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SOME SPHERE THEOREMS FOR SUBMANIFOLDS WITH POSITIVE BIORTHOGONAL CURVATURE

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Abstract. The purpose of this paper is to investigate sphere theorems for submanifolds with positive biorthogonal (sectional) curvature. We provide some upper bounds for the full norm of the second fundamental form under which a compact submanifold must be diffeomorphic to a sphere.

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1. Introduction. In the last decades many mathematicians investigated the topological and differentiable structures of submanifolds of spheres and Euclidean spaces. In this sense, in 1973, Lawson and Simons [**19**], by means of nonexistence for stable currents on compact submanifolds of a sphere, obtained a criterion for the vanishing of the homology groups of compact submanifolds of spheres. Leung [**20**] and Xin [**24**] were able to extend the results obtained by Lawson and Simons for compact submanifolds of Euclidean spaces, whereas Asperti and Costa [**1**] obtained an estimate for the Ricci curvature of submanifolds of a space form that improves Leung's estimates. As a consequence, Asperti and Costa obtained a new criterion for the vanishing of the homology groups of compact submanifolds of spheres and Euclidean spaces. In 2009, Xu and Zhao [**25**] investigated the topological and differentiable structures of submanifolds by imposing certain conditions on the second fundamental form.

From all that follows, we recall that a compact (without boundary) n-dimensional Riemannian manifold (M^n, g) is said to be δ-pinched if the sectional curvature K satisfies

$$1 \geq K \geq \delta. \tag{1}$$

If the strict inequality holds, we say that M^n is strictly δ-pinched.

In 1997, Xia [**12**] proved that an n-dimensional ($n \geq 4$) compact simply connected submanifold M^n, isometrically immersed in a δ-pinched ($\delta > \frac{1}{4}$) Riemannian manifold such that its second fundamental form α satisfies

$$\|\alpha(v, v)\|^2 < \frac{4}{9}\Big(\delta - \frac{1}{4}\Big),$$

for all unit tangent vectors v on M^n, must be homeomorphic to a sphere. Although

Xu and Zhao [25] have proved that an n-dimensional oriented compact submanifold in an $(n+m)$-dimensional $\delta(>\frac{1}{4})$-pinched Riemannian manifold satisfying

$$\|\alpha(v,v)\|^2 < \frac{4}{9}\big(\delta - \frac{1}{4}\big),\ \text{ for any unit } \ v \in T_pM,$$

must be diffeomorphic to a spherical space form. At same time, Gu and Xu [13] showed that a three-dimensional compact, simply connected submanifold M^3 in an n-dimensional Riemannian manifold $\overline{M}^n$ satisfying a suitable condition involving the mean curvature and scalar curvature, must be diffeomorphic to a sphere $\mathbb{S}^3$. Thus, investigating curvature conditions, which guarantee that a compact Riemannian manifold is diffeomorphic to a sphere is definitely an important issue.

Recently, Ribeiro and Costa [10] used the notion of biorthogonal (sectional) curvature to obtain some interesting sphere theorems. In particular, they were able to improve the pinching constants obtained in some previous works. In order to proceed let us recall the concept of biorthogonal (sectional) curvature. For each plane $P \subset T_xM$ at a point $x \in M^4$, the *biorthogonal (sectional) curvature* of P is given by the following average of the sectional curvatures:

$$K^{\perp}(P) = \frac{K(P) + K(P^{\perp})}{2}, \tag{2}$$

where $P^{\perp}$ is the orthogonal plane to P. In particular, for each point $x \in M^4$, we take the minimum of biorthogonal curvature to obtain the following function:

$$K^{\perp}_{min}(x) = \min\{K^{\perp}(P); P \text{ is a 2- plane in } T_xM\}. \tag{3}$$

As it was pointed out in [9] the sum of two sectional curvatures on two orthogonal planes appeared in works of Noronha [18] and Seaman [22]. We remark that the positivity of the biorthogonal curvature is an intermediate condition between positive sectional curvature and positive scalar curvature. There is considerable literature on the topic, for a comprehensive references on such a subject, we indicate, for instance [2, 9–11, 18, 21] and [22].

A famous result by Tachibana [23] asserts that a compact Einstein manifold (M^n, g) with positive curvature operator has constant sectional curvature. Furthermore, he also proved that if (M^n, g) has nonnegative curvature operator, then it is locally symmetric. We now recall that a Riemannian manifold N^n, $n \geq 4$, has *positive isotropic curvature,* if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0.$$

The notion of isotropic curvature was introduced by Micallef and Moore [17], where it is proved that a compact Riemannian manifold with positive isotropic curvature is homeomorphic to a sphere. Micallef and Wang [16] extended Tachibana's result for $n = 4$ showing that a four-dimensional Einstein manifold with nonnegative isotropic curvature is locally symmetric. Recently, these results have been extended in two remarkable ways. First, by means of a deep convergence analysis for the Ricci flow, Böhm and Wilking [8] proved for $n \geq 4$ that manifolds with positive curvature operator

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are space forms. Second Brendle [4] proved that if (M^n, g) is an Einstein manifold, $n \geq 4$, with nonnegative isotropic curvature then it is locally symmetric. Moreover, from Costa and Ribeiro Jr. [9], $\mathbb{S}^4$ and $\mathbb{CP}^2$ are the only compact simply connected four-dimensional manifolds with positive biorthogonal curvature that can have (weakly) 1/4-pinched biorthogonal curvature, or nonnegative isotropic curvature, or satisfy $K^{\perp} \geq \frac{s}{24} > 0$.

We proceed to state our results. Indeed, motivated by the ideas developed by Ribeiro and Costa [9, 10], Gu and Xu [13] as well as Xu and Zhao [25], we shall investigate sphere theorems for submanifolds under certain conditions involving the biorthogonal (sectional) curvature. After these preliminary remarks, we may announce our first result as follows.

THEOREM 1. *Let M^4 be a four-dimensional compact, simply connected submanifold, isometrically immersed in a δ-pinched Riemannian manifold. Then, M^4 is diffeomorphic to a sphere $\mathbb{S}^4$, provided that one of the following conditions holds:*

1. $\|\alpha(v,v)\|^2 < K^{\perp}_{min} - \frac{1}{3}(1-\delta)$, *for any unit* $v \in T_pM^4$; *or*
2. $\|\alpha\|^2 < 4K^{\perp}_{min} + 4\delta + 8H^2 - \frac{s}{3}$, *where H and s stand for the mean curvature and scalar curvature of M^4, respectively.*

As an immediate consequence of Theorem 1, we get the following corollary:

COROLLARY 1. *Let M^4 be a four-dimensional compact, simply connected Riemannian submanifold, isometrically immersed in to a standard sphere $\mathbb{S}^n$. Suppose that*

$$\|\alpha(v,v)\|^2 < K^{\perp}_{min},$$

for any unit tangent vector v on M^4. Then M^4 is diffeomorphic to sphere $\mathbb{S}^4$.

In Theorem 13 of [13], it was showed that a three-dimensional compact submanifold in an n-dimensional Riemannian manifold $\overline{M}^n$ satisfying

$$\|\alpha\|^2 < 2\overline{K}_{min} + \frac{9}{2}H^2,$$

must be diffeomorphic to a spherical space form. Motivated by this result, we shall consider a compact simply connected hypersurface M^3 of a compact four-dimensional manifold N^4 to establish the following theorem.

THEOREM 2. *Let M^3 be a compact simply connected hypersurface of a compact four-dimensional manifold N^4. Suppose that*

$$\|\alpha\|^2 < 4K^{\perp}_{min} - 2\overline{K}_{\max} + \frac{9}{2}H^2,$$

where $\overline{K}_{\max}$ is the maximum value of the sectional curvature of M^4. Then, M^3 is diffeomorphic to a sphere $\mathbb{S}^3$.

In [18], Noronha obtained some classification results for four-dimensional compact manifolds with nonnegative sectional curvature. For instance, she showed that if $\|W^-\|^2 \geq -\omega_1^- \frac{s}{2}$ and the self-dual part of the Weitzenböck operator E^+ has a negative eigenvalue at some point of M^4, then $W^- = 0$. In this case, the scalar curvature s cannot be constant; for more details see [18]. We shall combine Noronha's theorem with Brendle [**5**], Brendle and Schoen [**6**] to deduce the following classification result.

THEOREM 3. *Let M^4 be a four-dimensional compact submanifold in an n-dimensional Riemannian manifold $\overline{M}^n$ satisfying*

$$\|\alpha\|^2 \leq 2\left(\overline{K}_{min} + K^{\perp}_{min}\right) + \frac{16}{3}H^2 - \frac{s}{6}. \tag{4}$$

Then we have:

1. *If M^4 is oriented, $\Delta W = 0$ (cf. [**18**] for details on this condition) and $\| W^- \|^2 \geq -(\omega_1^-)\frac{s}{2}$, then one of the following assertions holds:*
   (a) *M^4 is conformally equivalent to S^4, or it is covered by either $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$ with their standard metrics.*
   (b) *M^4 is covered by $\mathbb{S}^2 \times \mathbb{S}^2$, where $\mathbb{S}^2$ has constant curvature.*
   (c) *M^4 is isometric to $\mathbb{CP}^2$.*
   (d) *M^4 is anti-self-dual and negative definite.*
   (e) *M^4 is self-dual and the scalar curvature is not constant.*
2. *If M^4 is locally irreducible, then one the following assertions holds:*
   (a) *M^4 is diffeomorphic to a spherical space form.*
   (b) *The universal cover of M^4 is a Kähler manifold biholomorphic to $\mathbb{CP}^2$.*
   (c) *The universal cover of M^4 is isometric to a compact symmetric space.*
3. *If inequality (4) is strict, then M^4 is diffeomorphic to a spherical space form. In addition, if M^4 is simply connected, then M^4 is diffeomorphic to $\mathbb{S}^4$.*

2. Preliminaries. In this section, we present a couple of formulae and notations that are essential for our purpose. It is well-known that four-dimensional manifolds are fairly special. For instance, the bundle of two-forms on a four-dimensional oriented Riemannian manifold can be invariantly decomposed as a direct sum

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{5}$$

where $\Lambda^{\pm}$ is the (± 1)-eigenspace of Hodge star operator $*$. The decomposition (5) is conformally invariant. Moreover, it allows us to conclude that the Weyl tensor W is an endomorphism of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ such that $W = W^+ \oplus W^-$. A manifold is conformally flat if $W = 0$, it is said to be *half conformally flat* if either $W^+ = 0$ or $W^- = 0$. In particular, an oriented four-dimensional manifold M^4 is *self-dual* if $W^- = 0$. We note that on a half-conformally flat manifold, self-duality is a property that depends on the orientation.

We fix a point and diagonalize $W^{\pm}$ such that $w_i^{\pm}$, $1 \leq i \leq 3$, are their respective eigenvalues. So, we point out that the eigenvalues of $W^{\pm}$ satisfy

$$w_1^{\pm} \leq w_2^{\pm} \leq w_3^{\pm} \quad \text{and} \quad w_1^{\pm} + w_2^{\pm} + w_3^{\pm} = 0. \tag{6}$$

In particular, (6) tells us that

$$|W^{\pm}|^2 \leq 6(w_1^{\pm})^2. \tag{7}$$

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From this, as it was detailed in [**9**], equation (3) provides us the following key identity:

$$K^{\perp}_{min} = \frac{w_1^+ + w_1^-}{2} + \frac{s}{12}. \tag{8}$$

In an analogous way, we have

$$K^{\perp}_{max} = \frac{w_3^+ + w_3^-}{2} + \frac{s}{12}, \tag{9}$$

where $K^{\perp}_{max}(p) = \max\{K^{\perp}(P); P \subset T_pM\}$. More details can be found in [**9**, **21**] and references therein.

In the sequel, let M^n be a submanifold in a Riemannian manifold $\overline{M}^N$. So, we shall adopt the following convention on the range of indices:

$$1 \le i, j, k, l \le n, \quad 1 \le A, B, C, D \le N, \quad n+1 \le \beta \le N.$$

For any arbitrary point $x \in M$, we choose an orthonormal frame $\{e_i, e_\beta\}$ of the $\overline{M}$ such that $\{e_i\}$ are tangent to M. Denote by $\{e^i\}$ the dual frame field of $\{e_i\}$. Let

$$Rm = \sum_{i,j,k,l} R_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l,$$

$$\overline{Rm} = \sum_{A,B,C,D} \overline{R}_{ABCD} e^A \otimes e^B \otimes e^C \otimes e^D;$$

be the Riemannian curvature tensor of M and $\overline{M}$, respectively. We consider α and $\overrightarrow{H}$ the second fundamental form and the mean curvature vector of M. We set

$$\alpha = \sum_{\beta,i,j} h^{\beta}_{ij} e^i \otimes e^j \otimes e_\beta\,, \qquad \overrightarrow{H} = \frac{1}{n}\sum_{\beta,i} h^{\beta}_{ii} e_\beta.$$

The squared norm of the second fundamental form $\|\alpha\|^2$ and the mean curvature H of M are given by

$$\|\alpha\|^2 := \sum_{\beta,i,j} (h^{\beta}_{ij})^2,$$

and

$$H := \frac{1}{n}\sqrt{\sum_{\beta}\Big(\sum_i h^{\beta}_{ii}\Big)^2}.$$

Therefore, from the Gauss equation we have

$$R_{ijkl} = \overline{R}_{ijkl} + \langle \alpha(e_i, e_k), \alpha(e_j, e_l)\rangle - \langle \alpha(e_i, e_l), \alpha(e_j, e_k)\rangle,$$

which can be succinctly written as

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_{\beta} h^{\beta}_{ik} h^{\beta}_{jl} - \sum_{\beta} h^{\beta}_{il} h^{\beta}_{jk}.$$

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3. Proof of the main results.

3.1. Proof of Theorem 1.

Proof. To begin with, we notice that

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} = K_{13} + K_{14} + K_{23} + K_{24} - 2R_{1234},$$

where $K_{ij} = K(e_i, e_j)$. Therefore, by setting $K_{ij}^{\perp} = K^{\perp}(e_i, e_j)$ we may use (2) and (3) to obtain

$$\begin{aligned} R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} &= 2K_{13}^{\perp} + 2K_{14}^{\perp} - 2R_{1234} \\ &\geq 4K_{\min}^{\perp} - 2R_{1234}. \end{aligned} \tag{10}$$

On the other hand, from the Gauss equation we deduce

$$-2R_{1234} = -2\big[\overline{R}_{1234} + \langle \alpha(e_1, e_3), \alpha(e_2, e_4)\rangle - \langle \alpha(e_2, e_3), \alpha(e_1, e_4)\rangle\big]. \tag{11}$$

Now, we invoke Berger's inequality **[3]** to infer

$$|\overline{R}_{ijkl}| \leq \frac{2}{3}(1 - \delta).$$

From here it follows that

$$\begin{aligned} 4K_{min}^{\perp} - 2R_{1234} \geq 4K_{\min}^{\perp} - \frac{4}{3}(1 - \delta) - 2\langle \alpha(e_1, e_3), \alpha(e_2, e_4)\rangle \\ +2\langle \alpha(e_2, e_3), \alpha(e_1, e_4)\rangle. \end{aligned} \tag{12}$$

In order to proceed, let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal frame in M^4. Then, as consequence of the Cauchy–Schwarz inequality, we get the following expressions:

$$\langle \alpha(e_i, e_j), \alpha(e_k, e_l)\rangle \geq -\frac{1}{2}\{| \alpha(e_i, e_j) |^2 + | \alpha(e_k, e_l) |^2\}, \tag{13}$$

and

$$-\langle \alpha(e_i, e_j), \alpha(e_k, e_l)\rangle \geq -\frac{1}{2}\{| \alpha(e_i, e_j) |^2 + | \alpha(e_k, e_l) |^2\}. \tag{14}$$

In particular, according to **[12]**, we also have

$$| \alpha(e_i, e_j) |^2 \leq \frac{1}{2}\Big\{ | \alpha(\frac{e_i + e_j}{\sqrt{2}}, \frac{e_i + e_j}{\sqrt{2}}) |^2 + | \alpha(\frac{e_i - e_j}{\sqrt{2}}, \frac{e_i - e_j}{\sqrt{2}}) |^2 \Big\}. \tag{15}$$

Therefore, we may use the above information into (12) to arrive at

$$\begin{aligned} 4K_{min}^{\perp} - 2R_{1234} \geq 4K_{\min}^{\perp} - \frac{4}{3}(1 - \delta) - \big\{ | \alpha(e_1, e_3) |^2 + | \alpha(e_2, e_4) |^2 \big\} \\ -\big\{ | \alpha(e_2, e_3) |^2 + | \alpha(e_1, e_4) |^2 \big\}. \end{aligned} \tag{16}$$

From this, we use our first assumption together with (15) to deduce that M^4 has positive isotropic curvature. Whence, it suffices to invoke a theorem of Hamilton **[14]** to conclude that M^4 is diffeomorphic to a sphere $\mathbb{S}^4$.

Proceeding, we treat our second assumption. For sake of simplicity, we now consider

$$A = K_{13} + K_{14} + K_{23} + K_{24} - 2R_{1234}.$$

Then, from the Gauss equation and insofar as N is $\delta-$pinched, we have

$$\begin{aligned} A &= \overline{K}_{13} + \overline{K}_{14} + \overline{K}_{23} + \overline{K}_{24} - 2R_{1234} + \sum_{\beta}\Big[h^{\beta}_{11}h^{\beta}_{33} + h^{\beta}_{22}h^{\beta}_{33} + \\ &\quad + h^{\beta}_{22}h^{\beta}_{44} + h^{\beta}_{11}h^{\beta}_{44} - (h^{\beta}_{13})^2 - (h^{\beta}_{23})^2 - (h^{\beta}_{24})^2 - (h^{\beta}_{14})^2\Big] \\ &\geq 4\delta - 2R_{1234} + \sum_{\beta}\Big[h^{\beta}_{11}h^{\beta}_{33} + h^{\beta}_{22}h^{\beta}_{33} + +h^{\beta}_{22}h^{\beta}_{44} + h^{\beta}_{11}h^{\beta}_{44} \\ &\quad - (h^{\beta}_{13})^2 - (h^{\beta}_{23})^2 - (h^{\beta}_{24})^2 - (h^{\beta}_{14})^2\Big]. \end{aligned} \tag{17}$$

On the other hand, using that $(a+b)^2 \leq 2a^2 + 2b^2$, we immediately get

$$\begin{aligned} 16H^2 &= \sum_{\beta}(h^{\beta}_{11} + h^{\beta}_{22} + h^{\beta}_{33} + h^{\beta}_{44})^2 \\ &= \sum_{\beta}[(h^{\beta}_{11} + h^{\beta}_{22})^2 + (h^{\beta}_{33} + h^{\beta}_{44})^2 + 2(h^{\beta}_{11} + h^{\beta}_{22})(h^{\beta}_{33} + h^{\beta}_{44})] \\ &\leq \sum_{\beta}\{2[(h^{\beta}_{11})^2 + (h^{\beta}_{22})^2] + 2[(h^{\beta}_{33})^2 + (h^{\beta}_{44})^2] + 2(h^{\beta}_{11}h^{\beta}_{33} + h^{\beta}_{11}h^{\beta}_{44} + h^{\beta}_{22}h^{\beta}_{33} + h^{\beta}_{22}h^{\beta}_{44})\}, \end{aligned}$$

which can be rewritten as

$$\sum_{\beta}[h^{\beta}_{11}h^{\beta}_{33} + h^{\beta}_{11}h^{\beta}_{44} + h^{\beta}_{22}h^{\beta}_{33} + h^{\beta}_{22}h^{\beta}_{44}] \geq 8H^2 - \sum_{\beta,i}(h^{\beta}_{ii})^2. \tag{18}$$

From (18) and (17) we infer

$$\begin{aligned} A &\geq 4\delta + 8H^2 - 2R_{1234} - \sum_{\beta,i}(h^{\beta}_{ii})^2 - \sum_{\beta}\Big[(h^{\beta}_{13})^2 + (h^{\beta}_{23})^2 + (h^{\beta}_{24})^2 + (h^{\beta}_{14})^2\Big] \\ &\geq 4\delta + 8H^2 - 2R_{1234} - \|\alpha\|^2. \end{aligned} \tag{19}$$

Now, from Seaman's inequality **[22]** (see also **[3]**) we see that

$$| R_{1234} | \leq \frac{2}{3}(K^{\perp}_{\max} - K^{\perp}_{\min}). \tag{20}$$

In particular, as a consequence of (9), it is easy to see that $K^{\perp}_{\max} \leq \frac{s}{4} - 2K^{\perp}_{\min}$, which combined with (20) gives

$$| R_{1234} | \leq \frac{2}{3}\left(\frac{s}{4} - 3K^{\perp}_{\min}\right). \tag{21}$$

Now, by (19) and (21) we have

$$A \geq 4\delta + 8H^2 + 4K^{\perp}_{\min} - \frac{s}{3} - \|\alpha\|^2.$$

So, it suffices to use our second assumption to conclude that M^4 has positive isotropic curvature. Finally, we invoke once more Hamilton's theorem [**14**] to finish the proof of the theorem. $\square$

3.2. Proof of Theorem 2.

Proof. First of all, let $v = e_3$ be a unit tangent vector in T_pM^3 such that $\{e_1, e_2, e_3\}$ is an orthonormal basis of T_pM^3 and $\{e_1, e_2, e_3, e_4\}$ an orthonormal basis of T_pM^4. We then use Gauss equation to infer

$$\begin{aligned} Ric(v) &= K_{13} + K_{23} \\ &= \overline{K}_{13} + \overline{K}_{23} + h_{11}h_{33} - (h_{13})^2 + h_{22}h_{33} - (h_{23})^2 \\ &= (\overline{K}_{13} + \overline{K}_{24}) + (\overline{K}_{23} + \overline{K}_{14}) - \overline{K}_{24} - \overline{K}_{14} + h_{11}h_{33} + h_{22}h_{33} - (h_{13})^2 - (h_{23})^2. \end{aligned} \tag{22}$$

Notice that

$$H = \frac{1}{3}\sqrt{(h_{11} + h_{22} + h_{33})^2}.$$

This immediately gives

$$\begin{aligned} 9H^2 &= (h_{11} + h_{22} + h_{33})^2 \\ &= ((h_{11} + h_{22})^2 + 2(h_{11} + h_{22})h_{33} + (h_{33})^2 \\ &\leq 2[(h_{11})^2 + (h_{22})^2] + 2h_{11}h_{33} + 2h_{22}h_{33} + 2(h_{33})^2, \end{aligned}$$

which can be written succinctly as

$$h_{11}h_{33} + h_{22}h_{33} \;\geq\; \frac{9}{2}H^2 - [(h_{11})^2 + (h_{22})^2 + (h_{33})^2]. \tag{23}$$

Next, from (22) and (23) we get

$$\begin{aligned} Ric(v) &\geq 2\overline{K}^{\perp}_{13} + 2\overline{K}^{\perp}_{14} + \frac{9}{2}H^2 - 2\overline{K}_{\max} - \|\alpha\|^2 \\ &\geq 4K^{\perp}_{\min} + \frac{9}{2}H^2 - 2\overline{K}_{\max} - \|\alpha\|^2 > 0, \end{aligned} \tag{24}$$

where we have used our assumption. Therefore, it suffices to apply Hamilton's theorem [**15**] to conclude the proof of the theorem. $\square$

3.3. Proof of Theorem 3.

Proof. To start with, we treat the first statement. To do so, let $\{e_1, e_2, e_3, e_4\}$ an orthonormal frame in M^4 and let $\lambda,\ \mu \in [-1, 1]$. Moreover, we set

$$B = K_{13} + \lambda^2 K_{14} + \mu^2 K_{23} + \lambda^2\mu^2 K_{24} - 2\lambda\mu R_{1234}.$$

From the Gauss equation, we infer

$$\begin{aligned} B = \overline{K}_{13} &+ \sum_\beta [h^\beta_{11}h^\beta_{33} - (h^\beta_{13})^2] + \lambda^2\{\overline{K}_{14} + \sum_\beta [h^\beta_{11}h^\beta_{44} - (h^\beta_{14})^2]\} - 2\lambda\mu R_{1234} \\ &+\mu^2\{\overline{K}_{23} + \sum_\beta [h^\beta_{22}h^\beta_{33} - (h^\beta_{23})^2]\} + \lambda^2\mu^2\{\overline{K}_{24} + \sum_\beta [h^\beta_{22}h^\beta_{44} - (h^\beta_{24})^2]\}. \end{aligned} \tag{25}$$

We now claim that

$$1 + \lambda^2 + \mu^2 + \lambda^2\mu^2 \geq 4\lambda\mu. \tag{26}$$

Indeed, we invoke Cauchy's inequality to deduce $\lambda^2 + \mu^2 \geq 2\lambda\mu$. Moreover, it is not difficult to check that $(\lambda\mu)^2 - 2\lambda\mu + 1 \geq 0$, for all λ, μ, which settles our claim (26).

Proceeding, from inequalities (25), (21) and (26) we obtain

$$\begin{aligned}
B &\geq (1 + \lambda^2 + \mu^2 + \lambda^2\mu^2)\overline{K}_{\min} - 4\lambda\mu\frac{1}{3}\Big(\frac{s}{4} - 3K^{\perp}_{\min}\Big) \\
&\quad + \sum_{\beta}\{h^{\beta}_{11}h^{\beta}_{33} - (h^{\beta}_{13})^2 + \lambda^2\mu^2[h^{\beta}_{22}h^{\beta}_{44} - (h^{\beta}_{24})^2] \\
&\quad + \mu^2[h^{\beta}_{22}h^{\beta}_{33} - (h^{\beta}_{23})^2] + \lambda^2[h^{\beta}_{11}h^{\beta}_{44} - (h^{\beta}_{14})^2]\} \\
&\geq (1 + \lambda^2 + \mu^2 + \lambda^2\mu^2)\Big[\overline{K}_{\min} - \frac{1}{3}\Big(\frac{s}{4} - 3K^{\perp}_{\min}\Big)\Big] \\
&\quad + \sum_{\beta}\{h^{\beta}_{11}h^{\beta}_{33} + \lambda^2\mu^2h^{\beta}_{22}h^{\beta}_{44} + \mu^2h^{\beta}_{22}h^{\beta}_{33} + \lambda^2h^{\beta}_{11}h^{\beta}_{44}\} \\
&\quad - \sum_{\beta}\big\{(h^{\beta}_{13})^2 + \lambda^2(h^{\beta}_{14})^2 + \mu^2(h^{\beta}_{23})^2 + \lambda^2\mu^2(h^{\beta}_{24})^2\big\}.
\end{aligned}$$

Now, from [**13**] (cf. Equation 4.8 in [**13**]) we already know that, for all $m \neq l$, we have

$$h^{\beta}_{mm}h^{\beta}_{ll} \geq \sum_{i<j}(h^{\beta}_{ij})^2 + \frac{1}{6}\Big(\sum_{i=1}^{4}h^{\beta}_{ii}\Big)^2 - \frac{1}{2}\sum_{i,j}\big(h^{\beta}_{ij}\big)^2. \tag{27}$$

Furthermore, it is well known that

$$\|\alpha\|^2 = \sum_{\beta,i,j}(h^{\beta}_{ij})^2 \quad \text{and} \quad H^2 = \frac{1}{16}\sum_{\beta}\Big(\sum_{i}h^{\beta}_{ii}\Big)^2. \tag{28}$$

From which it follows that

$$\sum_{\beta}h^{\beta}_{mm}h^{\beta}_{ll} \geq \sum_{\beta,i<j}(h^{\beta}_{ij})^2 + \frac{8}{3}H^2 - \frac{\|\alpha\|^2}{2}. \tag{29}$$

Therefore, a straightforward computation gives

$$\begin{aligned}
B &\geq (1 + \lambda^2 + \mu^2 + \lambda^2\mu^2)\Big[\overline{K}_{\min} - \frac{1}{3}\Big(\frac{s}{4} - 3K^{\perp}_{\min}\Big)\Big] \\
&\quad + (1 + \lambda^2 + \mu^2 + \lambda^2\mu^2)\Big(\sum_{\beta,i<j}(h^{\beta}_{ij})^2 + \frac{8}{3}H^2 - \frac{\|\alpha\|^2}{2}\Big) \\
&\quad - \sum_{\beta}[(h^{\beta}_{13})^2 + \lambda^2\mu^2(h^{\beta}_{24})^2 + \mu^2(h^{\beta}_{23})^2 + \mu^2h^{\beta}_{14})^2]
\end{aligned}$$

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$$
\begin{aligned}
&\geq \frac{(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{2}[2\overline{K}_{\min} - (\frac{s}{6} - 2K^{\perp}_{\min}) + \frac{16}{3}H^2 - \|\alpha\|^2] \\
&\quad + (1+\lambda^2+\mu^2+\lambda^2\mu^2)\{\sum_{\beta,i<j}(h^{\beta}_{ij})^2 - \sum_{\beta}[(h^{\beta}_{13})^2 + (h^{\beta}_{24})^2 + (h^{\beta}_{23})^2 + (h^{\beta}_{14})^2]\} \\
&= \frac{(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{2}[2\overline{K}_{\min} - (\frac{s}{6} - 2K^{\perp}_{\min}) + \frac{16}{3}H^2 - \|\alpha\|^2] \\
&\quad +(1+\lambda^2+\mu^2+\lambda^2\mu^2)[\sum_{\beta}(h^{\beta}_{12})^2 + (h^{\beta}_{34})^2] \\
&\geq \frac{(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{2}[2\overline{K}_{\min} - (\frac{s}{6} - 2K^{\perp}_{\min}) + \frac{16}{3}H^2 - \|\alpha\|^2].
\end{aligned}
$$

Whence our assumption assures that $\|\alpha\|^2 \leq 2\overline{K}_{\min} - (\frac{s}{6} - 2K^{\perp}_{\min}) + \frac{16}{3}H^2$ and then we immediately have $B \geq 0$. But, according to a theorem due to Brendle and Schoen, $B \geq 0$ implies that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature; for more details see Theorems 4.4–4.6 in **[7]** (see also the discussion on p. 70). From this, we deduce that M^4 has nonnegative sectional curvature (cf. **[7]**, Section 4, p. 71–72). We now invoke Theorem 4 in **[18]** to obtain the first assertion.

Next, we treat the second case. In this case, we may use the first part of the proof to arrive at

$$\|\alpha\|^2 \leq 2\overline{K}_{\min} - (\frac{s}{6} - 2K^{\perp}_{\min}) + \frac{16}{3}H^2,$$

which implies that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, for more details see Section 4 in **[7]**. Since M^4 is locally irreducible, we can invoke the Brendle–Schoen theorem **[6]** to conclude the proof of this case.

Finally, we shall prove the third assertion. To do so, we consider

$$C = K_{13} + \lambda^2 K_{14} + K_{23} + \lambda^2 K_{24} - 2\lambda R_{1234}.$$

Notice that from the Gauss equation, we infer

$$
\begin{aligned}
C &= \overline{K}_{13} + \sum_{\beta}[h^{\beta}_{11}h^{\beta}_{33} - (h^{\beta}_{13})^2] + \lambda^2\{\overline{K}_{14} + \sum_{\beta}[h^{\beta}_{11}h^{\beta}_{44} - (h^{\beta}_{14})^2]\} - 2\lambda R_{1234} \\
&\quad + \{\overline{K}_{23} + \sum_{\beta}[h^{\beta}_{22}h^{\beta}_{33} - (h^{\beta}_{23})^2]\} + \lambda^2\{\overline{K}_{24} + \sum_{\beta}[h^{\beta}_{22}h^{\beta}_{44} - (h^{\beta}_{24})^2]\} \\
&\geq (2+2\lambda^2)\overline{K}_{\min} + \sum_{\beta}[h^{\beta}_{11}h^{\beta}_{33} - (h^{\beta}_{13})^2] + \lambda^2[\sum_{\beta}[h^{\beta}_{11}h^{\beta}_{44} - (h^{\beta}_{14})^2] \\
&\quad + \sum_{\beta}[h^{\beta}_{22}h^{\beta}_{33} - (h^{\beta}_{23})^2] + \lambda^2\sum_{\beta}[h^{\beta}_{22}h^{\beta}_{44} - (h^{\beta}_{24})^2] - 2\lambda R_{1234}. \qquad (30)
\end{aligned}
$$

Taking into account that $|R_{1234}| \leq \frac{2}{3}(\frac{s}{4} - 3K^{\perp}_{\min})$, we can use (29) to get

$$
\begin{aligned}
C &\geq (2+2\lambda^2)\Big[\overline{K}_{\min} + \sum_{\beta,i<j}(h^{\beta}_{ij})^2 + \frac{8}{3}H^2 - \frac{\|\alpha\|^2}{2}\Big] - 4\lambda\frac{1}{3}(\frac{s}{4} - 3K^{\perp}_{\min}) \\
&\quad - (h^{\beta}_{13})^2 - (h^{\beta}_{23})^2 - \lambda^2(h^{\beta}_{14})^2 - \lambda^2(h^{\beta}_{24})^2. \qquad (31)
\end{aligned}
$$

But, since $2 + 2\lambda^2 \geq 4\lambda$, for all $\lambda \in \mathbb{R}$, we deduce

$$\begin{aligned} C &\geq (2+2\lambda^2)\Big[\overline{K}_{\min} + \frac{8}{3}H^2 - \frac{\|\alpha\|^2}{2} - \frac{1}{3}(\frac{s}{4} - 3K^{\perp}_{\min})\Big] \\ &\quad + (2+2\lambda^2)[\sum_{\beta}(\sum_{i<j}(h^{\beta}_{ij})^2 - (h^{\beta}_{13})^2 - (h^{\beta}_{23})^2 - (h^{\beta}_{14})^2 - (h^{\beta}_{24})^2)] \\ &\geq \frac{(2+2\lambda^2)}{2}\Big[2\overline{K}_{\min} + \frac{16}{3}H^2 - (\frac{s}{6} - 2K^{\perp}_{\min}) - \|\alpha\|^2\Big]. \end{aligned} \tag{32}$$

By using our assumption we arrive at

$$K_{13} + \lambda^2 K_{14} + K_{23} + \lambda^2 K_{24} - 2\lambda R_{1234} > 0, \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Finally, it suffices to invoke Brendle's theorem [**5**] to deduce that the normalized Ricci Flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)}$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Here, $r_{g(t)}$ stands for the mean value of the scalar curvature of $g(t)$. This tells us that M^4 is diffeomorphic to a spherical space form. In particular, if M^4 is simply connected, then M^4 is diffeomorphic to a sphere $\mathbb{S}^4$. So, the proof is completed. □

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REFERENCES

1. A. Asperti and E. Costa, Vanishing of homology groups, Ricci estimate for submanifolds and applications, *Kodai Math. J.* **24** (2001), 313–328.

2. R. Bettiol, Positive biorthogonal curvature on $\mathbb{S}^2 \times \mathbb{S}^2$, *Proc. Amer. Math. Soc.* **142** (2014), 4341–4353.

3. M. Berger, Sur quelques variétés riemanniennes suffisamment pincées, *Bull. Soc. Math. France* **88** (1960), 57–71.

4. S. Brendle, Einstein manifolds with nonnegative isotropic curvature are locally symmetric, *Duke Math. J.* **151** (2010), 1–21.

5. S. Brendle, A general convergence result for the Ricci flow in higher dimensions, *Duke Math. J.* **145** (2008), 585–601, MR 2462114 Zbl 1161.53052.

6. S. Brendle and R. Schoen, Classification of manifolds with weakly 1/4-pinched curvatures, *Acta Math.* **200** (2008), 1–13, MR 2386107.

7. S. Brendle and R. Schoen, *Sphere theorems in geometry*, *Surveys in differential geometry*, Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, *Surv. Differ. Geom.*, vol. 13 (International Press, Somerville, MA, 2009), 49–84.

8. C. Böhm and B. Wilking, Manifolds with positive curvature operators are space forms, *Ann. Math.* **167**(2) (2008), 1079–1097.

9. E. Costa and E. Ribeiro Jr, Four-dimensional compact manifolds with nonnegative biorthogonal curvature, *Michigan Math. J.* **63** (2014), 673–688.

10. E. Costa and E. Ribeiro Jr, Minimal volume invariants, topological sphere theorems and biorthogonal curvature on 4-manifolds, avaliable at: arXiv:1504.06212 [math.DG] (2015).

11. E. Costa, E. Ribeiro Jr and A. Santos, Modified Yamabe problem on four-dimensional compact manifolds, *Houston J. Math.* **42**(4) (2016), 1141–1152.

12. H. Chang-Yu Xia, A sphere theorem for submanifolds in a manifold with pinched positive curvature, *Monast. für Math.* **124** (1997), 365–368.

13. J.-R. Gu and H.-W. Xu, The sphere theorems for manifolds with positive scalar curvature, *J. Differ. Geom.* **92** (2012), 507–545.

14. R. Hamilton, Four-manifolds with positive curvature operator, *J. Differ. Geom.* **24** (1986), 153–179.

15. R. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differ. Geom.* **17** (1982), 255–306.

16. M. Micallef and M. Wang, Metrics with nonnegative isotropic curvature, *Duke Math. J.* **72** (1993), 649–672.

17. M. Micallef and J. Moore, Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes, *Ann. Math.* **127** (1988), 199–227.

18. M. H. Noronha, Positively curved 4-manifold and the nonnegativity of isotropic curvatures, *Michigan Math. J.* **44** (1997).

19. H. Lawson and J. Simons, On stable currents and their applications to global problems in real and complex geometry, *Ann. Math.* **98** (1973), 427–450.

20. P. Leung, On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold, *Proc. Edinburgh Math. Soc.* **28** (1985), 305–311.

21. E. Ribeiro Jr, Rigidity of four-dimensional compact manifolds with harmonic Weyl tensor, *Annali di Matematica Pura ed. Appl.* **195** (2016), 2171–2181.

22. W. Seaman, Orthogonally pinched curvature tensors and aplications, *Math. Scand.* **69** (1991), 5–14.

23. S. Tashibana, A theorem on Riemannian manifolds with positive curvature operator, *Proc. Japan Acad.* **50** (1974), 301–302.

24. Y. Xin, An application of integral currents to the vanishing theorems, *Sci. Sinica Ser. A* **27** (1984), 233–241.

25. H. Xu and E. Zhao, Topological and differentiable sphere theorems for complete submanifolds, *Comm. Analys. Geom.* **17** (2009), 565–585.