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A logarithmic lower bound for the second Bohr radius

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Abstract. The purpose of this note is to obtain an improved lower bound for the multidimensional Bohr radius introduced by L. Aizenberg (2000, *Proceedings of the American Mathematical Society* 128, 1147–1155), by means of a rather simple argument.

Bohr's theorem [4] states that for each bounded holomorphic self-mapping $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of the open unit disk $\mathbb{D}$, we have

$$\sum_{k=0}^{\infty} |a_k| \left(\frac{1}{3}\right)^k \le 1,$$

and this quantity 1/3 is the best possible. In an attempt to generalize this result in higher dimensions, the *first Bohr radius* $K(R)$ for a bounded complete Reinhardt domain $R \subset \mathbb{C}^n$ was defined in [3] by Boas and Khavinson. Namely, $K(R)$ is the supremum of all $r \in [0,1]$ such that for each holomorphic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ on R with $|f(z)| \le 1$ for all $z \in R$, we have

$$\sum_\alpha |a_\alpha z^\alpha| \le 1$$

for all $z \in rR$. Let us clarify here that a complete Reinhardt domain R in $\mathbb{C}^n$ is a domain such that if $z = (z_1, z_2, \ldots, z_n) \in R$, then $(\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n) \in R$ for all $\lambda_i \in \overline{\mathbb{D}}, 1 \le i \le n$. Of particular interest to us are the Reinhardt domains

$$B_{\ell_p^n} := \{z \in \mathbb{C}^n : \|z\|_p < 1\},$$

where ℓ_p^n is the Banach space $\mathbb{C}^n$ equipped with the p-norm $\|z\|_p := (\sum_{k=1}^n |z_k|^p)^{1/p}$ for $1 \le p < \infty$, while $\|z\|_\infty := \max_{1\le k\le n} |z_k|$. Also, we use the standard multi-index notation: α denotes an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers, $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and for $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, z^α is the product $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. Indeed, $K(\mathbb{D}) = 1/3$, and it is known from [3, Theorem 3] that $K(R) \ge K(B_{\ell_\infty^n})$ for any complete Reinhardt domain R. Through the substantial progress made in a series of papers from 1997 to 2011, it was finally concluded by Defant and Frerick in [5] that

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there exists a constant $c \geq 0$ such that for each $p \in [1, \infty]$,

$$\frac{1}{c}\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{p,2\}}} \leq K(B_{\ell_p^n}) \leq c\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{p,2\}}} \tag{0.1}$$

for all $n > 1$.

On the other hand, Aizenberg [1] introduced a *second Bohr radius* $B(R)$ for a bounded complete Reinhardt domain $R \subset \mathbb{C}^n$, which is the largest $r \in [0,1]$ such that for each holomorphic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ on R satisfying $|f(z)| \leq 1$ for all $z \in R$, we have

$$\sum_\alpha \sup_{z \in rR} |a_\alpha z^\alpha| \leq 1.$$

Clearly, $B(\mathbb{D}) = 1/3$ and $B(B_{\ell_\infty^n}) = K(B_{\ell_\infty^n})$. It was also shown in [1] that $B(R) \geq 1 - (2/3)^{1/n} > 1/(3n)$ for any bounded complete Reinhardt domain $R \subset \mathbb{C}^n (n \geq 2)$, and that

$$B(B_{\ell_1^n}) < \frac{0.446663}{n}. \tag{0.2}$$

Further advances were made by Boas in [2], showing that for all $p \in [1, \infty]$,

$$\frac{1}{3}\left(\frac{1}{n}\right)^{\frac{1}{2}+\frac{1}{\max\{p,2\}}} \leq B(B_{\ell_p^n}) < 4\left(\frac{\log n}{n}\right)^{\frac{1}{2}+\frac{1}{\max\{p,2\}}} \quad (n > 1). \tag{0.3}$$

To the best of our knowledge, except for the subsequent article [6], the problem of estimating $B(B_{\ell_p^n})$ has not been considered ever since. This is probably because no specific application of this second Bohr radius seems to be known. However, we believe that this is a problem of independent interest. Our aim is to point out that the results of [2, 5] readily yield a much refined lower bound for $B(B_{\ell_p^n})$. This bound shows that analogous to $K(B_{\ell_p^n})$, $B(B_{\ell_p^n})$ must also contain a $\log n$ term. It may also be noted that for a variety of bounded complete Reinhardt domains $R \subset \mathbb{C}^n$, parts of our arguments could be adopted to derive results for $B(R)$ from previously known results for $K(R)$.

To facilitate our discussion, let us now denote by $\chi_{\text{mon}}(\mathcal{P}(^m\ell_p^n))$ the unconditional basis constant associated with the basis consisting of the monomials z^α, for the space $\mathcal{P}(^m\ell_p^n)$ of m-homogeneous complex-valued polynomials P on ℓ_p^n. This space is equipped with the norm $\|P\| = \sup_{\|z\|_p \leq 1} |P(z)|$. We mention here that a Schauder basis (x_n) of a Banach space X is said to be unconditional if there exists a constant $c \geq 0$ such that

$$\left\|\sum_{k=1}^t \varepsilon_k \alpha_k x_k\right\| \leq c \left\|\sum_{k=1}^t \alpha_k x_k\right\|$$

for all $t \in \mathbb{N}$ and for all $\varepsilon_k, \alpha_k \in \mathbb{C}$ with $|\varepsilon_k| \leq 1, 1 \leq k \leq t$. The best constant c is called the unconditional basis constant of (x_n). Now, it is known from [6, p. 56] (see also Lemma 2.1 of [6]) that

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$$\chi_{\text{mon}}(\mathcal{P}(^m\ell_p^n)) = \frac{1}{(K_m(B_{\ell_p^n}))^m}, \tag{0.4}$$

where $K_m(B_{\ell_p^n})$ is the supremum of all $r \in [0,1]$ such that for each m-homogeneous complex-valued polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ with $|P(z)| \le 1$ for all $z \in B_{\ell_p^n}$, we have $\sum_{|\alpha|=m} |a_\alpha z^\alpha| \le 1$ for all $z \in rB_{\ell_p^n}$. Clearly, $K_m(B_{\ell_p^n}) \ge K(B_{\ell_p^n})$. These facts are instrumental in proving Theorem 0.1.

Theorem 0.1 *There exists a constant $C > 0$ such that for each $p \in [1, \infty]$,*

$$B(B_{\ell_p^n}) \ge C \frac{(\log n)^{1-\frac{1}{\min\{p,2\}}}}{n^{\frac{1}{2}+\frac{1}{\max\{p,2\}}}}$$

for all $n > 1$.

Proof It is observed in [2, p. 335] that

$$B(B_{\ell_p^n}) \ge \frac{B(B_{\ell_\infty^n})}{n^{\frac{1}{p}}}.$$

Since $B(B_{\ell_\infty^n}) = K(B_{\ell_\infty^n}) \ge C(\sqrt{\log n}/\sqrt{n})$ for some constant $C > 0$ (see (0.1)), the proof for the case $p \in [2, \infty]$ follows immediately from the above inequality.

For the case $p \in [1, 2)$, a little more work is needed. Given any holomorphic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ on $B_{\ell_p^n}$ with $|f(z)| \le 1$ for all $z \in B_{\ell_p^n}$, it is evident that for any fixed $z \in B_{\ell_p^n}$, $h(u) := f(uz) = a_0 + \sum_{m=1}^{\infty} \left(\sum_{|\alpha|=m} a_\alpha z^\alpha\right) u^m : \mathbb{D} \to \overline{\mathbb{D}}$ is a holomorphic function of $u \in \mathbb{D}$. The well-known Wiener's inequality asserts that

$$\sup_{\|z\|_p \le 1} \left| \sum_{|\alpha|=m} a_\alpha z^\alpha \right| \le 1 - |a_0|^2$$

for all $m \ge 1$. The definition of $\chi_{\text{mon}}(\mathcal{P}(^m\ell_p^n))$ guarantees that for the choices of ε_αs such that $\varepsilon_\alpha a_\alpha = |a_\alpha|$,

$$\begin{aligned}\left(\sum_{|\alpha|=m} |a_\alpha|\right) \frac{1}{n^{\frac{m}{p}}} = \sum_{|\alpha|=m} |a_\alpha| \left(\frac{1}{n^{\frac{1}{p}}}\right)^\alpha &\le \sup_{\|z\|_p \le 1} \left| \sum_{|\alpha|=m} \varepsilon_\alpha a_\alpha z^\alpha \right| \\ &\le \chi_{\text{mon}}(\mathcal{P}(^m\ell_p^n)) \sup_{\|z\|_p \le 1} \left| \sum_{|\alpha|=m} a_\alpha z^\alpha \right| \\ &\le (1 - |a_0|^2) \frac{1}{(K_m(B_{\ell_p^n}))^m}\end{aligned}$$

(see (0.4)). That is to say,

$$\sum_{|\alpha|=m} |a_\alpha| \le (1 - |a_0|^2) \frac{n^{\frac{m}{p}}}{(K_m(B_{\ell_p^n}))^m} \le (1 - |a_0|^2) \frac{n^{\frac{m}{p}}}{(K(B_{\ell_p^n}))^m}.$$

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A little computation reveals that

$$\sum_{\alpha} \sup_{z \in rB_{\ell_p^n}} |a_\alpha z^\alpha| = |a_0| + \sum_{m=1}^{\infty} r^m \sum_{|\alpha|=m} |a_\alpha| \left(\frac{\alpha^\alpha}{m^m}\right)^{\frac{1}{p}}$$

$$\leq |a_0| + \sum_{m=1}^{\infty} r^m \sum_{|\alpha|=m} |a_\alpha|$$

$$\leq |a_0| + (1 - |a_0|^2) \sum_{m=1}^{\infty} \left(\frac{rn^{\frac{1}{p}}}{K(B_{\ell_p^n})}\right)^m .$$

It is clear from the above inequality that $\sum_\alpha \sup_{z \in rB_{\ell_p^n}} |a_\alpha z^\alpha| \leq 1$ whenever

$$r \leq \frac{1}{3}\left(\frac{K(B_{\ell_p^n})}{n^{\frac{1}{p}}}\right),$$

i.e., $B(B_{\ell_p^n}) \geq K(B_{\ell_p^n})/(3n^{1/p})$. In view of the inequalities (0.1), this completes the proof. ■

Remark 0.2 It should be mentioned that the logarithmic term in the known upper bound for $B(B_{\ell_p^n})$ in (0.3) differs from the logarithmic term in the lower bound for $B(B_{\ell_p^n})$ obtained in Theorem 0.1. Hence, it remains unknown whether this lower bound is asymptotically optimal. Let us also note that for $p = 1$, (0.3) asserts that $B(B_{\ell_1^n})$ is bounded above by $(4 \log n)/n$, but from (0.2) it is clear that the $4 \log n$ term can be replaced by a constant less than 1. Therefore, it seems that there is room for improvement on the upper bound of $B(B_{\ell_p^n})$ in (0.3) as well, at least for a certain range of p.

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