be the geodesic with respect to ω_ε and $\widetilde{\omega}_\varepsilon$, respectively, and observe that $\varphi_t^\varepsilon \leqslant \tilde{\varphi}_t^\varepsilon \leqslant \varphi_t^{\varepsilon'}$. The same arguments as before give

$$|\dot{\varphi}_0^\varepsilon|^p \leqslant |\dot{\tilde{\varphi}}_0^\varepsilon|^p \leqslant |\dot{\varphi}_0^{\varepsilon'}|^p,$$

hence

$$\int_X |\dot{\varphi}_0^\varepsilon|^p (\omega_\varepsilon + dd^c\varphi_0)^n \leqslant \int_X |\dot{\tilde{\varphi}}_0^\varepsilon|^p (\tilde{\omega}_\varepsilon + dd^c\varphi_0)^n \leqslant \int_X |\dot{\varphi}_0^{\varepsilon'}|^p (\omega_{\varepsilon'} + dd^c\varphi_0)^n.$$

The latter tells us that the limit does not depend on ω_X. To get rid of the assumption $\varphi_0 \leqslant \varphi_1$, one can use the Pythagorean formula, as before. □

An adaptation of the classical Perron envelope technique yields the following result of Berndtsson [Ber15].

Proposition 1.17. *Assume that φ_0, φ_1 are bounded ω-psh functions. Then*

$$\varphi(x,z) := \sup\Big\{u(x,z) \mid u \in \mathrm{PSH}(X\times S,\omega) \text{ with } \lim_{t\to 0,1} u \leqslant \varphi_{0,1}\Big\}$$

is the unique bounded ω-psh function on $X \times S$, which is the solution of the Dirichlet problem $\varphi_{|X\times\partial S} = \varphi_{0,1}$, with

$$(\omega + dd^c_{x,z}\varphi)^{n+1} = 0 \quad \text{in } X\times S.$$

Moreover $\varphi(x,z) = \varphi(x,t)$ only depends on $\Re(z)$ and $|\dot{\varphi}| \leqslant \|\varphi_1 - \varphi_0\|_{L^\infty(X)}$.

The proof goes exactly as that of Proposition 1.4. The function φ (or rather the path $\varphi_t \subset \mathrm{PSH}(X,\omega)\cap L^\infty(X)$) is called a *bounded geodesic* in [Ber15]. We use the same terminology here, as it turns out that bounded geodesics are geodesics in the metric sense.

Proposition 1.18. *Bounded geodesics are metric geodesics. More precisely, if φ_0, φ_1 are bounded ω-psh functions and $\varphi(x,z) = \varphi_t(x)$ is the bounded geodesic joining φ_0 to φ_1, then for all $t,s \in [0,1]$,*

$$d_p(\varphi_t,\varphi_s) = |t-s|\, d_p(\varphi_0,\varphi_1).$$

Proof. Let $\varphi_0^j, \varphi_1^k \in \mathcal{H}_{\omega_\varepsilon}$ be sequences decreasing, respectively, to φ_0, φ_1. It follows from the comparison principle and the uniqueness in Proposition 1.17 that $\varphi_{t,j}$ decreases to φ_t as j increases to $+\infty$. From Definition 1.15, Proposition 1.16 and the fact that the identity in the statement holds in the Kähler setting for d_ε we obtain

$$\begin{aligned} d_p(\varphi_t,\varphi_s) &= \liminf_{\varepsilon\to 0}\ \liminf_{j,k\to+\infty} d_{p,\varepsilon}(\varphi_{t,j},\varphi_{s,k}) \\ &= |t-s| \liminf_{\varepsilon\to 0}\ \liminf_{j,k\to+\infty} d_{p,\varepsilon}(\varphi_0^j,\varphi_1^k) = |t-s| d_p(\varphi_0,\varphi_1). \end{aligned}$$ □

Remark 1.19. One can no longer expect that $d_p(\varphi_0,\varphi_1)^p = \int_X |\dot{\varphi}_t|^p \,\mathrm{MA}(\varphi_t)$ for a.e. $t\in[0,1]$, as simple examples show. One can, e.g., take $\varphi_0 \equiv 0$ and $\varphi_1 = \max(u,0)$, where u takes positive values, has isolated singularities, and solves $\mathrm{MA}(u) =$ Dirac mass at some point: in this case $\mathrm{MA}(\varphi_1)$ is concentrated on the contact set $(u=0)$ while $\dot{\varphi}_1 \equiv 0$ on this set, hence $\int_X |\dot{\varphi}_1|^p \,\mathrm{MA}(\varphi_1) = 0$. We thank Darvas for pointing this out to us.

As this remark points out, we do not have that $d_p^p(\varphi_0,\varphi_1)=I(0)=I(1)$ when φ_0,φ_1 are just bounded ω-psh functions. Nevertheless, we can still recover the formula in some special cases.

We start by recalling the following.

Theorem 1.20. *Let f be a continuous function, such that $dd^c f\leqslant C\omega_X$ on X, for some $C>0$. Then $P(f)$ has bounded Laplacian on* $\mathrm{Amp}(\{\omega\})$ *and*

$$(\omega+dd^c P_\omega(f))^n=\mathbb{1}_{\{P_\omega(f)=f\}}(\omega+dd^c f)^n. \tag{4}$$

The fact that $P(f)$ has a locally bounded Laplacian in $\mathrm{Amp}(\{\omega\})$ is essentially [Ber13, Theorem 1.2]. We do not assume here that f is smooth but one can check that the upper bound on $dd^c f$ is the only estimate needed to pursue Berman's approach. One can then argue as in [GZ17, Theorem 9.25] to get (4).

Set

$$\mathcal{H}_{bd}:=\{\varphi\in\mathrm{PSH}(X,\omega)\cap L^\infty(X),\varphi=P_\omega(f)\text{ for some }f\in C^0(X)\text{ with }dd^c f\leqslant C\omega_X, C>0\}.$$

Theorem 1.21. *Assume that $\varphi_0,\varphi_1\in\mathcal{H}_{bd}$. Let φ_t be the Mabuchi geodesic joining φ_0 and φ_1. Then*

$$d_p^p(\varphi_0,\varphi_1)=\int_X|\dot{\varphi}_0|^p\,\mathrm{MA}(\varphi_0)=\int_X|\dot{\varphi}_1|^p\,\mathrm{MA}(\varphi_1). \tag{5}$$

Proof. Set $\varphi_{0,\varepsilon}:=P_{\omega_\varepsilon}(f_0)$ and $\varphi_{1,\varepsilon}:=P_{\omega_\varepsilon}(f_1)$. Clearly, $\varphi_{i,\varepsilon}$ decreases pointwise to φ_i, $i=1,2$. Let φ_t^ε be the ω_ε-geodesic joining $\varphi_{0,\varepsilon}$ and $\varphi_{1,\varepsilon}$. Combining [Dar15, Theorem 3.5] with (4), we get

$$V_\varepsilon d_{p,\varepsilon}^p(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon})=\int_X|\dot{\varphi}_0^\varepsilon|^p(\omega_\varepsilon+dd^c\varphi_{0,\varepsilon})^n=\int_{\{\varphi_{0,\varepsilon}=f_0\}}|\dot{\varphi}_0^\varepsilon|^p(\omega_\varepsilon+dd^c f_0)^n.$$

Set $D_\varepsilon:=\{\varphi_{0,\varepsilon}=f_0\}$, $D_0:=\{\varphi_0=f_0\}$, and observe that $D_0\subseteq D_\varepsilon$. Since $\varphi_{0,\varepsilon}=P_{\omega_\varepsilon}(f)$ and $\varphi_0=P_\omega(f)$, Theorem 1.20 ensures that $(\omega_\varepsilon+dd^c\varphi_{0,\varepsilon})^n=g_\varepsilon\omega_X^n$ and $(\omega+dd^c\varphi_0)^n=g_0\omega_X^n$ where g_ε,g_0 are defined as

$$g_\varepsilon:=\begin{cases}0, & x\notin D_\varepsilon,\\ \dfrac{(\omega_\varepsilon+dd^c f_0)^n}{\omega_X^n}, & x\in D_\varepsilon,\end{cases}\qquad g_0:=\begin{cases}0, & x\notin D_0,\\ \dfrac{(\omega+dd^c f_0)^n}{\omega_X^n}, & x\in D_0.\end{cases}$$

We claim that g_ε converges pointwise to g_0. Indeed, when $x\in D_0\subseteq D_\varepsilon$, then $g_\varepsilon(x)=((\omega_\varepsilon+dd^c f_0)^n/\omega_X^n)(x)$ converges to $((\omega+dd^c f_0)^n/\omega_X^n)(x)=g_0(x)$ as ε goes to 0. In the case when $x\notin D_0$, i.e. $\varphi_0(x)<f_0(x)$, since $\varphi_\varepsilon(x)$ decreases to $\varphi_0(x)$ as ε goes to zero, we can infer that, for ε sufficiently small, we still have $\varphi_\varepsilon(x)<f_0(x)$, which means $x\notin D_\varepsilon$. Hence, $g_\varepsilon(x)=0=g_0(x)$. The claim is then proved.

Since $\mathbb{1}_{D_\varepsilon}\varphi_0^\varepsilon=f_0=\mathbb{1}_{D_0}\varphi_0$, the same arguments in Theorem 1.13 show that $|\mathbb{1}_{D_\varepsilon}\dot{\varphi}_0^\varepsilon-\mathbb{1}_{D_0}\dot{\varphi}_0|$ converges pointwise to 0 as ε goes to zero.

We thus infer that $\mathbb{1}_{D_\varepsilon}|\dot{\varphi}_0^\varepsilon|^p g_\varepsilon$ converges pointwise to $\mathbb{1}_{D_0}|\dot{\varphi}_0|^p g_0$ as $\varepsilon\to 0$. The dominated convergence theorem yields

$$\lim_{\varepsilon\to 0}d_{p,\varepsilon}^p(\varphi_0,\varphi_1)=\lim_{\varepsilon\to 0}\int_X\mathbb{1}_{D_\varepsilon}|\dot{\varphi}_0^\varepsilon|^p(\omega_\varepsilon+dd^c\varphi_{0,\varepsilon})^n=\int_X\mathbb{1}_{D_0}|\dot{\varphi}_0|^p(\omega+dd^c\varphi_0)^n,$$

hence the conclusion. □

Observe that if $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$, then $\varphi_0 \vee \varphi_1 \in \mathcal{H}_{bd}$. Indeed, since φ_0, φ_1 are smooth, the functions $-\varphi_0, -\varphi_1$ are quasi-plurisubharmonic, i.e. there exists $C > 0$ such that $dd^c(-\varphi_i) \geqslant -C\omega_X$ for any $i = 1, 2$. Thus, $\min(\varphi_0, \varphi_1) = -\max(-\varphi_0, -\varphi_1)$ is such that

$$dd^c \min(\varphi_0, \varphi_1) = -dd^c \max(-\varphi_0, -\varphi_1) \leqslant C\omega_X.$$

In particular, (5) holds for $d_p(\varphi_0, \varphi_0 \vee \varphi_1)$ and $d_p(\varphi_1, \varphi_0 \vee \varphi_1)$.

2. Finite-energy classes

We define in this section the set $\mathcal{E}(\alpha)$ (respectively $\mathcal{E}^p(\alpha)$) of positive closed currents $T = \omega + dd^c\varphi$ with full Monge–Ampère mass (respectively finite weighted energy) in α, by defining the corresponding class $\mathcal{E}(X, \omega)$ (respectively $\mathcal{E}^p(X, \omega)$) of finite-energy potentials φ.

2.1 The space $\mathcal{E}(\alpha)$

2.1.1 *Quasi-plurisubharmonic functions.* Recall that a function is quasi-plurisubharmonic if it is locally given as the sum of a smooth and a psh function. In particular quasi-psh (*qpsh* for short) functions are upper semi-continuous and integrable.

Definition 2.1. We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-plurisubharmonic functions. These are quasi-psh functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that

$$\omega + dd^c\varphi \geqslant 0$$

in the weak sense of currents.

The set $\mathrm{PSH}(X, \omega)$ is a closed subset of $L^1(X)$, for the L^1-topology.

2.1.2 *The class $\mathcal{E}(X, \omega)$.* Given $\varphi \in \mathrm{PSH}(X, \omega)$, we consider

$$\varphi_j := \max(\varphi, -j) \in \mathrm{PSH}(X, \omega) \cap L^\infty(X).$$

It follows from the Bedford–Taylor theory [BT82] that the $\mathrm{MA}(\varphi_j)$ are well-defined probability measures. Moreover, the sequence $\mu_j := \mathbb{1}_{\{\varphi > -j\}} \mathrm{MA}(\varphi_j)$ is increasing [GZ07, p. 445]. Since the μ_j all have total mass bounded from above by 1, we consider

$$\mu_\varphi := \lim_{j \to +\infty} \mu_j,$$

which is a positive Borel measure on X, with total mass $\leqslant 1$.

Definition 2.2. We set

$$\mathcal{E}(X, \omega) := \{\varphi \in \mathrm{PSH}(X, \omega) \mid \mu_\varphi(X) = 1\}.$$

For $\varphi \in \mathcal{E}(X, \omega)$, we set $\mathrm{MA}(\varphi) := \mu_\varphi$.

The latter can be characterized as the largest class for which the complex Monge–Ampère mass is well defined and the maximum principle holds [GZ07, Theorem 1.5]. We further note that the *domination principle* holds ([BEGZ10, Corollary 2.5], [DDL18, Proposition 2.4]).

Proposition 2.3. *If $\varphi, \psi \in \mathcal{E}(X,\omega)$ are such that*

$$\varphi(x) \leqslant \psi(x) \quad \text{for } \mathrm{MA}(\psi)\text{-a.e. } x,$$

then $\varphi(x) \leqslant \psi(x)$ for all $x \in X$.

It follows from the $\partial\overline{\partial}$-lemma that any positive closed current $T \in \alpha$ can be written $T = \omega + dd^c\varphi$ for some function $\varphi \in \mathrm{PSH}(X,\omega)$ that is unique up to an additive constant.

Definition 2.4. We let $\mathcal{E}(\alpha)$ denote the set of all positive currents in α, $T = \omega + dd^c\varphi$, with $\varphi \in \mathcal{E}(X,\omega)$.

Note that this definition does not depend on the choice of ω, nor does it depend on the choice of φ.

2.2 The class $\mathcal{E}^1(X,\omega)$

2.2.1 *The Aubin–Mabuchi functional.* Each tangent space $T_\varphi\mathcal{H}_\omega$ admits the following orthogonal decomposition

$$T_\varphi\mathcal{H}_\omega = \{\psi \in C^\infty(X); \beta_\varphi(\psi) = 0\} \oplus \mathbb{R},$$

where $\beta = \mathrm{MA}$ is the 1-form defined on $\mathcal{H}$ by

$$\beta_\varphi(\psi) = \int_X \psi\, \mathrm{MA}(\varphi).$$

It is a classical observation of Mabuchi that the 1-form β is closed. Therefore, there exists a unique function E defined on the convex open set $\mathcal{H}_\omega$, such that $\beta = dE$ and $E(0) = 0$. It is often called the *Aubin–Mabuchi functional* and can be expressed (after integration along affine paths) by

$$E(\varphi) = \frac{1}{(n+1)V_\alpha} \sum_{j=0}^{n} \int_X \varphi\, (\omega + dd^c\varphi)^j \wedge \omega^{n-j}.$$

Lemma 2.5. *The Aubin–Mabuchi functional E is concave along Euclidean segments, non-decreasing, and satisfies the cocycle condition*

$$E(\varphi) - E(\psi) = \frac{1}{(n+1)V_\alpha} \sum_{j=0}^{n} \int_X (\varphi - \psi)\, (\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}\,.$$

It is affine along geodesics and convex along subgeodesics in $\mathcal{H}_\omega$.

Proof. These properties are well known when ω is in a Kähler class.

The monotonicity property follows from the definition since the first derivative of E is $dE = \beta = \mathrm{MA} \geqslant 0$, a probability measure: if φ_t is an arbitrary path, then

$$\frac{d}{dt}E(\varphi_t) = \int_X \dot{\varphi}_t\, \mathrm{MA}(\varphi_t).$$

It follows from Stokes theorem that

$$\begin{aligned}\frac{d^2}{dt^2}E(\varphi_t) &= \int_X \ddot{\varphi}_t\, \mathrm{MA}(\varphi_t) + \frac{n}{V_\alpha}\int_X \dot{\varphi}_t\, dd^c\dot{\varphi}_t \wedge \omega_\varphi^{n-1} \\ &= \int_X \left\{\ddot{\varphi}_t\, \mathrm{MA}(\varphi_t) - \frac{n}{V_\alpha}\, d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1}\right\}.\end{aligned}$$

https://doi.org/10.1112/S0010437X18007170 Published online by Cambridge University Press

Thus, E is concave along Euclidean segments ($\ddot{\varphi}_t = 0$), affine along Mabuchi geodesics, and convex along Mabuchi subgeodesics. The cocycle condition follows by differentiating $E(t\varphi + (1-t)\psi)$.

These computations are merely heuristic as $t \to \varphi_t(x)$ is poorly regular when φ_t is a geodesic or subgeodesic. We can, however, approximate ω by $\omega_\varepsilon = \omega + \varepsilon\omega_X$; consider (φ_t^ε) the corresponding geodesic

$$E_{\omega_\varepsilon}(\varphi_t^\varepsilon) = \frac{1}{(n+1)V_\varepsilon} \sum_{j=0}^{n} \int_X \varphi_t^\varepsilon (\omega_\varepsilon + dd^c \varphi_t^\varepsilon) \wedge \omega_\varepsilon^{n-j}. \tag{6}$$

It follows from Proposition 1.6 that $\varepsilon \mapsto \varphi_t^\varepsilon$ decreases to φ_t; hence, $t \mapsto E(\varphi_t)$ is affine, where the limit of the affine maps $t \mapsto E_{\omega_\varepsilon}(\varphi_t^\varepsilon)$.

For subgeodesics, we again approximate ω by ω_ε and we proceed as in the Kähler case. □

Observe that $E(\varphi + t) = E(\varphi) + t$. Given $\varphi \in \mathcal{H}_\omega$ there exists a unique $c \in \mathbb{R}$ such that $E(\varphi + c) = 0$. The restriction of the Mabuchi metric to the fiber $E^{-1}(0)$ induces a Riemannian structure on the quotient space $\mathcal{H}_\alpha = \mathcal{H}_\omega/\mathbb{R}$ and allows decomposition of $\mathcal{H}_\omega = \mathcal{H}_\alpha \times \mathbb{R}$ as a product of Riemannian manifolds.

Definition 2.6. For $\varphi \in \mathrm{PSH}(X, \omega)$, we set

$$E(\varphi) := \inf\{E(\psi); \varphi \leqslant \psi \text{ and } \psi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)\} \in [-\infty, +\infty[$$

and $\mathcal{E}^1(X, \omega) := \{\varphi \in \mathrm{PSH}(X, \omega); E(\varphi) > -\infty\}$.

2.2.2 *Strong topology on* $\mathcal{E}^1(\alpha)$. Set

$$I(\varphi, \psi) = \int_X (\varphi - \psi)(\mathrm{MA}(\psi) - \mathrm{MA}(\varphi)).$$

It has been shown in [BBEGZ] that I defines a complete metrizable uniform structure on $\mathcal{E}^1(\alpha)$. More precisely, we identify $\mathcal{E}^1(\alpha)$ with the set

$$\mathcal{E}^1_{\mathrm{norm}}(X, \omega) = \Big\{\varphi \in \mathcal{E}^1(X, \omega) \mid \sup_X \varphi = 0\Big\}$$

of normalized potentials. Then:

(a) I is symmetric and positive on $\mathcal{E}^1_{\mathrm{norm}}(X, \omega)^2 \backslash \{\text{diagonal}\}$;
(b) I satisfies a quasi-triangle inequality [BBEGZ, Theorem 1.8];
(c) I induces a uniform structure that is metrizable [Bou07];
(d) the metric space $(\mathcal{E}^1(\alpha), d_I)$ is complete [BBEGZ, Proposition 2.4], where d_I denotes one of the distances induced by the uniform structure I.

Definition 2.7. The strong topology on $\mathcal{E}^1(\alpha)$ is the metrizable topology defined by I.

The corresponding notion of convergence is the *convergence in energy* previously introduced in [BBGZ13] (see [BBEGZ, Proposition 2.3]). It is the coarsest refinement of the weak topology such that E becomes continuous. In particular, if $T_j \longrightarrow T$ in $(\mathcal{E}^1(\alpha), d_I)$, then

$$T_j \longrightarrow T \quad \text{weakly and} \quad T_j^n \longrightarrow T^n$$

in the weak sense of Radon measures, while the Monge–Ampère operator is usually discontinuous for the weak topology of currents.

2.2.3 *Yet another distance.* To fit in with the notation of the next section, we introduce yet another notion of convergence in $\mathcal{E}^1(X,\omega)$. We set

$$I_1(\varphi,\psi) := \int_X |\varphi-\psi| \left[\frac{\mathrm{MA}(\varphi)+\mathrm{MA}(\psi)}{2}\right].$$

This symmetric quantity is non-negative. It follows from Proposition 2.3 that it only vanishes on the diagonal of $\mathcal{E}^1(X,\omega)^2$, while Theorem 3.6 will insure that it satisfies a quasi-triangle inequality. Hence, I_1 induces a uniform structure, which is metrizable [Bou07].

For $C>0$, we set

$$\mathcal{E}^1_C(X,\omega) := \{\varphi\in\mathcal{E}^1(X,\omega); E(\varphi)\geqslant -C \text{ and } \varphi\leqslant C\}.$$

It follows from Hartogs' lemma, the upper semi-continuity, and the concavity of E along Euclidean segments (Lemma 2.5) that this set is a compact and convex subset of $\mathrm{PSH}(X,\omega)$, when endowed with the L^1-topology (see [BBGZ13, Lemma 2.6]).

PROPOSITION 2.8. *For all $\varphi,\psi\in\mathcal{E}^1(X,\omega)$, $I(\varphi,\psi)\leqslant 2I_1(\varphi,\psi)$. Conversely, for each $C>0$, there exists $A>0$ such that, for all $\varphi,\psi\in\mathcal{E}^1_C(X,\omega)$,*

$$I_1(\varphi,\psi)\leqslant \int_X [2\max(\varphi,\psi)-(\varphi+\psi)]\,\mathrm{MA}(0) + A\,I(\varphi,\psi)^{1/2^n}. \tag{7}$$

In particular, the topologies induced by I, I_1 on $\mathcal{E}^1_{\mathrm{norm}}(X,\omega)$ are the same.

Observe that I_1 induces a distance on $\mathcal{E}^1(X,\omega)$, but I is merely defined on $\mathcal{E}^1_{\mathrm{norm}}(X,\omega)$, as $I(\varphi+c,\psi+c')=I(\varphi,\psi)$, for any $c,c'\in\mathbb{R}$.

Proof. The first inequality is obvious, as

$$I(\varphi,\psi)=\int_X(\varphi-\psi)(\mathrm{MA}(\psi)-\mathrm{MA}(\varphi))\leqslant\int_X|\varphi-\psi|(\mathrm{MA}(\psi)+\mathrm{MA}(\varphi)).$$

It follows from Proposition 2.13 that

$$I_1(\varphi,\psi)=I_1(\varphi,\max(\varphi,\psi))+I_1(\max(\varphi,\psi),\psi),$$

hence it suffices to establish the second inequality when $\varphi\leqslant\psi$. In this case

$$I_1(\varphi,\psi)\leqslant\int_X(\psi-\varphi)\,\mathrm{MA}(\varphi),$$

by Lemma 2.12, while the Cauchy–Schwarz inequality yields

$$\begin{aligned}\int_X(\psi-\varphi)\,\mathrm{MA}(\varphi)&=\int_X(\psi-\varphi)\,\mathrm{MA}(0)+\int_X d(\varphi-\psi)\wedge d^c\varphi\wedge S_\varphi\\&\leqslant\int_X(\psi-\varphi)\,\mathrm{MA}(0)+I(\varphi,0)^{1/2}\bigg(\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge S_\varphi\bigg)^{1/2},\end{aligned}$$

where we have set $S_\varphi:=\sum_{j=0}^{n-1}\omega_\varphi^j\wedge\omega^{n-1-j}$. Observing that $S_\varphi\leqslant 2^{n-1}\omega_{\varphi/2}^{n-1}$, we can invoke [BBEGZ, Lemma 1.9] to obtain

$$\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge S_\varphi\leqslant c_nI(\varphi,\psi)^{1/2^{n-1}}\bigg\{I\Big(\varphi,\frac{\varphi}{2}\Big)^{1-1/2^{n-1}}+I\Big(\psi,\frac{\varphi}{2}\Big)^{1-1/2^{n-1}}\bigg\}.$$

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Now $I(\varphi,\varphi/2) \leqslant a_n I(\varphi,0) \leqslant C'$ and [BBEGZ, Theorem 1.3] yields

$$I(\psi,\varphi/2) \leqslant b_n\{I(\psi,0)+I(\varphi/2,0)\} \leqslant b_n'\{I(\psi,0)+I(\varphi,0)\} \leqslant C''.$$

We thus get (7).

To prove the last statement, we need to show that, given a sequence $\varphi_j \in \mathcal{E}^1_{\mathrm{norm}}(X,\omega)$ converging to ψ with respect to I, then it also converges to ψ with respect to I_1, and vice versa. We first note that the I-convergence implies the L^1-convergence of the potentials [GZ17, Theorem 10.37]. This ensures that

$$\int_X [2\max(\varphi_j,\psi)-(\varphi_j+\psi)]\,\mathrm{MA}(0) \to 0 \quad \text{as } j \to +\infty,$$

and moreover we have that $\varphi_j, \psi \in \mathcal{E}^1_C(X,\omega)$ for some $C>0$ [GZ17, Lemma 10.33 and Definition 10.34]. The I_1-convergence would then follow from (7). Moreover, since $I(\varphi_j,\psi) \leqslant 2I_1(\varphi_j,\psi)$, we conclude that the I_1-convergence implies the I-convergence. □

2.3 The complete metric spaces $\mathcal{E}^p(\alpha)$

Fix $p \geqslant 1$. Following [GZ07, BEGZ10], we consider the following finite-energy classes.

Definition 2.9. We set

$$\mathcal{E}^p(X,\omega) := \{\varphi \in \mathcal{E}(X,\omega)/|\varphi|^p \in L^1(\mathrm{MA}(\varphi))\}$$

and let $\mathcal{E}^p(\alpha) = \{T=\omega+dd^c\varphi \mid \varphi \in \mathcal{E}^p(X,\omega)\}$ denote the corresponding sets of finite-energy currents.

On the class $\mathcal{E}^p(X,\omega)$, $p \geqslant 1$, we define

$$I_p(\varphi,\psi) := \left(\int_X |\varphi-\psi|^p \left[\frac{\mathrm{MA}(\varphi)+\mathrm{MA}(\psi)}{2}\right]\right)^{1/p}.$$

This quantity is well defined by [GZ07, Proposition 3.6]. It is obviously non-negative and symmetric. It follows from the domination principle (Proposition 2.3) that

$$I_p(\varphi,\psi)=0 \Longrightarrow \varphi=\psi.$$

Moreover, it will follow from Theorem 3.6 (which shows in particular that I_p satisfies a quasi-triangle inequality) that I_p induces a uniform structure. We can then define the following:

Definition 2.10. The strong topology on $\mathcal{E}^p(\alpha)$ is the one induced by I_p.

By [BEGZ10, Theorem 2.17], a decreasing sequence converges strongly. We also have good convergence properties if we approximate by slightly larger finite-energy classes $\mathcal{E}^p(X,\omega_\varepsilon)$.

Proposition 2.11. *Fix $\omega_\varepsilon = \omega+\varepsilon\omega_X$, $\varepsilon>0$. If $\varphi,\psi \in \mathcal{E}^p(X,\omega)\cap L^\infty(X)$, then $\varphi,\psi \in \mathcal{E}^p(X,\omega_\varepsilon)\cap L^\infty(X)$ and $I_{p,\omega_\varepsilon}(\varphi,\psi) \to I_{p,\omega}(\varphi,\psi)$ as $\varepsilon \to 0$.*

Moreover, if $\varphi,\psi \in \mathcal{E}^p(X,\omega)$ and φ_j,ψ_j are sequences of smooth ω_{ε_j}-psh functions decreasing to φ,ψ with $\varepsilon_j \to 0$, then

$$I_{p,\omega_{\varepsilon_j}}(\varphi_j,\psi_j) \to I_{p,\omega}(\varphi,\psi)$$

as j goes to $+\infty$.

https://doi.org/10.1112/S0010437X18007170 Published online by Cambridge University Press

Proof. Note that φ, ψ belong to any energy class with respect to any Kähler form since they are bounded. In particular, $\varphi, \psi \in \mathcal{E}^p(X, \omega_\varepsilon)$. The first assertion follows from the fact that $(\omega_\varepsilon + dd^c\varphi)^n$ and $(\omega_\varepsilon + dd^c\psi)^n$ converge weakly to $(\omega + dd^c\varphi)^n$ and $(\omega + dd^c\psi)^n$ as $\varepsilon \to 0$, respectively. For the second statement, we observe that, by symmetry, it suffices to prove that

$$\int_X |\varphi_j - \psi_j|^p (\omega_{\varepsilon_j} + dd^c\varphi_j)^n \to \int_X |\varphi - \psi|^p (\omega + dd^c\varphi)^n \quad \text{as } j \to +\infty.$$

Given a bounded function f on X, we set

$$|f|_p := \left(\int_X |f|^p (\omega_{\varepsilon_j} + dd^c\varphi_j)^n \right)^{1/p}.$$

The triangle inequality yields

$$|\varphi_j - \psi_j|_p \leqslant |\varphi - \psi|_p + |(\varphi_j - \varphi)| + |(\psi - \psi_j)|_p$$

and similarly

$$|\varphi_j - \psi_j|_p \geqslant |\varphi - \psi|_p - |(\varphi_j - \varphi)| - |(\psi - \psi_j)|_p.$$

Since $\varphi - \psi$ is a positive quasi-continuous uniformly bounded function on X, it follows from [GZ17, Theorem 4.26] that

$$|\varphi - \psi|_p^p = \int_X |\varphi - \psi|^p (\omega_{\varepsilon_j} + dd^c\varphi_j)^n \to \int_X |\varphi - \psi|^p (\omega + dd^c\varphi)^n$$

as $j \to +\infty$. Moreover, we claim that the terms $|(\varphi_j - \varphi)|_p$ and $|(\psi - \psi_j)|_p$ go to 0 as $j \to +\infty$. Lemma 2.12, together with the fact that $\omega_{\varepsilon_j} \leqslant \omega + \omega_X$, yields

$$\int_X (\varphi_j - \varphi)^p (\omega_{\varepsilon_j} + dd^c\varphi_j)^n \leqslant \int_X (\varphi_j - \varphi)^p (\omega + \omega_X + dd^c\varphi)^n.$$

Note that $\varphi_j, \varphi \in \mathcal{E}^p(X, \omega + \omega_X)$ (since they are bounded). Hence [GZ07, Theorem 3.8] ensures that the integral at the right-hand side of this inequality is finite.

Since φ_j is decreasing to φ, it then follows from the dominated convergence theorem that $|(\varphi_j - \varphi)|_p^p \to 0$ as $j \to +\infty$. Fix $j_0 < j$. Then

$$\int_X (\psi_j - \psi)^p (\omega_{\varepsilon_j} + dd^c\varphi_j)^n \leqslant \int_X (\psi_{j_0} - \psi)^p (\omega + \omega_X + dd^c\varphi_j)^n.$$

It follows again from the continuity of the Monge–Ampère operator along the decreasing sequence, [Kol05, Corollary 1.14], and the dominated convergence theorem that letting $j \to +\infty$ and then $j_0 \to +\infty$ we get

$$\int_X (\psi_{j_0} - \psi)^p (\omega + \omega_X + dd^c\varphi_j)^n \to 0.$$

Thus, $|(\psi_j - \psi)|_p^p \to 0$ as $j \to +\infty$. Hence the conclusion. □

It follows from Hölder's inequality that the strong topology on $\mathcal{E}^p(\alpha)$ is stronger than the one on $\mathcal{E}^1(\alpha)$: if a sequence $(\varphi_j) \in \mathcal{E}^p(X, \omega)$ is a Cauchy sequence for I_p, then it is a Cauchy sequence in $(\mathcal{E}^1(X, \omega), d_I)$, since

$$0 \leqslant I(\varphi, \psi) = \int_X (\varphi - \psi)[\mathrm{MA}(\psi) - \mathrm{MA}(\varphi)] \leqslant 2^{1/p} I_p(\varphi, \psi).$$

Since $(\mathcal{E}^1(X,\omega), d_I)$ is complete, there is $\varphi \in \mathcal{E}^1(X,\omega)$ such that $d_I(\varphi_j,\varphi) \to 0$. Now $I_p(\varphi_j,0)$ is bounded and $\mathrm{MA}(\varphi_j)$ converges to $\mathrm{MA}(\varphi)$ (by [BBGZ13, Proposition 5.6]). Thus, $\varphi \in \mathcal{E}^p(X,\omega)$ by Fatou's and Hartogs' lemmas.

One would now like to prove that $I_p(\varphi_j,\varphi) \to 0$ and conclude that the space $(\mathcal{E}^p(X,\omega), I_p)$ is complete, arguing as in [BBEGZ, Proposition 2.4]. We refer the reader to Theorem 4.2 for a neat treatment.

Lemma 2.12. *Let φ,ψ be bounded ω-psh functions and S be a positive closed current of bidimension $(1,1)$ on X. If $\varphi \leqslant \psi$, then*

$$\int_X (\psi-\varphi)^p \omega_\psi \wedge S \leqslant \int_X (\psi-\varphi)^p \omega_\varphi \wedge S.$$

In particular, $V_\alpha^{-1}\int_X (\psi-\varphi)^p \omega_\psi^j \wedge \omega_\varphi^{n-j} \leqslant \int_X (\psi-\varphi)^p \,\mathrm{MA}(\varphi)$.

Proof. By Stokes' theorem,

$$\int_X (\psi-\varphi)^p \omega_\varphi \wedge S - \int_X (\psi-\varphi)^p \omega_\psi \wedge S = p\int_X (\psi-\varphi)^{p-1} d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge S$$

is non-negative if $(\psi-\varphi) \geqslant 0$.

The second assertion follows by applying the first one inductively. □

We now establish a few useful properties of I_p that will notably allow us to compare I_p with d_p in the next section.

Proposition 2.13. *For $\varphi,\psi \in \mathcal{E}^p(X,\omega)$,*

$$I_p(\varphi,\psi)^p = I_p(\varphi,\max(\varphi,\psi))^p + I_p(\max(\varphi,\psi),\psi)^p.$$

Proof. Recall that the maximum principle ensures that

$$\mathbb{1}_{\{\varphi<\psi\}}\,\mathrm{MA}(\max(\varphi,\psi)) = \mathbb{1}_{\{\varphi<\psi\}}\,\mathrm{MA}(\psi),$$

while $(\varphi-\max(\varphi,\psi))^p = 0$ on $(\varphi \geqslant \psi)$; thus,

$$2I_p(\varphi,\max(\varphi,\psi))^p = \int_{\{\varphi<\psi\}} |\varphi-\psi|^p[\mathrm{MA}(\varphi)+\mathrm{MA}(\psi)].$$

Similarly, $2I_p(\psi,\max(\varphi,\psi))^p = \int_{\{\varphi>\psi\}} |\varphi-\psi|^p[\mathrm{MA}(\varphi)+\mathrm{MA}(\psi)]$ and the result follows, since

$$I_p(\varphi,\psi)^p = \frac{1}{2}\int_{\{\varphi\neq\psi\}} |\varphi-\psi|^p[\mathrm{MA}(\varphi)+\mathrm{MA}(\psi)].$$ □

Corollary 2.14. *For all $\varphi,\psi \in \mathcal{E}^p(X,\omega)$,*

$$I_p\left(\frac{\varphi+\psi}{2},\psi\right) \leqslant I_p(\varphi,\psi).$$

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Proof. By approximating φ, ψ from the above by a decreasing sequence, it suffices to treat the case when $\varphi, \psi \in \mathcal{H}_\omega$. Changing ω in ω_ψ, we can further assume that $\psi = 0$. It follows from Proposition 2.13 that

$$I_p(0, \varphi/2)^p = I_p(0, \max(0, \varphi/2))^p + I_p(\max(0, \varphi/2), \varphi/2)^p.$$

It follows from Lemma 2.12 that

$$\begin{aligned} I_p(0, \max(0, \varphi/2))^p &\leqslant \int_X \max(0, \varphi/2)^p \operatorname{MA}(0) \\ &= 2^{-p} \int_X \max(0, \varphi)^p \operatorname{MA}(0) \leqslant I_p(0, \max(0, \varphi))^p. \end{aligned}$$

We claim that, for all $0 \leqslant j \leqslant n$,

$$\int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi^j \wedge \omega^{n-j} \leqslant \int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi^n.$$

Assuming this for the moment, it follows again from Lemma 2.12 that

$$\begin{aligned} I_p(\max(0, \varphi/2), \varphi/2)^p &\leqslant \int_X (\max(0, \varphi/2) - \varphi/2)^p \operatorname{MA}(\varphi/2) \\ &= \frac{1}{2^{n+p} V_\alpha} \sum_{j=0}^n C_n^j \int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi^j \wedge \omega^{n-j} \\ &\leqslant \frac{1}{2} \int_X (\max(0, \varphi) - \varphi)^p \operatorname{MA}(\varphi) \leqslant I_p(\varphi, \max(0, \varphi))^p. \end{aligned}$$

We infer

$$I_p(0, \varphi/2)^p \leqslant I_p(0, \max(0, \varphi))^p + I_p(\max(0, \varphi), \varphi)^p = I_p(0, \varphi)^p,$$

by using Proposition 2.13 again.

It remains to justify our claim. Set $S = \omega^{j-1} \wedge \omega_\varphi^{n-j}$. It suffices, by induction, to establish the following inequality:

$$\begin{aligned} \int_X (\max(0, \varphi) - \varphi)^p \omega \wedge S &= \int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi \wedge S - \int_X (\max(0, \varphi) - \varphi)^p \, dd^c \varphi \wedge S \\ &\leqslant \int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi \wedge S. \end{aligned}$$

This follows by observing that

$$\begin{aligned} -\int_X (\max(0, \varphi) - \varphi)^p \, dd^c \varphi \wedge S &= p \int_X (\max(0, \varphi) - \varphi)^{p-1} \, d(\max(0, \varphi) - \varphi) \wedge d^c \varphi \wedge S \\ &= -p \int_{\{\varphi < 0\}} (-\varphi)^{p-1} \, d\varphi \wedge d^c \varphi \wedge S \leqslant 0. \end{aligned}$$

□

3. Comparing distances

In this section, we show that I_p is equivalent to d_p (Theorem 3.6). Recall that

$$\mathcal{H}_{bd} := \{\varphi \in \operatorname{PSH}(X, \omega) \cap L^\infty(X), \varphi = P_\omega(f) \text{ for some } f \in C^0(X) \text{ with } dd^c f \leqslant C\omega_X, C > 0\}.$$

In the following, we are going to use several times and in a crucial way the fact that Theorem 1.21 ensures

$$d_p^p(\varphi_0, \varphi_1) = \int_X |\dot{\varphi}_0|^p \frac{(\omega + dd^c \varphi_0)^n}{V} = \int_X |\dot{\varphi}_1|^p \frac{(\omega + dd^c \varphi_1)^n}{V}, \quad \forall \varphi_0, \varphi_1 \in \mathcal{H}_{bd}.$$

https://doi.org/10.1112/S0010437X18007170 Published online by Cambridge University Press

3.1 Kiselman transform and geodesics

Let $(\varphi_t)_{0\leqslant t\leqslant 1}$ be the Mabuchi geodesic. For all $x \in X$, $t \in [0,1] \mapsto \varphi_t(x) \in \mathbb{R}$ is convex. It is natural to consider its Legendre transform, $u_s(x) : s \mapsto \sup_{t\in[0,1]}\{st - \varphi_t(x)\}$. This function is convex in s, but the dependence in x is $-\omega$-psh, so we rather consider $-u_s$. We finally change s in $-s$ to obtain a more elegant formula,

$$\psi_s(x) := \inf_{0\leqslant t\leqslant 1}\{st + \varphi_t(x)\}.$$

Proposition 3.1. *The functions $x \mapsto \psi_s(x)$ are ω-plurisubharmonic. In particular, $x \mapsto \psi_0(x) = \inf_{0\leqslant t\leqslant 1}\varphi_t(x)$ is ω-psh.*

This is the minimum principle of Kiselman [Kis78]. For $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$, we let $\varphi_0 \vee \varphi_1$ denote the greatest ω-psh function that lies below φ_0 and φ_1. In the notation of Berman and Demailly [BD12]

$$\varphi_0 \vee \varphi_1 = P(\min(\varphi_0, \varphi_1)),$$

while $\varphi_0 \vee \varphi_1$ is denoted $P(\varphi_0, \varphi_1)$ in [Dar17c].

An important consequence of Kiselman's minimum principle [Kis78] is the following observation of Darvas and Rubinstein [DR16].

Proposition 3.2. *The function $\varphi_0 \vee \varphi_1$ is a bounded ω-psh, which has a locally bounded Laplacian on the ample locus of $\alpha = \{\omega\}$, and its Monge–Ampère measure* $\mathrm{MA}(\varphi_0 \vee \varphi_1)$ *is supported on the coincidence set*

$$\{x \in X \mid \varphi_0 \vee \varphi_1(x) = \min(\varphi_0, \varphi_1)(x)\}.$$

Moreover, $\mathrm{MA}(\varphi_0 \vee \varphi_1) = \mathbb{1}_{\{\varphi_0\vee\varphi_1=\varphi_0\}}\mathrm{MA}(\varphi_0) + \mathbb{1}_{\{\varphi_0\vee\varphi_1=\varphi_1<\varphi_0\}}\mathrm{MA}(\varphi_1)$.

Let (φ_t) be the Mabuchi geodesic joining φ_0 and φ_1. Then, for all $x \in X$,

$$\varphi_0 \vee \varphi_1(x) = \inf_{t\in[0,1]}\varphi_t(x).$$

Proof. It follows from a classical balayage procedure that goes back to Bedford and Taylor [BT82] that $\mathrm{MA}(\varphi_0 \vee \varphi_1)$ is supported on the coincidence set $\{x \in X \mid \varphi_0 \vee \varphi_1(x) = \min(\varphi_0, \varphi_1)(x)\}$. This holds true more generally for the Monge–Ampère measure of any envelope, namely

$$\mathbb{1}_{\{P(h)<h\}}\mathrm{MA}(P(h)) \equiv 0,$$

where h is a bounded lower semi-continuous function.

We have observed in Proposition 3.1 that $x \mapsto \inf_{t\in[0,1]}\varphi_t(x)$ is a ω-psh function. Since it lies both below φ_0 and φ_1, we infer

$$\inf_{t\in[0,1]}\varphi_t \leqslant \varphi_0 \vee \varphi_1.$$

Conversely, $(t,x) \mapsto \varphi_0 \vee \varphi_1(x)$ is a subgeodesic (independent of t); hence, for all t, x, $\varphi_0 \vee \varphi_1(x) \leqslant \varphi_t(x)$. Thus, $\psi := \varphi_0 \vee \varphi_1 = \inf_{t\in[0,1]}\varphi_t$; hence, ψ is bounded, thanks to Proposition 1.4.

By Proposition 3.1, ψ is ω-psh, hence $A\omega_X$-psh for some Kähler form ω_X and $A > 0$. Thus, $\sup_X \Delta_{\omega_X}\psi \geqslant -C$ for some $C > 0$.

It follows from the work of Berman and Demailly [BD12] (see also [Ber13, Theorem 1.2]) that for any compact subset $K \subset \mathrm{Amp}(\alpha)$, there exists $C_K > 0$, such that, for all $t \in [0,1]$,

$$\sup_K \Delta_{\omega_X} \varphi_t < C_K n.$$

Thus $(-\varphi_t)$ is a family of $C_K\omega_X$-psh functions in a neighborhood of K, which are uniformly bounded from above. Thus,

$$-\psi = \sup_{0\leqslant t\leqslant 1} (-\varphi_t) = -\inf_{0\leqslant t\leqslant 1} \varphi_t$$

is $C_K\omega_X$-psh near K, in particular $\Delta_{\omega_X}\psi < C_K n$. This means that ψ has a locally bounded Laplacian on $\mathrm{Amp}(\alpha)$.

It follows then from classical arguments that the measure $\mathrm{MA}(\varphi_0 \vee \varphi_1)$ is absolutely continuous with respect to the Lebesgue measure. Since $\varphi_0 \vee \varphi_1, \varphi_0$ (respectively $\varphi_0 \vee \varphi_1, \varphi_1$) have locally bounded Laplacians in $\mathrm{Amp}(\alpha)$, it follows from [GT83, Lemma 7.7] that their second partial derivatives agree on $\{\varphi_0 \vee \varphi_1 = \varphi_0\}$ (respectively on $\{\varphi_0 \vee \varphi_1 = \varphi_1\}$), hence

$$\mathrm{MA}(\varphi_0 \vee \varphi_1) = \mathbb{1}_{\{\varphi_0\vee\varphi_1=\varphi_0\}} \mathrm{MA}(\varphi_0) + \mathbb{1}_{\{\varphi_0\vee\varphi_1=\varphi_1<\varphi_0\}} \mathrm{MA}(\varphi_1).$$

We have used here the fact that none of the measures $\mathrm{MA}(\varphi_0 \vee \varphi_1), \mathrm{MA}(\varphi_0)$, $\mathrm{MA}(\varphi_1)$ charges the pluripolar set $X \setminus \mathrm{Amp}(\alpha)$. □

A basic observation that we shall use on several occasions is the following.

Lemma 3.3. *Assume $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$ and let $(\varphi_t)_{0\leqslant t\leqslant 1}$ be the Mabuchi geodesic joining φ_0 to φ_1. Then*

$$d_p(\varphi_0, \varphi_1) \leqslant \|\varphi_1 - \varphi_0\|_{L^\infty(X)}.$$

Moreover,

(i) *if $\varphi_0(x) \leqslant \varphi_1(x)$ for some $x \in X$, then $\dot\varphi_1(x) \geqslant 0$;*
(ii) *if $\varphi_0(x) \leqslant \varphi_1(x)$ for all $x \in X$ then $\dot\varphi_t(x) \geqslant 0$ for all $x \in X$ and a.e. $t \in [0,1]$.*

By symmetry, if $\varphi_1(x) \leqslant \varphi_0(x)$, it follows that $\dot\varphi_0(x) \leqslant 0$. Moreover, if $\varphi_1(x) \leqslant \varphi_0(x)$ for all $x \in X$ then $\dot\varphi_t(x) \leqslant 0$ for a.e. x, t. Here, and in the following, $\dot\varphi_0$, $\dot\varphi_1$ denote the right and left derivatives, respectively, while we recall that $\dot\varphi_t(x)$ is well defined for a.e. (x,t).

Proof. From Theorem 1.21 we know that $d_p^p(\varphi_0, \varphi_1) = \int_X |\dot\varphi_0|^p \,\mathrm{MA}(\varphi_0)$. Moreover, Proposition 1.4 ensures that $|\dot\varphi_0| \leqslant \|\varphi_1 - \varphi_0\|_{L^\infty(X)}$. Hence, the first statement.

Assume $\dot\varphi_1(x) < 0$. Since $t \mapsto \varphi_t(x)$ is convex, we infer $\dot\varphi_t(x) \leqslant \dot\varphi_1(x) < 0$. Thus, $t \mapsto \varphi_t(x)$ is decreasing, hence $\varphi_1(x) < \varphi_0(x)$, a contradiction. This proves part (i).

Assume now that $\varphi_0(x) \leqslant \varphi_1(x)$ for all $x \in X$. Then

$$\varphi_0 \leqslant \varphi_t \leqslant \varphi_1.$$

The first of these inequalities follows from the fact that, by Proposition 1.4,

$$\varphi = \sup\{u \ u \in \mathrm{PSH}(M, \omega) : u \leqslant \varphi_{0,1} \text{ on } M\},$$

with $\varphi(x, t+is) = \varphi_t(x)$, and that $\varphi_0(x, t+is) = \varphi_0(x)$ is a subsolution (i.e. a candidate in the envelope). The other inequality follows from the fact that $\varphi_1(x, t+is) = \varphi_1(x)$ is a supersolution of (3) since $(\omega + dd^c_{x,z}\varphi_1)^{n+1} = 0$ and $\varphi_1 \geqslant \varphi_{0,1}$. The same argument shows that $\varphi_0 \leqslant \varphi_s \leqslant \varphi_t$ for all $0 < s < t$ and $x \in X$, hence $\dot\varphi_t(x) \geqslant 0$ for all $x \in X$ and a.e. $t \in [0,1]$, since the derivative in time of φ_t is well defined for a.e. t. □

We now establish a very useful relation established by Darvas [Dar17c, Proposition 8.1] when ω is Kähler (see also [Dar15, Corollary 4.14]).

Proposition 3.4. *Assume $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$. Then, for all $p \geqslant 1$,*

$$d_p^p(\varphi_0, \varphi_1) = d_p^p(\varphi_0, \varphi_0 \vee \varphi_1) + d_p^p(\varphi_0 \vee \varphi_1, \varphi_1).$$

Proof. We proceed by approximation, so as to reduce to the Kähler case. The identity is known to hold for $d_{p,\varepsilon}$ and $\varphi_0 \vee_\varepsilon \varphi_1$, where $d_{p,\varepsilon}$ denotes the distance associated with the Kähler form $\omega_\varepsilon = \omega + \varepsilon\omega_X$ and $\varphi_0 \vee_\varepsilon \varphi_1$ is the greatest ω_ε-psh function that lies below $\min(\varphi_0, \varphi_1)$.

Using Theorem 1.21 and the triangle inequality, the proof boils down to checking that $d_{p,\varepsilon}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1) \to 0$ as $\varepsilon \to 0$. The same arguments used in the proof of Proposition 1.16 yield

$$d_{p,\varepsilon}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1) \leqslant d_{p,\varepsilon'}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1), \quad \varepsilon < \varepsilon'.$$

We claim that $d_{p,\varepsilon'}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1)$ goes to zero as ε goes to zero, since $\varphi_0 \vee_\varepsilon \varphi_1$ decreases to $\varphi_0 \vee \varphi_1$ as $\varepsilon \to 0$. Indeed, observe that $\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1 \in \mathcal{E}^p(X, \omega'_\varepsilon) \cap L^\infty(X)$ and that by Proposition 3.8 we know that

$$d_{p,\varepsilon'}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1) \leqslant 2I_{p,\varepsilon'}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1).$$

The same arguments in the proof of Proposition 2.11 then show that $I_{p,\varepsilon'}(\varphi_0 \vee \varphi_1, \varphi_0 \vee_\varepsilon \varphi_1) \to 0$ as ε goes to zero. The conclusion then follows. □

We note for later use the following consequence.

Corollary 3.5. *If $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$, then*

$$d_p(\varphi_0, \varphi_0 \vee \varphi_1) \leqslant d_p(\varphi_0, \varphi_1).$$

3.2 Comparing d_p and I_p

The goal of this section is to establish that d_p and I_p are equivalent, extending [Dar15, Theorem 5.5].

Theorem 3.6. *For all $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$,*

$$2^{-1} d_p(\varphi_0, \varphi_1) \leqslant I_p(\varphi_0, \varphi_1) \leqslant 2^{4+(2n-1)/p} d_p(\varphi_0, \varphi_1).$$

It follows from Definition 1.10 and Proposition 2.11 that

$$d_p(\varphi_0, \varphi_1) = \lim_{\varepsilon \to 0} d_{p,\varepsilon}(\varphi_0, \varphi_1) \quad \text{and} \quad I_p(\varphi_0, \varphi_1) = \lim_{\varepsilon \to 0} I_{p,\varepsilon}(\varphi_0, \varphi_1),$$

so it suffices to establish these inequalities when ω is a Kähler form.

We nevertheless give a direct proof, valid when ω is merely semi-positive, with several intermediate results of independent interest. Several of these results have been obtained by Darvas [Dar17b, Dar17c, Dar15] when ω is Kähler.

Lemma 3.7. *Assume that $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$ satisfy $\varphi_0 \leqslant \varphi_1$.*

(i) $d_p(\varphi_1, (\varphi_0 + \varphi_1)/2) \leqslant d_p(\varphi_0, \varphi_1)$.
(ii) $d_p(\varphi_0, \varphi_1) \leqslant 2^{1+n/p} d_p(\varphi_0/2, \varphi_1/2)$.

(iii) *If* $\varphi_1 = 0$ *then* $d_p(\varphi_0, 0) \geqslant 2d_p(\varphi_0/2, 0)$.

(iv) *If* $\psi \in \mathcal{H}_{bd}$ *is such that* $\varphi_0 \leqslant \psi \leqslant \varphi_1$, *then*

$$\max\{d_p(\varphi_0, \psi); d_p(\psi, \varphi_1)\} \leqslant d_p(\varphi_0, \varphi_1).$$

Proof. Let φ_t (respectively ψ_t) denote the Mabuchi geodesic joining φ_0 (respectively $(\varphi_0+\varphi_1)/2$) to φ_1. Since $\varphi_0 \leqslant \varphi_1$, it follows from Lemma 3.3(ii) that $t \mapsto \varphi_t$, $t \mapsto \psi_t$ are increasing and $\varphi_t \leqslant \psi_t$, hence

$$\frac{\varphi_t - \varphi_1}{t-1} \geqslant \frac{\psi_t - \psi_1}{t-1},$$

since $\varphi_1 = \psi_1$. Therefore, $\dot{\varphi}_1 \geqslant \dot{\psi}_1 \geqslant 0$ and we infer

$$\int_X |\dot{\psi}_1|^p \operatorname{MA}(\psi_1) = d_p\left(\varphi_1, \frac{\varphi_0+\varphi_1}{2}\right)^p \leqslant d_p(\varphi_0, \varphi_1)^p = \int_X |\dot{\varphi}_1|^p \operatorname{MA}(\varphi_1).$$

This, together with Theorem 1.21, proves part (i).

Now let (φ_t) (respectively (ψ_t)) denote the geodesic joining φ_0 to φ_1 (respectively $\varphi_0/2$ to $\varphi_1/2$). Observe that $t \mapsto \varphi_t, \psi_t$ are increasing, hence $\dot{\varphi}_0 \geqslant 0$. The family $(\varphi_t/2)$ is a subgeodesic joining $\varphi_0/2$ to $\varphi_1/2$, hence $\varphi_t/2 \leqslant \psi_t$ and

$$0 \leqslant \frac{\dot{\varphi}_0}{2} \leqslant \dot{\psi}_0 \Longrightarrow |\dot{\varphi}_0|^p \leqslant 2^p |\dot{\psi}_0|^p.$$

Moreover, $\operatorname{MA}(\varphi_0) \leqslant 2^n \operatorname{MA}(\varphi_0/2)$, so we infer

$$d_p(\varphi_0, \varphi_1)^p = \int_X |\dot{\varphi}_0|^p \operatorname{MA}(\varphi_0) \leqslant 2^{n+p} d_p(\varphi_0/2, \varphi_1/2)^p,$$

which proves part (ii). A similar argument shows that

$$0 \leqslant \dot{\psi}_1 \leqslant \frac{\dot{\varphi}_1}{2} \Longrightarrow |\dot{\psi}_1|^p \leqslant 2^{-p} |\dot{\varphi}_1|^p.$$

Now $\operatorname{MA}(\varphi_1/2) = \operatorname{MA}(\varphi_1) = \operatorname{MA}(0)$ when $\varphi_1 = 0$, hence

$$d_p(\varphi_0, 0)^p = \int_X |\dot{\varphi}_1|^p \operatorname{MA}(0) \geqslant 2^p d_p(\varphi_0/2, 0)^p,$$

which yields part (iii).

It remains to prove part (iv). Let $(\varphi_t)_{0 \leqslant t \leqslant 1}$ (respectively $(\psi_t)_{0 \leqslant t \leqslant 1}$) be the geodesic joining φ_0 to φ_1 (respectively φ_0 to ψ). Observe that $\varphi_0 = \psi_0$ and $\psi_t \leqslant \varphi_t$, hence $\dot{\psi}_0 \leqslant \dot{\varphi}_0$. Moreover, $0 \leqslant \dot{\psi}_0$ since $t \mapsto \psi_t(x)$ is increasing. We infer

$$d_p(\varphi_0, \psi)^p = \int_X |\dot{\psi}_0|^p \operatorname{MA}(\varphi_0) \leqslant \int_X |\dot{\varphi}_0|^p \operatorname{MA}(\varphi_0) = d_p(\varphi_0, \varphi_1)^p.$$

The other inequality is proved similarly. □

Proposition 3.8. *For all* $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$,

$$0 \leqslant d_p(\varphi_0, \varphi_1) \leqslant 2I_p(\varphi_0, \varphi_1).$$

Moreover, if $\varphi_0 \leqslant \varphi_1$ *then* $I_p(\varphi_0, \varphi_1) \leqslant (\int_X (\varphi_1 - \varphi_0)^p \operatorname{MA}(\varphi_0))^{1/p}$ *and*

$$d_p(\varphi_0, \varphi_1) \leqslant \left(\int_X (\varphi_1 - \varphi_0)^p \operatorname{MA}(\varphi_0)\right)^{1/p} \leqslant 2^{1+n/p} d_p(\varphi_0, \varphi_1).$$

Proof. We first assume that $\varphi_0 \leqslant \varphi_1$. The inequality

$$I_p(\varphi_0, \varphi_1) \leqslant \left(\int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0) \right)^{1/p}$$

follows from Lemma 2.12. Let (φ_t) be the geodesic joining φ_0 to φ_1. It follows from Lemma 3.3 that $0 \leqslant \dot{\varphi}_0 \leqslant \varphi_1 - \varphi_0 \leqslant \dot{\varphi}_1$, hence

$$\int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_1) \leqslant \int_X (\dot{\varphi}_1)^p \, \mathrm{MA}(\varphi_1) = d_p(\varphi_0, \varphi_1)^p \tag{8}$$

and, similarly, $d_p(\varphi_0, \varphi_1)^p \leqslant \int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0)$.

We now show that $\int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0) \leqslant 2^{n+p} d(\varphi_0, \varphi_1)^p$. Observe that $(\varphi_0 + \varphi_1)/2 \in \mathcal{H}_{bd}$ with $\mathrm{MA}(\varphi_0) \leqslant 2^n \, \mathrm{MA}((\varphi_0 + \varphi_1)/2)$, hence

$$\begin{aligned} \int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0) &= 2^p \int_X \left(\frac{\varphi_0 + \varphi_1}{2} - \varphi_0 \right)^p \mathrm{MA}(\varphi_0) \\ &\leqslant 2^{n+p} \int_X \left(\frac{\varphi_0 + \varphi_1}{2} - \varphi_0 \right)^p \mathrm{MA}\left(\frac{\varphi_0 + \varphi_1}{2} \right) \\ &\leqslant 2^{n+p} d_p\left(\varphi_0, \frac{\varphi_0 + \varphi_1}{2} \right)^p, \end{aligned}$$

as follows from the first step of the proof, since $\varphi_0 \leqslant \varphi_1$. Lemma 3.7(iv) yields

$$d_p\left(\varphi_0, \frac{\varphi_0 + \varphi_1}{2} \right) \leqslant d_p(\varphi_0, \varphi_1),$$

hence $\int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0) \leqslant 2^{n+p} d_p(\varphi_0, \varphi_1)^p$.

We finally treat the first upper bound of the proposition, which does not require φ_0 to lie below φ_1. It follows from the triangle inequality that

$$\begin{aligned} d_p(\varphi_0, \varphi_1) &\leqslant d_p(\varphi_0, \max(\varphi_0, \varphi_1)) + d_p(\max(\varphi_0, \varphi_1), \varphi_1) \\ &\leqslant \left(\int_{\{\varphi_0 < \varphi_1\}} (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0) \right)^{1/p} + \left(\int_{\{\varphi_0 > \varphi_1\}} (\varphi_0 - \varphi_1)^p \, \mathrm{MA}(\varphi_1) \right)^{1/p} \\ &\leqslant 2^{1-1/p} \left(\int_X |\varphi_1 - \varphi_0|^p [\mathrm{MA}(\varphi_0) + \mathrm{MA}(\varphi_1)] \right)^{1/p} \\ &= 2 \left(\int_X |\varphi_1 - \varphi_0|^p \frac{[\mathrm{MA}(\varphi_0) + \mathrm{MA}(\varphi_1)]}{2} \right)^{1/p} \end{aligned}$$

by using the elementary inequality $a^{1/p} + b^{1/p} \leqslant 2^{1-1/p} (a + b)^{1/p}$. □

Remark 3.9. Working with $\psi = t\varphi_0 + (1 - t)\varphi_1$, $0 < t < 1$, instead of $(\varphi_0 + \varphi_1)/2$, one can improve this inequality and obtain

$$\left(\int_X (\varphi_1 - \varphi_0)^p \, \mathrm{MA}(\varphi_0) \right)^{1/p} \leqslant \frac{(n + p)^{1 + n/p}}{p \, n^{n/p}} d_p(\varphi_0, \varphi_1).$$

We now extend Lemma 3.7(i), following [Dar15, Lemma 5.3].

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Lemma 3.10. *For all $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$,*

$$d_p\left(\varphi_0, \frac{\varphi_0 + \varphi_1}{2}\right) \leqslant 2^{2+n/p} d_p(\varphi_0, \varphi_1).$$

Proof. When $\varphi_0 \leqslant \varphi_1$, this follows from Lemma 3.7(i). Replacing ω with $\omega + dd^c\varphi_0$, we can assume, without loss of generality, that $\varphi_0 = 0$. The triangle inequality yields

$$d_p\left(0, \frac{\varphi_1}{2}\right) \leqslant d_p\left(0, 0 \vee \frac{\varphi_1}{2}\right) + d_p\left(0 \vee \frac{\varphi_1}{2}, \frac{\varphi_1}{2}\right).$$

Observe that $0 \vee \varphi_1 \leqslant 0 \vee \varphi_1/2 \leqslant \min(0, \varphi_1/2)$. It follows, therefore, from Lemma 3.7(iv) that

$$d_p\left(0, 0 \vee \frac{\varphi_1}{2}\right) + d_p\left(0 \vee \frac{\varphi_1}{2}, \frac{\varphi_1}{2}\right) \leqslant d_p(0, 0 \vee \varphi_1) + d_p\left(0 \vee \varphi_1, \frac{\varphi_1}{2}\right).$$

Since $0 \vee \varphi_1 \leqslant 0$ and $0 \vee \varphi_1 \leqslant \varphi_1/2$, we can invoke Proposition 3.8 to obtain

$$\begin{aligned} d_p(0, 0 \vee \varphi_1) + d_p\left(0 \vee \varphi_1, \frac{\varphi_1}{2}\right) &\leqslant \left(\int_X |0 \vee \varphi_1|^p \operatorname{MA}(0 \vee \varphi_1)\right)^{1/p} \\ &\quad + \left(\int_X \left|0 \vee \varphi_1 - \frac{\varphi_1}{2}\right|^p \operatorname{MA}(0 \vee \varphi_1)\right)^{1/p} \\ &\leqslant 2^{1-1/p}\left(\int_X \left[|0 \vee \varphi_1|^p + \left|0 \vee \varphi_1 - \frac{\varphi_1}{2}\right|^p\right] \operatorname{MA}(0 \vee \varphi_1)\right)^{1/p}. \end{aligned}$$

Recall now that the measure $\operatorname{MA}(0 \vee \varphi_1)$ is supported on the contact set $S := \{x \in X; 0 \vee \varphi_1(x) = \min(0, \varphi_1)(x)\}$. On this set, we have

$$|0 \vee \varphi_1|^p + \left|0 \vee \varphi_1 - \frac{\varphi_1}{2}\right|^p \leqslant 2|\varphi_1|^p = 2[|0 \vee \varphi_1|^p + |0 \vee \varphi_1 - \varphi_1|^p],$$

while Proposition 3.8 yields

$$\begin{aligned} &\int_X [|0 \vee \varphi_1|^p + |0 \vee \varphi_1 - \varphi_1|^p] \operatorname{MA}(0 \vee \varphi_1) \\ &\quad \leqslant 2^{p+n}[d_p(0, 0 \vee \varphi_1)^p + d_p(0 \vee \varphi_1, \varphi_1)^p] = 2^{p+n} d_p(0, \varphi_1)^p, \end{aligned}$$

where the last equality follows from Proposition 3.4. Altogether, this yields $d_p(0, \varphi_1/2) \leqslant 2^{2+n/p} d_p(0, \varphi_1)$, as claimed. □

We are now ready to prove Theorem 3.6.

Proof. We have already observed that $d_p(\varphi_0, \varphi_1) \leqslant 2I_p(\varphi_0, \varphi_1)$ in Proposition 3.8, so we focus on the reverse control. Lemma 3.10 and Proposition 3.4 yield

$$\begin{aligned} 2^{2p+n} d_p^p(\varphi_0, \varphi_1) &\geqslant d_p^p\left(\varphi_0, \frac{\varphi_0 + \varphi_1}{2}\right) \\ &= d_p^p\left(\varphi_0, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right) + d_p^p\left(\frac{\varphi_0 + \varphi_1}{2}, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right). \end{aligned}$$

It follows from (8) together with the fact that $2^n \operatorname{MA}((\varphi_0 + \varphi_1)/2) \geqslant \operatorname{MA}(\varphi_0)$ that

$$d_p^p\left(\varphi_0, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right) \geqslant \int_X \left(\varphi_0 - \frac{\varphi_0 + \varphi_1}{2} \vee \varphi_0\right)^p \operatorname{MA}(\varphi_0)$$

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and

$$d_p^p\Big(\frac{\varphi_0+\varphi_1}{2},\varphi_0\vee\frac{\varphi_0+\varphi_1}{2}\Big)\geqslant 2^{-n}\int_X\Big(\frac{\varphi_0+\varphi_1}{2}-\varphi_0\vee\frac{\varphi_0+\varphi_1}{2}\Big)^p\,\mathrm{MA}(\varphi_0).$$

Hence

$$\begin{aligned}d_p^p(\varphi_0,\varphi_1)&\geqslant 2^{-2(p+n)}\int_X\Big[\Big(\varphi_0-\frac{\varphi_0+\varphi_1}{2}\vee\varphi_0\Big)^p+\Big(\frac{\varphi_0+\varphi_1}{2}-\frac{\varphi_0+\varphi_1}{2}\vee\varphi_0\Big)^p\Big]\,\mathrm{MA}(\varphi_0)\\&\geqslant 2^{1-3p-2n}\int_X\Big|\varphi_0-\frac{\varphi_0+\varphi_1}{2}\Big|^p\,\mathrm{MA}(\varphi_0)\\&=2^{1-4p-2n}\int_X|\varphi_0-\varphi_1|^p\,\mathrm{MA}(\varphi_0),\end{aligned}$$

where in the last inequality we used the fact that $|a-b|^p\leqslant 2^{p-1}(a^p+b^p)$, for any $a,b\in\mathbb{R}^+$.

Reversing the roles of φ_0 and φ_1, we get

$$d_p^p(\varphi_0,\varphi_1)\geqslant 2^{1-4p-2n}\int_X|\varphi_1-\varphi_0|^p\,\mathrm{MA}(\varphi_1),$$

from which it follows that $d_p^p(\varphi_0,\varphi_1)\geqslant 2^{1-4p-2n}I_p^p(\varphi_0,\varphi_1)$. □

3.3 Controlling the supremum

It follows from previous results that the supremum of a bounded potential with locally bounded Laplacian in $\mathrm{Amp}(\alpha)$ is controlled by the distance to the base point.

Lemma 3.11. *There exists $C>0$, such that for all $\varphi\in\mathcal{H}_{bd}$,*

$$-2^{4+2n}d_1(0,\varphi)\leqslant\sup_X\varphi\leqslant 2^{4+2n}(n+1)d_1(0,\varphi)+C.$$

Proof. If $\sup_X\varphi\leqslant 0$, then $\sup_X\varphi\leqslant 0\leqslant(n+1)d_1(0,\varphi)+C$, while

$$-d_1(0,\varphi)=E(\varphi)\leqslant\sup_X\varphi,$$

as follows from Proposition 3.12. We therefore assume in what follows that $\sup_X\varphi\geqslant 0$. If $\varphi\geqslant 0$, Proposition 3.12 yields

$$\frac{1}{n+1}\int_X\varphi\,\mathrm{MA}(0)\leqslant E(\varphi)=d_1(0,\varphi).$$

It is a classical consequence of the ω-plurisubharmonicity [GZ05, Proposition 2.7] that there exists $C>0$ such that, for all $\varphi\in\mathrm{PSH}(X,\omega)$,

$$\sup_X\varphi\leqslant\int_X\varphi\,\mathrm{MA}(0)+C.$$

Thus, $\sup_X\varphi\leqslant(n+1)d_1(0,\varphi)+C$.

When $\sup_X\varphi\geqslant 0$ but φ takes both positive and negative values, we set $\psi=\max(0,\varphi)$ and observe that $\sup_X\psi=\sup_X\varphi$. Using Propositions 2.13 and 3.8 and Theorem 3.6, we obtain

$$d_1(0,\max(0,\varphi))\leqslant 2I_1(0,\max(0,\varphi))\leqslant 2I_1(0,\varphi)\leqslant 2^{5-(2n-1)/p}d_1(0,\varphi).$$

The conclusion, therefore, follows from the previous case. □

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PROPOSITION 3.12. *Assume $\varphi, \psi \in \mathcal{H}_{bd}$. Then*

$$d_1(\varphi,\psi) = E(\varphi) + E(\psi) - 2E(\varphi \vee \psi).$$

Proof. We proceed by approximation, so as to reduce to the Kähler case. By [Dar15, Corollary 4.14], we know that

$$d_{1,\varepsilon}(\varphi,\psi) = E_{\omega_\varepsilon}(\varphi) + E_{\omega_e}(\psi) - 2E_{\omega_\varepsilon}(\varphi \vee_\varepsilon \psi),$$

where $\omega_\varepsilon := \omega + \varepsilon\omega_X$, $\varphi \vee_\varepsilon \psi$ is the greatest ω_ε-psh function that lies below $\min(\varphi,\psi)$, and E_{ω_ε} is as in (6). Since $(\omega_\varepsilon + dd^c\varphi)^n$ converges weakly to $(\omega + dd^c\varphi)^n$, we have that $E_{\omega_\varepsilon}(\varphi)$ converges to $E(\varphi)$ as ε goes to 0. The same holds for $E_{\omega_\varepsilon}(\psi)$. We then need to ensure that $E_{\omega_\varepsilon}(\varphi \vee_{\omega_\varepsilon} \psi)$ converges to $E(\varphi \vee \psi)$. Denote $\phi_\varepsilon := \varphi \vee_{\omega_\varepsilon} \psi$ and $\phi := \varphi \vee \psi$. Fix $\varepsilon' > \varepsilon$. Using Lemma 2.5 and the fact that ϕ_ε is decreasing to ϕ, we get

$$\begin{aligned} 0 \geqslant E_{\omega_\varepsilon}(\phi_\varepsilon) - E_{\omega_\varepsilon}(\phi) &= \frac{1}{(n+1)V_\varepsilon}\sum_{j=0}^{n}\int_X (\phi_\varepsilon - \phi)(\omega_\varepsilon + dd^c\phi_\varepsilon)^j \wedge (\omega_\varepsilon + dd^c\phi)^{n-j} \\ &\geqslant \frac{1}{(n+1)V_\varepsilon}\sum_{j=0}^{n}\int_X (\phi_{\varepsilon'} - \phi)(\omega + \omega_X + dd^c\phi_\varepsilon)^j \wedge (\omega + \omega_X + dd^c\phi)^{n-j}. \end{aligned}$$

Letting first ε to zero and the ε', we get the result. The conclusion then follows from the previous arguments and Proposition 1.16. ☐

4. The complete geodesic space $(\mathcal{E}^p(X,\omega), d_p)$

4.1 Metric completion

For $\varphi, \psi \in \mathcal{E}^p(X,\omega)$, we let φ_j, ψ_k denote sequences of elements in $\mathcal{H}_{bd}$ decreasing to φ, ψ, respectively, and set

$$D_p(\varphi,\psi) := \liminf_{j,k\to+\infty} d_p(\varphi_j,\psi_k).$$

We list in the following proposition various properties of this extension.

PROPOSITION 4.1.

(i) *D_p is a distance on $\mathcal{E}^p(X,\omega)$, which coincides with d_p on $\mathcal{H}_{bd}$.*
(ii) *The definition of D_p is independent of the choice of the approximants.*
(iii) *D_p is continuous along decreasing sequences in $\mathcal{E}^p(X,\omega)$.*

Moreover all previous inequalities comparing d_p and I_p on $\mathcal{H}_{bd}$ extend to inequalities between D_p and I_p on $\mathcal{E}^p(X,\omega)$.

In the following, therefore, we will denote D_p by d_p.

Proof. It is a tedious exercise to verify that D_p defines a 'semi-distance', i.e. satisfies all properties of a distance but for the separation property. It follows from the definition of D_p and Proposition 2.11 that Theorem 3.6 extends in a natural way to potentials in $\mathcal{E}^p(X,\omega)$. If $D_p(\varphi,\psi) = 0$, it therefore follows that $I_p(\varphi,\psi) = 0$, hence $\varphi = \psi$ by the domination principle.

One can check that D_p coincides with d_p on $\mathcal{H}_{bd}$ as follows: using part (ii), one can use the constant sequences $\varphi_j \equiv \varphi$ and $\psi_k \equiv \psi$ to obtain this equality.

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We now prove part (ii). Let φ_j, u_j (respectively ψ_k, v_k) denote two sequences of elements of $\mathcal{H}_{bd}$ decreasing to φ (respectively ψ). We can assume without loss of generality that these sequences are intertwining, i.e. for all $j, k \in \mathbb{N}$, there exists $\ell, q \in \mathbb{N}$, such that $\varphi_j \leqslant u_\ell$ and $\psi_k \leqslant v_q$, with similar reverse inequalities. It follows from Proposition 3.8 and the triangle inequality that

$$\begin{aligned}|d_p(\varphi_j, \psi_k) - d_p(u_\ell, v_q)| &\leqslant d_p(\varphi_j, u_\ell) + d_p(\psi_k, v_q)\\ &\leqslant 2I_p(\varphi_j, u_\ell) + 2I_p(\psi_k, v_q).\end{aligned}$$

Now, again by Proposition 3.8, we get

$$I_p(\varphi_j, u_\ell)^p \leqslant \int_X (u_\ell - \varphi_j)^p \,\mathrm{MA}(\varphi_j) \leqslant (p+1)^n \int_X (u_\ell - \varphi)^p \,\mathrm{MA}(\varphi),$$

where the last inequality follows from [GZ07, Lemma 3.5]. The monotone convergence theorem, therefore, yields $I_p(\varphi_j, u_\ell) + I_p(\psi_k, v_q) \to 0$ as $\ell, q \to +\infty$, proving part (ii).

One shows part (iii) with similar arguments. The extension of the inequalities comparing d_p and I_p follows from [BEGZ10, Theorem 2.17]. □

PROPOSITION 4.2. *The metric spaces $(\mathcal{E}^p_{\mathrm{norm}}(X, \omega), d_p)$ and $(\mathcal{E}^p(X, \omega), d_p)$ are complete. The Mabuchi topology d_p dominates the topology induced by I: if a sequence converges for d_p, it converges in energy.*

Proof. Let $(\varphi_j) \in \mathcal{E}^p(X, \omega)^{\mathbb{N}}$ be a Cauchy sequence for d_p. We claim that there exists $\psi \in \mathcal{E}^p(X, \omega)$, such that

$$d_p(\varphi_j, \psi) \to 0 \quad \text{and} \quad I(\psi, \varphi_j) \to 0.$$

Extracting and relabelling, we can assume that

$$d_p(\varphi_j, \varphi_{j+1}) \leqslant 2^{-j}, \quad j \geqslant 1.$$

Set $\varphi_{-1} \equiv 0$ and for $k \geqslant j$, $\psi_{j,k} := \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_k$, and observe that $\psi_{j,k} := \varphi_j \vee \psi_{j,k+1}$. Hence, the Pythagorean formula gives

$$d_p(\varphi_j, \psi_{j,k}) \leqslant d_p(\varphi_j, \psi_{j+1,k}) \leqslant 2^{-j} + d_p(\varphi_{j+1}, \psi_{j+1,k}).$$

Repeating this argument, we get $d_p(\varphi_j, \psi_{j,k}) \leqslant 2^{-j+1}$. We then have

$$\begin{aligned}d_p(0, \psi_{j,k}) &\leqslant \sum_{\ell=-1}^{j-1} d_p(\varphi_\ell, \varphi_{\ell+1}) + d_p(\varphi_j, \psi_{j,k})\\ &\leqslant \sum_{\ell=-1}^{j} d_p(\varphi_\ell, \varphi_{\ell+1}) + d_p(\varphi_{j+1}, \psi_{j+1,k})\\ &\leqslant d_p(0, \varphi_1) + 2 + 2^{-j+1}.\end{aligned}$$

It follows from Theorem 3.6 that $I_p(0, \psi_{j,k})$ is uniformly bounded, hence its decreasing limit $\psi_j := \lim_{k \to +\infty} \psi_{j,k} \in \mathcal{E}^p(X, \omega)$ [BEGZ10, Proposition 2.19]. From the above, we also have

$$d_p(0, \psi_j) \leqslant d_p(0, \varphi_1) + 2 + 2^{-j+1}.$$

Lemma 3.11 then ensures that $(\sup_X \psi_j)_j$ is uniformly bounded, hence ψ_j increases a.e. towards $\psi \in \mathrm{PSH}(X, \omega)$. Also, $\psi \in \mathcal{E}^p(X, \omega)$ thanks to [BEGZ10, Proposition 2.4]. Moreover, [BEGZ10, Theorem 2.17] yields

$$I(\psi, \psi_j) + I_p(\psi_j, \psi) \longrightarrow 0.$$

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It follows, therefore, from Proposition 3.8 that $d_p(\psi, \psi_j) \to 0$ and

$$d_p(\psi, \varphi_j) \leqslant d_p(\psi, \psi_j) + d_p(\psi_j, \varphi_j) \leqslant d_p(\psi, \psi_j) + 2^{-j+1} \to 0.$$

Recalling that $\psi_j \leqslant \varphi_j$, it follows from the quasi-triangle inequality, Proposition 2.8, and Theorem 3.6 that

$$I(\psi, \varphi_j) \leqslant c_n\{I(\psi, \psi_j) + I(\psi_j, \varphi_j)\} \leqslant c_{n,p}\{I(\psi, \psi_j) + d_p(\psi_j, \varphi_j)\} \to 0. \qquad \square$$

Recall that the *precompletion* of a metric space (X, d) is the set of all Cauchy sequences C_X of X, together with the semi-distance

$$\delta(\{x_j\}, \{y_j\}) = \lim_{j \to +\infty} d(x_j, y_j).$$

The metric *completion* $(\overline{X}, d)$ of (X, d) is the quotient space $C_X/\sim$, where

$$\{x_j\} \sim \{y_j\} \Longleftrightarrow \delta(\{x_j\}, \{y_j\}) = 0,$$

equipped with the induced distance, which we still denote d.

We are now taking advantage of the fact that $\mathcal{H}_{bd}$ lives inside the complete metric space $(\mathcal{E}^p(\alpha), d_p)$ to conclude the following.

Theorem 4.3. *The metric completion of $(\mathcal{H}_{bd}, d_p)$ is isometric to $(\mathcal{E}^p(X, \omega), d_p)$.*

Thanks to Theorem 3.6, an equivalent formulation of this statement is that the metric completion of $(\mathcal{H}_{bd}, d_p)$ is bi-Lipschitz equivalent to $(\mathcal{E}^p(X, \omega), I_p)$.

Proof. We work at the level of normalized potentials,

$$\mathcal{E}^p_0(X, \omega) = \{\varphi \in \mathcal{E}^p(X, \omega) \mid E(\varphi) = 0\}$$

and $\mathcal{H}_0 := \{\varphi \in \mathcal{H}_{bd} \mid \omega + dd^c\varphi \geqslant 0 \text{ and } E(\varphi) = 0\}$.

Since $(\mathcal{E}^p_0(X, \omega), d_p)$ is a complete metric space that contains $\mathcal{H}_0$, it suffices to show that the latter is dense in $\mathcal{E}^p_0(X, \omega)$. Fix $\varphi \in \mathcal{E}^p_0(X, \omega)$ and let $(\varphi_j) \in \mathcal{H}_0^{\mathbb{N}}$ be a sequence quasi-decreasing to φ: the normalization condition $E(\varphi_j) = 0$ prevents us from getting a truly decreasing sequence; however, $\varphi_j + \varepsilon_j$ is decreasing where ε_j is a sequence of real numbers decreasing to zero. It follows from Proposition 3.8 that

$$d_p(\varphi_{j+\ell} + \varepsilon_{j+l}, \varphi_j + \varepsilon_j)^p \leqslant \int_X (\varphi_j - \varphi_{j+\ell})^p \operatorname{MA}(\varphi_{j+\ell}) + \varepsilon_j.$$

Now [GZ07, Lemma 3.5] shows that the latter is bounded from above by

$$(p+1)^n \int_X (\varphi_j - \varphi)^p \operatorname{MA}(\varphi) + \varepsilon_j,$$

which converges to zero as $j \to +\infty$, as follows from the monotone convergence theorem. Therefore (φ_j) is a Cauchy sequence in $(\mathcal{H}_0, d_p)$ that converges to φ, since

$$0 \leqslant d_p(\varphi, \varphi_j + \varepsilon_j) \leqslant \liminf_{\ell \to +\infty} d_p(\varphi_{j+\ell}, \varphi_j) \leqslant 2(1+p)^{n/p} I_p(\varphi_j, \varphi) + \varepsilon_j^{1/p} \to 0,$$

by Proposition 3.8 and [BEGZ10, Theorem 2.17].

We note the following alternative approach of independent interest. One first shows that $\mathcal{H}_0$ is dense in the set of all bounded ω-psh functions. Given $\varphi \in \mathcal{E}_0^p(X,\omega)$, one then considers its 'canonical approximants',

$$\varphi_j = \max(\varphi, -j) + \varepsilon_j \in \mathrm{PSH}_0(X,\omega) \cap L^\infty(X)\,,$$

which decrease towards $\varphi \in \mathcal{E}_0^p(X,\omega)$. It follows from Proposition 3.8 that

$$\begin{aligned} d_p(\varphi_{j+\ell},\varphi_j)^p &\leqslant o(1) + \int_X (\varphi_j - \varphi_{j+\ell})^p\, \mathrm{MA}(\varphi_{j+\ell}) \\ &= o(1) + \int_{(\varphi \leqslant -j-\ell)} \ell^p\, \mathrm{MA}(\varphi_{j+\ell}) + \int_{(-j-\ell<\varphi<-j)} (\varphi_j - \varphi_{j+\ell})^p\, \mathrm{MA}(\varphi) \\ &= o(1) + \int_{(\varphi \leqslant -j-\ell)} \ell^p\, \mathrm{MA}(\varphi) + \int_{(-j-\ell<\varphi<-j)} (\varphi_j - \varphi_{j+\ell})^p\, \mathrm{MA}(\varphi) \\ &\leqslant o(1) + \int_{(\varphi<-j)} \varphi^p\, \mathrm{MA}(\varphi), \end{aligned}$$

where we have used the maximum principle, together with the fact that

$$\int_{(\varphi \leqslant -k)} \mathrm{MA}(\varphi_k) = \int_X \mathrm{MA}(\varphi_k) - \int_{(\varphi>-k)} \mathrm{MA}(\varphi_k) = \int_{(\varphi \leqslant -k)} \mathrm{MA}(\varphi),$$

since $\varphi \in \mathcal{E}(X,\omega)$, as follows again from the maximum principle. We infer that (φ_j) is a Cauchy sequence, which converges to φ. □

We are now in a position to prove Theorem B of the introduction.

Corollary 4.4. *Assume $\omega = \pi^*\omega_Y$, where ω_Y is a Hodge form. Then the metric completion $(\overline{\mathcal{H}}_\alpha, d_p)$ is isometric to $(\mathcal{E}^p(\alpha), d_p)$. Similarly, the metric completion $(\overline{\mathcal{H}}_\omega, d_p)$ is isometric to $(\mathcal{E}^p(X,\omega), d_p)$.*

Proof. Thanks to [CGZ13, Corollary C] we can ensure that the space $\mathcal{H}_\omega$ is dense in $\mathcal{H}_{bd}$. The result then follows from Theorem 4.3. □

4.2 Weak geodesics

4.2.1 *Finite-energy geodesics.* We now define finite-energy geodesics joining two finite-energy endpoints $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\omega)$. Fix $j \in \mathbb{N}$ and consider φ_0^j, φ_1^j bounded ω-psh functions decreasing to φ_0, φ_1. We let $\varphi_{t,j}$ denote the bounded geodesic joining φ_0^j to φ_1^j. It follows from the maximum principle that $j \mapsto \varphi_{t,j}$ is non-increasing. We can thus set

$$\varphi_t := \lim_{j \to +\infty} \varphi_{t,j}.$$

Definition 4.5. The map $(t,x) \mapsto \varphi_t(x)$ is the (finite-energy) Mabuchi geodesic joining φ_0 to φ_1.

The φ_t indeed form a family of finite-energy functions: since $t \mapsto E(\varphi_{t,j})$ is affine (Lemma 2.5), we infer, for all $j \in \mathbb{N}$,

$$(1-t)E(\varphi_0) + tE(\varphi_1) \leqslant (1-t)E(\varphi_0^{(j)}) + tE(\varphi_1^{(j)}) = E(\varphi_{t,j}),$$

hence $\varphi_t \in \mathcal{E}^1(X,\omega)$ with $(1-t)E(\varphi_0) + tE(\varphi_1) = E(\varphi_t)$.

It follows from the maximum principle that φ_t is independent of the choice of the approximants φ_0^j, φ_1^j: if we set $\varphi(x,z) := \varphi_t(x)$, $z = t + is$, then φ is a maximal ω-psh function in $X \times S$, as a decreasing limit of maximal ω-psh functions. It is thus the unique maximal ω-psh function in $X \times S$ with boundary values φ_0, φ_1.

When φ_0, φ_1 belong to $\mathcal{E}^p(X,\omega)$, these weak geodesics are again *metric geodesics* in the complete metric space $(\mathcal{E}^p(X,\omega), d_p)$.

PROPOSITION 4.6. *Given $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\omega)$, the Mabuchi geodesic φ joining φ_0 to φ_1 lies in $\mathcal{E}^p(X,\omega)$ and satisfies, for all $t, s \in [0,1]$,*

$$d_p(\varphi_t, \varphi_s) = |t-s|\, d_p(\varphi_0, \varphi_1).$$

Thus, $(\mathcal{E}^p(X,\omega), d_p)$ is a geodesic space.

Proof. We can assume, without loss of generality, that $\varphi_0, \varphi_1 \leqslant 0$. Fix $j \in \mathbb{N}$ and consider φ_0^j, φ_1^j bounded ω-psh functions decreasing to φ_0, φ_1. We let $\varphi_{t,j}$ denote the bounded geodesic joining φ_0^j to φ_1^j, which decreases towards φ_t as j increases to $+\infty$. Observe that

$$\varphi_0 \vee \varphi_1 \leqslant \varphi_0^j \vee \varphi_1^j \leqslant \varphi_{t,j}.$$

It therefore follows from [GZ07, Lemma 3.5] and Lemma 4.7 that

$$\int_X (-\varphi_{t,j})^p \,\mathrm{MA}(\varphi_{t,j}) \leqslant (p+1)^n \int_X (-\varphi_0 \vee \varphi_1)^p \,\mathrm{MA}(\varphi_0 \vee \varphi_1) < +\infty\,,$$

hence the monotone convergence theorem yields $\int_X (-\varphi_t)^p \,\mathrm{MA}(\varphi_t) < +\infty$, for all t, i.e. $\varphi_t \in \mathcal{E}^p(X,\omega)$.

The remaining assertion is proved as in the case of bounded geodesics (Proposition 1.18). □

LEMMA 4.7. *Assume $0 \geqslant \varphi_0, \varphi_1 \in \mathcal{E}^p(X,\omega)$. Then $\varphi_0 \vee \varphi_1 \in \mathcal{E}^p(X,\omega)$ and*

$$\int_X (-\varphi_0 \vee \varphi_1)^p \,\mathrm{MA}(\varphi_0 \vee \varphi_1) \leqslant \int_X (-\varphi_0)^p \,\mathrm{MA}(\varphi_0) + \int_X (-\varphi_1)^p \,\mathrm{MA}(\varphi_1).$$

Proof. It suffices to establish the claimed inequality when $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$ and then proceed by approximation. It follows from Proposition 3.2 that

$$\mathrm{MA}(\varphi_0 \vee \varphi_1) \leqslant \mathbb{1}_{\{\varphi_0 \vee \varphi_1 = \varphi_0\}} \mathrm{MA}(\varphi_0) + \mathbb{1}_{\{\varphi_0 \vee \varphi_1 = \varphi_1\}} \mathrm{MA}(\varphi_1).$$

The inequality follows, since $\varphi_0, \varphi_1 \leqslant 0$. □

4.2.2 *(Non-)uniqueness of geodesics.* Fix $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\omega)$. If the sets $(\varphi_0 < \varphi_1)$ and $(\varphi_0 > \varphi_1)$ are both non-empty, the function $\varphi_0 \vee \varphi_1$ differs from φ_0 and φ_1 and it follows from Proposition 3.4 that

$$d_1(\varphi_0, \varphi_1) = d_1(\varphi_0, \varphi_0 \vee \varphi_1) + d_1(\varphi_0 \vee \varphi_1, \varphi_1),$$

thus the concatenation of the geodesic joining φ_0 to $\varphi_0 \vee \varphi_1$ and of that joining $\varphi_0 \vee \varphi_1$ to φ_1 gives another minimizing path joining φ_0 to φ_1.

When $\varphi_0 \leqslant \varphi_1$, this argument no longer works, but there are nevertheless very many minimizing paths, as shown by the following result.

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Lemma 4.8. *Assume $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$ are such that $\varphi_0 \leqslant \varphi_1$. Let $(\psi_t)_{0\leqslant t\leqslant 1}$ be a path joining φ_0 to φ_1. Then*

$$\ell_1(\psi) = d_1(\varphi_0,\varphi_1) \Longleftrightarrow \dot\psi_t(x) \geqslant 0 \quad \text{for a.e. } t, x.$$

In particular, $t \mapsto t\varphi_1(x) + (1-t)\varphi_0$ is a minimizing path for d_1, which is not a Mabuchi geodesic, unless $\varphi_1 - \varphi_0$ is constant.

Proof. Observe that

$$\begin{aligned}\ell_1(\psi) &= \int_0^1 \int_X |\dot\psi_t(x)|\, \mathrm{MA}(\psi_t)\, dt \geqslant \left| \int_0^1 \int_X \dot\psi_t(x)\, \mathrm{MA}(\psi_t)\, dt \right| \\ &= \left| \int_0^1 \frac{d}{dt} E(\psi_t)\, dt \right| = |E(\varphi_1) - E(\varphi_0)| = d_1(\varphi_0, \varphi_1),\end{aligned}$$

where the last identity follows from Proposition 3.12. There is equality if and only if $|\dot\psi_t(x)| = \dot\psi_t(x) \geqslant 0$ for a.e. (t,x) (the sign has to be positive because $\psi_0 = \varphi_0 \leqslant \varphi_1 = \psi_1$).

In particular, $t \mapsto \psi_t = t\varphi_1(x) + (1-t)\varphi_0$ has this property, since $\dot\psi_t = \varphi_1 - \varphi_0 \geqslant 0$. We recall that, since ψ_t is a smooth path, the geodesic equation can be written as

$$\ddot\psi_t\, \mathrm{MA}(\psi_t) = \frac{n}{V}\, d\dot\psi_t \wedge d^c\dot\psi_t \wedge \omega_{\psi_t}^{n-1}$$

(see § 1.1.1). Now $\ddot\psi_t = 0$, hence $t \mapsto \psi_t$ is not a Mabuchi geodesic, unless $d(\varphi_1 - \varphi_0) \wedge d^c(\varphi_1 - \varphi_0) \wedge \omega_{\psi_t}^{n-1} = 0$ for all t, i.e. $\varphi_1 - \varphi_0$ is constant. □

Conversely, it follows from the work of Darvas [Dar17c, Lemma 6.12] (based on [CC02, § 2.4]) that geodesics are unique in $\mathcal{E}^2(X,\omega)$.

Theorem 4.9. *Assume $\omega = \pi^*\omega_Y$, where ω_Y is a Hodge form. Then the space $(\mathcal{E}^2(X,\omega), d_2)$ is a* CAT(0) *space.*

Complete CAT(0) spaces are also called Hadamard spaces. Recall that a CAT(0) space is a geodesic space that has non-positive curvature in the sense of Alexandrov. Hadamard spaces enjoy many interesting properties (uniqueness of geodesics, contractibility, convexity properties, etc., see [BH99]).

Proof. By Corollary 4.4 we know that $(\mathcal{E}^2(X,\omega), d_2)$ is the completion of $(\mathcal{H}_\omega, d_2)$ and by Proposition 4.6 that it is a geodesic metric space. [BH99, Exercise 1.9.1.c (p. 163)] ensures that $(\mathcal{E}^2(X,\omega), d_2)$ is a CAT(0) space if and only if the CN inequality of Bruhat and Tits [BT72] holds, i.e. $\forall P,Q,R \in \mathcal{E}^2(X,\omega)$ and, for any $M \in \mathcal{E}^2(X,\omega)$ such that $d_2(Q,M) = d_2(R,M) = d_2(Q,R)/2$ (in other words $M = \varphi_t^{QR}|_{t=1/2}$ where φ_t^{QR} is the geodesic joining Q,R), one has

$$d_2(P,M)^2 \leqslant \tfrac{1}{2} d_2(P,Q)^2 + \tfrac{1}{2} d_2(P,R)^2 - \tfrac{1}{4} d_2(Q,R)^2. \tag{9}$$

Assume first that $P,Q,R \in \mathcal{H}_\omega \subset \mathcal{H}_{\omega_\varepsilon}$. Then by [CC02, § 2.4] (see also [Dar17c, Lemma 6.12]), we have that

$$d_{2,\varepsilon}(P,M_\varepsilon)^2 \leqslant \tfrac{1}{2} d_{2,\varepsilon}(P,Q)^2 + \tfrac{1}{2} d_{2,\varepsilon}(P,R)^2 - \tfrac{1}{4} d_{2,\varepsilon}(Q,R)^2\,,$$

where M_ε is the point of ε-geodesic joining Q,R such that $d_{2,\varepsilon}(Q,M) = d_{2,\varepsilon}(R,M) = d_{2,\varepsilon}(Q,R)/2$. Thanks to Theorem 1.13, the right-hand side in the inequality converges to the right-hand side of (9) as ε goes to zero. We claim that $d_{2,\varepsilon}(P,M_\varepsilon)$ converges to $d(P,M)$.

Observe first that M_ε decreases to M, since ε-geodesics decreases as ε decrease to zero (Proposition 1.6). Moreover, the triangle inequality yields $|d_{2,\varepsilon}(P, M_\varepsilon) - d_{2,\varepsilon}(P, M)| \leqslant d_{2,\varepsilon}(M, M_\varepsilon)$. Since M, M_ε are both bounded, it follows from Theorem 3.6 and Proposition 2.11 that $d_{2,\varepsilon'}(M, M_\varepsilon) \to 0$ as $\varepsilon \to 0$. This proves the claim.

If $P, Q, R \in \mathcal{E}^2(X, \omega)$, we choose smooth approximants $P_k, Q_k, R_k \in \mathcal{H}_\omega$ decreasing to P, Q, R. The above arguments ensure that

$$d_2(P_k, M_k)^2 \leqslant \tfrac{1}{2} d_2(P_k, Q_k)^2 + \tfrac{1}{2} d_2(P_k, R_k)^2 - \tfrac{1}{4} d_2(Q_k, R_k)^2. \qquad (10)$$

The comparison principle implies that M_k decreases to M as k goes to $+\infty$. It then follows from Propositions 3.8 and 4.1 that $d_2(M, M_k) \to 0$ as k goes to $+\infty$. This, together with Proposition 4.1, gives (9) when letting $k \to +\infty$. □

5. Singular Kähler–Einstein metrics of positive curvature

The existence of singular Kähler–Einstein metrics of non-positive curvature has been established in [EGZ09], generalizing the fundamental work of Aubin [Aub78] and Yau [Yau78]. They always exist, provided the underlying variety has mild singularities and the first Chern class is non-positive.

Singular Kähler–Einstein metrics of positive curvature are more difficult to construct. It is already so in the smooth case [CDS15]. Their first properties have been obtained in [BBGZ13, BBEGZ]. In § 5.3, pushing further these works, we provide a necessary and sufficient analytic condition for their existence, generalizing a result of Tian [Tia97] and Phong *et al.* [PSSW08].

5.1 Log terminal singularities

A *pair* (Y, D) is the data of a connected normal compact complex variety Y and an effective $\mathbb{Q}$-divisor D, such that $K_Y + D$ is $\mathbb{Q}$-Cartier. We write

$$Y_0 := Y_{\text{reg}} \backslash \operatorname{Supp} D.$$

Given a log resolution $\pi : X \to Y$ of (Y, D) (which may be chosen to be an isomorphism over Y_0), there exists a unique $\mathbb{Q}$-divisor $\sum_i a_i E_i$, whose push-forward to Y is $-D$, such that

$$K_X = \pi^*(K_Y + D) + \sum_i a_i E_i.$$

DEFINITION 5.1. The pair (Y, D) is *klt* if $a_j > -1$ for all j.

The same condition will then hold for all log resolutions of Y. When $D = 0$, one says that Y is *log terminal* when the pair $(Y, 0)$ is klt. We have the following analytic interpretation. Fix $r \in \mathbb{N}^*$ such that $r(K_Y + D)$ is Cartier. If σ is a nowhere-vanishing section of the corresponding line bundle over a small open set U of Y, then

$$(i^{rn^2} \sigma \wedge \bar{\sigma})^{1/r} \qquad (11)$$

defines a smooth, positive volume form on $U_0 := U \cap Y_0$. If f_j is a local equation of E_j around a point of $\pi^{-1}(U)$, then

$$\pi^*(i^{rn^2} \sigma \wedge \bar{\sigma})^{1/r} = \prod_i |f_i|^{2a_i}\, dV$$

https://doi.org/10.1112/S0010437X18007170 Published online by Cambridge University Press

locally on $\pi^{-1}(U)$ for some local volume form dV. Since $\sum_i E_i$ has normal crossings, this shows that (Y, D) is klt if and only if each volume form of the form of (11) has locally finite mass near singular points of Y.

The previous construction globalizes as follows.

Definition 5.2. Let (Y, D) be a pair and let ϕ be a smooth Hermitian metric on the $\mathbb{Q}$-line bundle $-(K_Y + D)$. The corresponding *adapted measure* mes_ϕ on Y_{reg} is locally defined by choosing a nowhere-zero section σ of $r(K_Y + D)$ over a small open set U and setting

$$\mathrm{mes}_\phi := (i^{rn^2}\sigma \wedge \overline{\sigma})^{1/r}/|\sigma|^{2/r}_{r\phi}.$$

The point is that the measure mes_ϕ does not depend on the choice of σ; hence, it is globally defined. This discussion shows that

$$(Y, D) \text{ is klt} \iff \mathrm{mes}_\phi \text{ has finite total mass on } Y,$$

in which case we view it as a Radon measure on the whole of Y.

5.2 Kähler–Einstein metrics on log Fano pairs

Definition 5.3. A *log Fano pair* is a klt pair (Y, D) such that Y is projective and $-(K_Y + D)$ is ample.

Let (Y, D) be a log Fano pair. Fix a reference smooth strictly psh metric ϕ_0 on $-(K_Y + D)$, with curvature ω_0 and adapted measure $\mu_0 = \mathrm{mes}_{\phi_0}$. We normalize ϕ_0 so that μ_0 is a probability measure. The volume of (Y, D) is

$$V := c_1(Y, D)^n = \int_X \omega_0^n.$$

Definition 5.4. A *Kähler–Einstein metric* T for the log Fano pair (Y, D) is a finite-energy current $T \in c_1(Y, D)$. such that $T^n = V \cdot \mu_T$.

We now list some important properties of these objects, established in [BBGZ13, Ber15, BBEGZ].

(i) A Kähler–Einstein metric ω is automatically smooth on Y_0, with continuous potentials on Y, and it satisfies
$$\mathrm{Ric}(\omega_{KE}) = \omega_{KE} + [D] \quad \text{on } Y_{\mathrm{reg}}.$$
(ii) The definition of a log Fano pair requires the singularities to be klt. This condition is, in fact, necessary to obtain K–E metrics on Y_{reg}.
(iii) The Kähler–Einstein equation reads $(\omega_0 + dd^c\phi)^n = e^{-\phi+c}\mu_0$ for some constant $c \in \mathbb{R}$. If we choose a log resolution, the equation becomes $(\omega + dd^c\varphi)^n = e^{-\varphi+c}\widetilde{\mu}_0$, where $\omega = \pi^*\omega_0$ is semi-positive and big and $\widetilde{\mu}_0 = \prod_i |f_i|^{2a_i}\, dV$.
(iv) The potential φ belongs to $\mathcal{H}_\omega$ and maximizes the functional
$$\mathcal{F}(\varphi) := E(\varphi) + \log\left[\int_{\tilde{X}} e^{-\varphi} d\widetilde{\mu}_0\right].$$
Conversely, any maximizer of $\mathcal{F}$ is a Kähler–Einstein metric.

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(v) Two Kähler–Einstein metrics are connected by the flow of a holomorphic vector field that leaves D invariant.

(vi) If the functional $\mathcal{F}$ is *proper* (i.e. if $E(\varphi_j) \to -\infty \Rightarrow \mathcal{F}(\varphi_j) \to -\infty$), then there exists a unique Kähler–Einstein metric.

Here, $[D]$ is the integration current on $D|_{Y_{\mathrm{reg}}}$. Writing $\mathrm{Ric}(\omega_{KE})$ on Y_{reg} implicitly means that the positive measure $\omega_{KE}^n|_{Y_{\mathrm{reg}}}$ corresponds to a singular metric on $-K_{Y_{\mathrm{reg}}}$, whose curvature is then $\mathrm{Ric}(\omega_{KE})$ by definition.

5.3 The analytic criterion

Following an idea of Darvas and Rubinstein [DR17], we now extend [Tia97, Theorem 1.6] and [PSSW08] by proving the following.

Theorem 5.5. *Let (Y, D) be a log Fano pair. It admits a unique Kähler–Einstein metric if and only if there exists $\varepsilon, M > 0$ such that, for all $\varphi \in \mathcal{H}_{\mathrm{norm}}$,*

$$\mathcal{F}(\varphi) \leqslant -\varepsilon d_1(0, \varphi) + M.$$

This is Theorem D of the introduction.

Proof. We are going to use Theorem B. Note that $\omega_Y \in c_1(-K_X - D)$ is a Hodge form. One implication is given by [BBEGZ, Theorems 4.8 and 5.4]: if

$$\mathcal{F}(\varphi) \leqslant -\varepsilon d_1(0, \varphi) + M,$$

then $\mathcal{F}$ is proper, hence there exists a unique Kähler–Einstein metric.

So we assume now that there exists ω, a unique Kähler–Einstein metric, which we take as our base point of $\mathcal{H}_\omega$. It is the unique maximizer of $\mathcal{F}$ on $\mathcal{E}^1(X, \omega)$,

$$\mathcal{F}(0) = \sup_{\varphi \in \mathcal{E}^1(X,\omega)} \mathcal{F}(\varphi),$$

as follows from [BBGZ13, Theorem 6.6], [BBEGZ, Theorems 4.8 and 5.3].

Note that $\mathcal{F}$ is invariant by translations, so we actually consider the restriction of $\mathcal{F}$ on $\mathcal{E}^1_{\mathrm{norm}}(X, \omega) = \{\varphi \in \mathcal{E}^1(X, \omega), \sup_X \varphi = 0\}$. Assume for contradiction that there is no $\varepsilon > 0$ such that $\mathcal{F}(\varphi) \leqslant -\varepsilon d_1(0, \varphi) + M$ for all $\varphi \in \mathcal{H}_{\mathrm{norm}}$, where we set $M := \mathcal{F}(0) + 1$. Then we can find a sequence $(\varphi_j) \in \mathcal{H}_\omega^{\mathbb{N}}$ such that $\sup_X \varphi_j = 0$ and

$$\mathcal{F}(\varphi_j) > -\frac{d_1(0, \varphi_j)}{j+1} + \mathcal{F}(0) + 1.$$

If $E(\varphi_j)$ does not blow up to $-\infty$, we reach a contradiction: up to extracting and relabelling, we can assume that $E(\varphi_j)$ is bounded and φ_j converges to some $\psi \in \mathcal{E}^1(X, \omega)$. Since $\mathcal{F}$ is upper semi-continuous, we infer $\mathcal{F}(\psi) \geqslant \mathcal{F}(0) + 1$, a contradiction.

So we assume now that $E(\varphi_j) \to -\infty$. It follows from Lemma 3.12 that $d_j := d_1(0, \varphi_j) = -E(\varphi_j) \to +\infty$. We let $(\varphi_{t,j})_{0 \leqslant t \leqslant d_j}$ denote the Mabuchi geodesic with unit speed joining 0 to φ_j and set $\psi_j := \varphi_{1,j}$. Note that the arguments in Lemma 3.3 show that $t \mapsto \varphi_{t,j}$ is decreasing, hence $\varphi_j \leqslant \psi_j \leqslant 0$. In particular, $\sup_X \psi_j = 0$, while, by definition, $d_1(0, \psi_j) = 1 = -E(\psi_j)$.

It now follows from Berndtsson's convexity result [Ber15, § 6.2] and its generalization to the singular context [BBEGZ, Theorem 11.1] that the map $t \mapsto \mathcal{F}(\varphi_{t,j})$ is concave. We infer

$$0 \geqslant \mathcal{F}(\varphi_{1,j}) - \mathcal{F}(\varphi_{0,j}) \geqslant \frac{\mathcal{F}(\varphi_{d_j,j}) - \mathcal{F}(\varphi_{0,j})}{d_j} > -\frac{1}{j+1} + \frac{1}{d_j},$$

thus $\mathcal{F}(\psi_j) \to \mathcal{F}(0)$. This shows that (ψ_j) is a maximizing sequence for $\mathcal{F}$, which therefore strongly converges to 0, by [BBEGZ, Theorem 5.3.3]. This yields a contradiction, since $d_1(0, \psi_j) = 1$. $\square$

ACKNOWLEDGEMENTS

We thank Darvas, Lu and Zeriahi for useful conversations.

E.D.N. was supported by a Marie Sklodowska Curie individual fellowship 660940KRFCY (MSCAIF). V.G. was partially supported by the ANR project GRACK.

This paper is based on work supported also by the NSF Grant DMS-1440140, while E.D.N. was in residence at the MSRI, during the Spring 2016 semester. It is partially based on lecture notes of V.G. [Gue14], after a series of lectures he gave at KIAS in April 2013. The authors thank J.-M. Hwang and M. Paun for their invitation and the staff of KIAS and MSRI for providing excellent conditions of work.

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