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# ENDS OF SPACES RELATED BY A COVERING MAP

BY

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**Introduction.** Consider a covering $p : X \longrightarrow B$ of connected topological spaces. If $B$ is a compact polyhedron, a classical result of H. Hopf [4] says that the end space $E(X)$ of $X$ is an invariant of the group $G$ of covering transformations. Thus it becomes meaningful to define the end space of the finitely generated group $G$ as $E(G) := E(X)$.

If $B$ is not compact, then $E(X)$ does not depend on $G$ in such a simple way; e.g. consider the covering $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Yet, under certain assumptions, $E(X)$ is completely determined by end space data of $B$ and of $G$, and the main purpose of this paper is to make this relation explicit. See Theorem (3.8).

The key steps towards Theorem (3.8) are the following:

STEP 1. For every group $H$ we define functorially an end space $E(H)$, which is homeomorphic to Hopf's if $H$ is finitely generated. See §1.

STEP 2. There is a continuous map $k : E(G) \longrightarrow E(X)$. If $B$ is compact then $k$ is a homeomorphism. This is merely a restatement of Hopf's result [4]. In general, $k$ need be neither $1-1$ nor onto. However, if an end of $X$ does not belong to $\operatorname{im} k$, then it can be related to an end of $B$. Thus $E(X)$ is the union (not necessarily disjoint) of im $k$ and 'fibers' of ends of $X$ over the ends of $B$. See §2.

STEP 3. $G$ also acts on $E(X)$ and the 'fibers' of ends of $X$ over the ends of $B$ are invariant subspaces under this action. In many cases, if $\underline{\epsilon} \in E(B)$, the 'fiber' $X(\underline{\epsilon})$ of ends of $X$ over $\underline{\epsilon}$ is a single orbit under this action. The isotropy groups of ends $\epsilon \in X(\underline{\epsilon})$ are related to the contributions of the fundamental groups of neighbourhoods of $\underline{\epsilon}$ to $G$. In §3 we make this relation explicit and show that it completely determines the end space structure of $X$. See Theorem (3.8).

We also draw the reader's attention to Theorem (2.8) which gives a relation between the end space of a countable group and the distribution of its infinite cyclic subgroups.

Our results constitute an extension of the pioneering work of H. Hopf [4]. We give some applications in §4.

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§1. **Ends of groups.** It is implicit in Hopf's work [4] that assigning to a f.g. group $H$ its end space $E(H)$ actually determines a covariant functor $E$ from the category

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of f.g. groups and homomorphisms with finite kernel to the category of topological spaces and continuous maps. For a combinatorial interpretation of this, see also [3] and [7].

For every group $G$, $G = \varinjlim G_\lambda$, the direct limit of the system of its f.g. subgroups. Every homomorphism of this limiting system is a monomorphism. Thus

DEFINITION 1.1. *The end space of an arbitrary group $G$ is*

$$E(G) := \varinjlim E(G_\lambda),$$

*where the limit is taken over the system of all finitely generated subgroups $G_\lambda$ of $G$.*

REMARKS 1.2. (i) $E$ is a covariant functor from the category of groups and homomorphisms whose kernel has finite intersection with all f.g. subgroups of the domain group to the category of topological spaces and continuous maps.

(ii) If $G$ is f.g. then the directed system of its f.g. subgroups terminates in $G$. Consequently, $E(G)$ as defined in (1.1) is homeomorphic to Hopf's end space of $G$.

(iii) If $G$ is not f.g. then $E(G)$ may not be compact; see 2.11.

(iv) D. Cohen [1] has used a combinatorial method to define the number of ends of an arbitrary group $G$. If $G$ is f.g. this number corresponds to the cardinality of $E(G)$. However, for non f.g. groups this correspondence may fail. E.g. the Cohen invariant of the rationals is 1, whereas $E(\mathbb{Q}) = E(\mathbb{Z})$ is the 2-element discrete space. Compare also Stallings [7].

(iv) The end spaces of f.g. groups are completely classified; see [4], [1], [7]. Therefore, the chances of computing end spaces of more general groups are quite good. E.g. $E(\mathbb{R})$ = 1-point space; $E(G \times H)$ = 1-point space if $E(G)$ and $E(H)$ are both not empty, etc.

§2. **Relating $E(G)$ to $E(X)$.** The remainder of this paper is dedicated to the investigation of the end space of $X$, where $p : X \longrightarrow B$ is a regular covering with group of covering transformations $G$. We assume throughout that $X$ and $B$ are connected topological spaces of one of the following two types:

TYPE 1. Locally finite simplicial complexes with at most countably many simplices.

TYPE 2. Locally finite $CW$-complexes of finite dimension with at most countably many cells.

In this set-up we exhibit a continuous map $k : E(G) \longrightarrow E(X)$ and study some of its properties; see in particular Proposition 2.5 and Theorem 2.8.

We recall that a proper map $f : X \longrightarrow Y$ has a continuous extension $\bar{f} : \bar{X} \longrightarrow \bar{Y}$ over the Freudenthal compactifications $\bar{X}, \bar{Y}$ of $X, Y$. Further, $\bar{f}$ restricts to $\hat{f} : E(X) \longrightarrow E(Y)$; see [2].

DEFINITION 2.1. (i) *$\epsilon \in E(X)$ is a vertical end of $X$ :⇔ there is a sequence $(g_n)$ in $G$ such that $g_n.x \longrightarrow \epsilon$ for some $x \in X$ (and, hence, for all $x \in X$); compare Hopf* [4].

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(ii) $\epsilon \in E(X)$ *is a horizontal end of* $X$ :⇔ *there is a proper map* $e : R \longrightarrow X$ *with* $\hat{e}(\infty) = \epsilon$ *and such that the sequence* $(pe(n))$ *converges to some end* $\underline{\epsilon} \in E(B)$. *Here,* $R := [0, \infty[$ *is the ray of non-negative real numbers.*

For (2.1.ii) it is relevant that $E(X)$ is in bijective correspondence with the set of all proper maps $R \longrightarrow X$ subject to the following equivalence relation. Two proper maps $e_i : R \times \{i\} \longrightarrow X$, $i = 0, 1$, are equivalent :⇔ there is a proper map $f : L \longrightarrow X$ with $f_{|R\times\{i\}} = e_i$, where $L$ is the infinite ladder $R \times \{0, 1\} \cup \mathbb{N}_0 \times [0, 1]$.

DEFINITION 2.2. *Let* $\underline{\epsilon} \in E(B)$. *The 'fiber'* $X(\underline{\epsilon})$ *is the set of all horizontal ends* $\epsilon$ *of* $X$ *having a proper map* $e : R \longrightarrow X$ *with* $\hat{e}(\infty) = \epsilon$ *and* $pe(n) \longrightarrow \underline{\epsilon}$, *for* $n \longrightarrow \infty$.

Writing 'fiber' is a safety indicator for the fact that 'fibers' over distinct ends of $B$ need not be disjoint; consider e.g., the covering $p : \mathbb{R} \times \mathbb{R} \longrightarrow (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where the one and only end of the total space belongs to the 'fibers' of both ends of $B$.

REMARK 2.3. If $\underline{\epsilon} \in E(B)$, then $X(\underline{\epsilon}) \neq \emptyset$. This follows by considering lifts of proper maps $e : R \longrightarrow B$ with $\hat{e}(\infty) = \underline{\epsilon}$. □

PROPOSITION 2.4. *There is a continuous map* $k : E(G) \longrightarrow E(X)$.

PROPOSITION 2.5. $E(X)$ *is the union of* $\operatorname{im} k$ *and the horizontal ends of* $X$.

PROOF OF 2.4. The map $k$ comes from the defining universal property of $E(G)$; see 1.1: For each f.g. subgroup $G_\lambda$ of $G$ take a representation of $G$ as the group of covering transformations of a regular covering $X_\lambda \longrightarrow B_\lambda$, where $X_\lambda$, $B_\lambda$ are connected 1-dimensional locally finite $CW$-complexes, and $B_\lambda$ is compact. Using Hopf's work [4], the inclusion $G_\lambda \longrightarrow G$ induces a proper equivariant map $u_\lambda : X_\lambda \longrightarrow X$. The map $\hat{u}_\lambda : E(G_\lambda) = E(X_\lambda) \longrightarrow E(X)$ depends only on $G_\lambda$ and is functorial with respect to the homomorphisms in the limiting diagram of f.g. subgroups of $G$. This implies 2.4. □

We remark that the image of $k$ consists of all those vertical ends $\epsilon \in E(X)$ for which there exists a f.g. subgroup $G_\lambda$ of $G$ and a sequence $(h_n)$ in $G_\lambda$ so that $h_n.x \longrightarrow \epsilon$ for all $x \in X$.

PROOF OF 2.5. Suppose $\epsilon \in E(X)$ is not in $\operatorname{im} k$. Let $e : R \longrightarrow X$ be a proper map with $\hat{e}(\infty) = \epsilon$. Then $pe(R)$ can not be contained in any compact subspace $C$ of $B$. For $\pi_1 C$ is finitely generated, implying that $\epsilon \in \operatorname{im} k$; contradiction. Consequently, there is a proper strictly increasing sequence $(t_n)$ in $R$ such that $(pe(t_n))$ converges to some end $\underline{\epsilon}$ of $B$. Thus $\epsilon \in X(\underline{\epsilon})$. □

REMARK 2.6. If $\epsilon \in E(X)$ is not vertical, then $\epsilon$ is actually a lift of some end $\underline{\epsilon} \in E(B)$ in the following sense. For any proper map $e : R \longrightarrow X$ with $\hat{e}(\infty) = \epsilon$, $pe : R \longrightarrow B$ is also proper. Further, if $\epsilon \in X(\underline{\epsilon}')$, then $\underline{\epsilon} = \underline{\epsilon}'$.

Proof. If $pe$ is not proper, there is some closed cell $c$ of $B$ such that $e(R) \cap p^{-1}(c)$ is not compact. Subjecting $e$ to a suitable proper homotopy, if necessary, we see that $pe$ is an infinite sequence of based loops in $B$. These yield an infinite sequence $(g_n)$ in $G$ satisfying $g_n.\tilde{*} \longrightarrow \epsilon$, for a suitable lift $\tilde{*}$ of the base point of $B$. Thus $\epsilon$ is a vertical end; contradiction.

To see the uniqueness of $\underline{\epsilon}$, apply a similar argument to a proper map $f : L \longrightarrow X$, with $\hat{f}(\infty) = \epsilon$.

For later use, we need the following definition, based on 2.6.

Definition 2.7. (i) *$\epsilon \in E(X)$ has property (L) $:\Leftrightarrow$ there is a proper map $e : R \longrightarrow B$ with $\hat{\tilde{e}}(\infty) = \epsilon$, for some lift $\tilde{e}$ of $e$.*

(ii) *$p$ has property (L) $:\Leftrightarrow$ every horizontal end of $X$ has property (L).*

A monomorphism $a : \mathbb{Z} \longrightarrow G$ induces the map $\hat{a} : E(\mathbb{Z}) \longrightarrow E(G)$. We call an end $\epsilon \in E(G)$ primitive, if it is in the image of some map $\hat{a} : E(\mathbb{Z}) \longrightarrow E(G)$, where $a : \mathbb{Z} \longrightarrow G$ is a monomorphism. The following theorem relates in some sense the distribution of infinite cyclic subgroups of $G$ to the end space of $G$.

Theorem 2.8. *If $G$ has an element of infinite order, then the space of primitive vertical ends of $X$ is dense in the space of vertical ends of $X$.*

Corollary 2.9. *If $G$ has an element of infinite order, then*

(i) *The image of $k : E(G) \longrightarrow E(X)$ is dense in the space of vertical ends of $X$.*

(ii) *If in addition $E(G)$ is compact, then the image of $k : E(G) \longrightarrow E(X)$ is equal to the space of vertical ends of $X$.* □

Proof of 2.8. Let $\epsilon$ be a vertical end of $X$ and let $(g_n)$ be a sequence in $G$ with $g_n.x \longrightarrow \epsilon$, for all $x \in X$. For a neighbourhood $V$ of $\epsilon$ in $E(X)$, there is a compact connected set $C$ in $X$ such that

(i) all connected components $U_1, \ldots, U_k$ of $X - C$ are unbounded;

(ii) all ends of the closure $\bar{U}_1$ of $U_1$ in $X$ correspond to ends of $X$ in $V$ under the inclusion $\bar{U}_1 \longrightarrow X$.

As $C$ is compact, there is $N \in \mathbb{N}$ with $g_n.C \subset U_1$, for all $n \geqq N$. Consequently, $g_N.C \cap C = \emptyset = g_N^{-1}.C \cap C$. As $C$ is connected, $g_N^{-1}.C$ is contained in precisely one of the components $U_1, \ldots, U_k$.

Case 1. $g_N^{-1}.C \cap U_1 = \emptyset$. Then $C \cap g_N.U_1 = \emptyset$. As $U_1$ is connected, $g_N.U_1$ is contained in precisely one of the components $U_1, \ldots, U_K$. As $\partial U_1 \subset C$ and $g_N.C \subset U_1$, it follows that $g_N.U_1 \subset U_1$.

Now $g := g_N$ has infinite order. If not, $g^r = 1$ for some $r \geqq 2$. As $g^{-1} = g^{r-1}$, we get $\emptyset = g^{-1}.C \cap U_1 = (g_N^{r-1}.C) \cap U_1 \neq \emptyset$; contradiction. Thus $(g^n)$ determines a primitive vertical end of the closure of $U_1$ in $X$ and, hence, one of $V$.

Case 2. $g_N^{-1}.C \subset U_1$. By hypothesis, there is an element $t \in G$ of infinite order. Now, $t$ determines a primitive vertical end $\epsilon_+$ of $X$. If $\epsilon_+$ belongs to $V$, then we are done. Else, $\epsilon_+$ is an end of the closure of $U_2$ in $X$, say. As $g_N^{-1}.C \subset U_1$, $g_N.U_2 \cap C = \emptyset$.

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Consequently, $g_N.U_2 \subset U_1$. In particular, $\hat{g}_N(\epsilon_+) \in V$. This means that $g_N t^n.x \longrightarrow \hat{g}_N(\epsilon_+)+ : \epsilon'$, for all $x \in X$. In particular, $g_N t^n.(g_N^{-1}.x) \longrightarrow \epsilon'$, for all $x \in X$. Thus $g := g_N t g_N^{-1}$ determines a primitive vertical end of $V$. □

The following example (2.10) shows that $X$ may have vertical ends which are not in the image of $k : E(G) \longrightarrow E(X)$.

EXAMPLE 2.10. Let $G$ be the free group with basis $g_1, g_2, \ldots$. Let $B$ denote the non-negative real ray with a loop attached at every integer. Take $p : X \longrightarrow B$ to be the universal covering. Choosing $0 \in B$ as a base point, there is a representation of $g_n$ by the path in $B$ that runs from 0 to $n$, then once around the loop at $n$ (according to a chosen orientation), then back from $n$ to 0. Then the sequence $(g_n)$ determines a vertical end of $X$ which is not in $\operatorname{im} k$.

As the group $G$ of (2.10) satisfies the hypotheses of (2.9)ii, we get

REMARK 2.11. $E(G)$ is not always compact.

§3. **The structure of** $E(X)$. Given a regular covering $p : X \longrightarrow B$, as in §2, we now describe the end space $E(X)$ in terms of the end space $E(G)$ of the group of covering transformations of $X$ and end data of $B$. From 2.5, we already know that $E(X)$ is the union of the image of $k : E(G) \longrightarrow E(X)$ and the 'fibers' $X(\underline{\epsilon})$ over the ends of $B$. Accordingly, we seek to understand the behaviour of the map $k$ and the structure of the 'fibers' $X(\underline{\epsilon})$. We now sketch our approach.

Corresponding to a neighbourhood bases $U_1 \supset U_2 \supset \cdots$, see [2], of an end $\underline{\epsilon}$ of $B$, there is an inverse sequence $p^{-1}U_1 \supset p^{-1}U_2 \supset \cdots$. The path connectedness relation yields an inverse system $\mathrm{Comp}(p^{-1}U_1) \longleftarrow \mathrm{Comp}(p^{-1}U_2) \longleftarrow \cdots$ of discrete spaces, and there is an equivariant map from the resulting inverse limit $\Lambda$ to $X(\underline{\epsilon})$. We then show:

(1). If the fundamental groups of the sequence $(U_n)$ contribute to $G$ in a stable way, then $\Lambda$ is homeomorphic to the quotient of $G$ by any of certain conjugate subgroups $\{H_\mu\}$ of $G$.

If, in addition, all ends of $X(\underline{\epsilon})$ have property $(L)$, then (2) and (3) below also hold.

(2). The map $\Lambda \longrightarrow X(\underline{\epsilon})$ is onto.

(3). For $\epsilon, \epsilon' \in E(G)$, $k(\epsilon) = k(\epsilon') \Leftrightarrow$ there is an $\underline{\epsilon} \in E(B)$ with $k(\epsilon) = k(\epsilon') \in X(\underline{\epsilon}) \Leftrightarrow$ for some $\mu, \epsilon$ and $\epsilon'$ are in the image of the map $E(H_\mu) \longrightarrow E(G)$.

We also give conditions for the map $\Lambda \longrightarrow X(\underline{\epsilon})$ to be a bijection. A precise formulation of these results is given in Theorem 3.8.

LEMMA 3.1. *Suppose that $k(\epsilon) = k(\epsilon')$ for two distinct ends $\epsilon, \epsilon' \in E(G)$. Then $k(\epsilon)$ is also a horizontal end of $X$.*

PROOF. The assumption that $k(\epsilon)$ is not a horizontal end of $X$ leads to the existence of a f.g. subgroup $H$ of $G$ and an end $\delta$ of $H$ such that the induced map $E(H) \longrightarrow E(G)$ sends $\delta$ to $\epsilon$ as well as to $\epsilon'$. This is absurd. □

As the homeomorphisms by which $G$ acts on $X$ extend to homeomorphisms of the Freudenthal compactification of $X$, $G$ also acts on $E(X)$. Clearly,

LEMMA 3.2. *For every* $\underline{\epsilon} \in E(B)$, $X(\underline{\epsilon})$ *is invariant under the action of* $G$ *on* $E(X)$. $\square$

In order to exploit further the action of $G$ on $X$, we need some preparations: If $\alpha, \beta$ are composible (homotopy classes of) paths in a space, we write $\alpha\beta$ for the path running first along $\alpha$, then along $\beta$. Associated with the fundamental groupoid of a space $Y$ is the fundamental system $\sigma Y$. Its objects are the groups $\pi_1(Y, y)$, $y \in Y$. Its (iso-)morphisms ${}_\alpha\varphi : \pi_1(Y, y) \longrightarrow \pi_1(Y, y')$ are induced by homotopy classes of paths $\alpha$ joining $y$ to $y'$. Thus $\sigma Y$ is also a groupoid. The pull back $f^*\sigma Y$ of $\sigma Y$ along a map $f : W \longrightarrow Y$ has as its objects all pairs $(\pi_1(Y, f(w)), w)$, $w \in W$. The (iso-)morphisms of $f^*\sigma Y$ are all ${}_{(f\alpha)}\varphi : (\pi_1(Y, f(w)), w) \longrightarrow (\pi_1(Y, f(w')), w')$, $\alpha$ joining $w$ to $w'$. Thus $f^*\sigma Y$ is also a groupoid.

Using standard covering space theory, we see that our covering $p : X \longrightarrow B$ determines a 'representation' $\tau : p^*\sigma B \longrightarrow G$. The 'image' of $\tau$ is $G$. The 'kernel' of $\tau$ is $\sigma X$.

A subspace $S$ of $B$ yields the map $p_S : p^{-1}S \longrightarrow S$. By composition we get a representation

$$\tau_S : p_S^*\sigma S \longrightarrow p^*\sigma B \longrightarrow G.$$

If $S$ is connected, the 'image' of $\tau_S$ is a class of conjugate subgroups of $G$.

DEFINITION 3.3. *An end* $\underline{\epsilon}$ *of* $B$ *is* $G$*-stable* $:\Leftrightarrow$ *there is a neighbourhood basis* $(U_n)$ *of* $\underline{\epsilon}$ *such that the decreasing sequence* $G \supset \operatorname{im}\tau_{U_1} \supset \operatorname{im}\tau_{U_2} \supset \cdots$ *of classes of conjugate subgroups of* $G$ *becomes constant.*

Thus $\underline{\epsilon}$ is $G$-stable if and only if there is a connected open neighbourhood $U$ of $\underline{\epsilon}$ with compact frontier such that for any other such neighbourhood $U' \subset U$, $\operatorname{im}\tau_{U'} = \operatorname{im}\tau_U$. We refer to $U$ as a $G$-stable neighbourhood of $\underline{\epsilon}$ if it has this property.

We also remark that if $B$ is semistable at $\underline{\epsilon}$, then $\underline{\epsilon}$ is $G$-stable; compare [6].

Given a $G$-stable neighbourhood $U$ of $\underline{\epsilon}$, consider a connected component $V$ of the open subset $p^{-1}U$ of $X$. Since $G$ acts transitively on the set $\operatorname{Comp}(p^{-1}U)$ of connected components of $p^{-1}U$, we have

LEMMA 3.4. $\operatorname{Comp}(p^{-1}U)$ *is in bijective correspondence with* $G/G_V$, *where* $G_V$ *is the maximal subgroup of* $G$ *having* $V$ *as an invariant subspace.* $\square$

We now relate these facts to the horizontal ends of $X$. Let $\underline{\epsilon} \in E(B)$ be a $G$-stable end with $G$-stable neighbourhood $U$. We know, see [5], that $E(X) = \varprojlim \operatorname{Comp}(X - C_1) \longleftarrow \operatorname{Comp}(X - C_2) \longleftarrow \cdots$, where $C_1 \subset C_2 \subset \cdots$ is an expanding sequence of compact connected subspaces of $X$ whose union is $X$. For large $n, \partial U \subset p(C_n)$. Hence, any unbounded component of $B - p(C_n)$ is either contained in $U$ or has empty intersection with $U$. Therefore, every connected component of $p^{-1}(U - pC_n)$ is contained in exactly one component of $X - C_n$. Thus we get a commuting diagram ($V$ is a connected component of $p^{-1}U$);

3.5
$$\begin{tikzcd} G/G_V = \text{Comp } p^{-1}(U - pC_n) \arrow[d, "\lambda_n"] & \text{Comp } p^{-1}(U - pC_{n+1}) \arrow[l, "\cong"'] \arrow[d, "\lambda_{n+1}"] & \cdots \arrow[l, "\cong"'] \\ \text{Comp } (X - C_n) & \text{Comp } (X - C_{n+1}) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

inducing a map $\lambda : G/G_V \longrightarrow E(X)$. Note that $G_V$ represents the conjugacy class im $\tau_U$ in $G$.

LEMMA 3.6. (i) *The map* $\lambda$ *constructed above takes values in* $X(\underline{\epsilon})$ *and is equivariant.*
(ii) *If all ends of* $X(\underline{\epsilon})$ *have property* (*L*), *then* im $\lambda = X(\underline{\epsilon})$.

PROOF. This follows straight from the definitions. □

REMARK 3.7. If $G_V$ has finite index in $G$, then all ends of $X(\underline{\epsilon})$ have property $(L)$.

PROOF. For $k \geqq n$, let $A_k \subset \text{Comp}(X - C_k)$ be the set of connected components which are neighbourhoods of some $\epsilon \in X(\underline{\epsilon})$. Then im $\lambda_k = A_k$. If $(Y_k)$ is the neighbourhood basis in $X$ of a fixed $\epsilon \in X(\underline{\epsilon})$, corresponding to the sequence $(C_k)$, let $Z_k := \lambda_k^{-1}(Y_k)$. Then $Z_k \supset Z_{k+1} \supset \cdots$ and, since each $Z_k$ is finite and not empty, $\cap Z_k \neq \emptyset$. If $z$ belongs to that intersection, $\lambda(z) = \epsilon$, by commutativity of (3.5). This implies 3.7. □

We are now ready to state the main result of this chapter.

THEOREM 3.8. *Let* $p : X \longrightarrow B$ *be a regular covering of the kind described in the beginning of* §2. *Suppose that p has property* (*L*) *and that every end of B is G-stable. Let* $\underline{\epsilon} \in E(B)$ *with G-stable neighbourhood U and let* $\epsilon \in X(\underline{\epsilon})$. *Then*

(i) $X(\underline{\epsilon})$ *is the orbit of* $\epsilon$ *under the action of G on* $E(X)$.

(ii) *Let V be a connected component of* $p^{-1}U$ *such that* $\lambda(V) = \epsilon$. *Then* $G_V$ *is contained in the isotropy group* $G_\epsilon$ *of* $\epsilon$.

(iii) *If* $G_V$ *is finite, then* $G_V = G_\epsilon$.

(iv) *Let* $\epsilon'$ *be a vertical end of X. Then* $\epsilon = \epsilon' \Leftrightarrow$ *there is a sequence* $(h_n)$ *in* $G_V$ *such that* $h_n.x \longrightarrow \epsilon'$, *for all* $x \in X$. *Thus* $k^{-1}\{\epsilon'\} = \text{im}(E(G_V) \longrightarrow E(G))$.

PROOF. (i) and (ii) follow from (3.6).

(iii) If $G_V$ is finite, it follows that the connected components of $p^{-1}U$ have compact boundary, hence are neighbourhoods of distinct ends of $X(\underline{\epsilon})$. Therefore, $\lambda$ is a bijection. As $\lambda$ is equivariant, $G_V = G_\epsilon$.

(iv) "$\Rightarrow$" Let $e : R \longrightarrow B$ be a proper map with $\hat{e}(\infty) = \underline{\epsilon}$. As $p$ has property $(L)$, there is a lift $\tilde{e} : R \longrightarrow X$ with $\tilde{e}(t) \in V$, for large $t$. Thus $\hat{\tilde{e}}(\infty) = \epsilon$. As $\epsilon = \epsilon'$ is vertical, there is a sequence $(g_n)$ in $G$ with $g_n.x \longrightarrow \epsilon$, for all $x \in X$. Hence, there is a proper map $f : L \longrightarrow X$ ($L$ is the infinite ladder $[0, \infty[\times\{0, 1\} \cup \mathbb{N}_0 \times I$) with $f_{|[0,\infty[\times\{0\}} = e$ and $f(n, 1) = g_n.e(0)$, for all $n$. For technical convenience, we may assume that $e(0) \notin U$ and that $e$ and $f$ take values in the 1-skeleta of $B$, respectively $X$. (If $X, B$ are $CW$-complexes, it is at this point where we need the "finite dimension" hypothesis.)

It follows that for $n$ sufficiently large, $pf\{n\} \times I \cap \partial U \neq \emptyset$. $\partial U$ is compact and can be assumed to be a finite subcomplex of $B$. Thus there exists a vertex $v \in \partial U$, an infinite strictly increasing sequence $n_i \in \mathbb{N}$ and a corresponding sequence $t_i \in I$, such that $pf(n_i, t_i) = v$. The sequence $f(n_i, t_i)$ converges to $\epsilon$. Further, $pf$ yields loops $l_i$ in $U$, based at $v$, corresponding to a sequence $(h_i)$ in $\pi_1(U, v)$ with $h_i.f(n_0, t_0) = f(n_i, t_i)$.

"$\Leftarrow$" Let $(h_n)$ be a sequence in $G_V$ with $h_n.x \longrightarrow \epsilon'$, for all $x \in X$. Let $e : [0, \infty[ \longrightarrow V$ be a proper map, with $pe$ proper and $p\hat{e}(\infty) = \underline{\epsilon}$. To show that $\epsilon = \epsilon'$, we construct a proper map $f : L \longrightarrow X$ with $f(n, 1) = h_n.f(0, 1)$ and $f_{|[0,\infty[\times\{0\}} = e$.

Let $U =: U_0 \supset U_1 \supset U_2 \supset \cdots$ be a neighbourhood basis of $\underline{\epsilon}$. Then $\bar{V} =: \bar{V}_0 \supset \bar{V}_1 \supset \bar{V}_2 \supset \cdots$ is a decreasing sequence of closed subsets of $X$, with empty intersection; where $\bar{V}_k := p^{-1}\bar{U}_k \cap \bar{V}$. These induce the sequence of maps of end spaces $E(X) \longleftarrow E(\bar{V}) \longleftarrow E(\bar{V}_1) \longleftarrow \cdots$. There exists a proper increasing sequence $(t_k)$ in $[0, \infty[$ such that $e(t_k) \in p^{-1}(U_k) \cap V$. For every $k$, there is a subsequence $(h_n^k)$ of $(h_n)$ such that $h_n^k.e(t_k)$ converges to some end $\epsilon'_k \in E(\bar{V}_k)$, for $n \longrightarrow \infty$. Hence, there exist proper maps $e_k : [0, \infty[\times\{k\} \longrightarrow \bar{V}_k$ with $e_k(n) = h_n^k.e(t_k)$. Since $\bar{V}_0 \supset \bar{V}_1 \supset \cdots$ induces $\epsilon' \longleftarrow\!\shortmid \epsilon'_1 \longleftarrow\!\shortmid \epsilon'_2 \longleftarrow\!\shortmid \cdots$, there exist proper maps $f_k : [0, \infty[\times\{t_k, t_{k+1}\} \cup \mathbb{N}_0 \times [t_k, t_{k+1}] \longrightarrow \bar{V}_k$, extending the $e_k$'s, with $f_{k|\{0\}\times[t_k,t_{k+1}]} = e_{|[t_k,t_{k+1}]}$.

The maps $f_k$ combine to a proper map $F : [0, \infty[\times\{t_k : k \in \mathbb{N}_0\} \cup \mathbb{N}_0 \times [0, \infty[ \longrightarrow X$. The desired map $f$ can now be derived from $F$. □

§4. **Applications, Examples.** We give some applications of the general results of Section 3.

Example 4.1. Consider the covering $p : X := \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R} =: B$. Then $X$ has exactly one end and this end is both vertical and horizontal. We wish to see how Theorem 3.8 explains this from the covering space point of view.

Clearly, $G = \mathbb{Z}$. Now, $B$ has two ends $\pm\underline{\epsilon}$ corresponding to the ends $\pm\infty$ of $\mathbb{R}$. So $E(X) = \mathrm{im}(k : E(\mathbb{Z}) \longrightarrow E(X)) \cup X(+\underline{\epsilon}) \cup X(-\underline{\epsilon})$. Both ends $\pm\underline{\epsilon}$ are $G$-stable and the stable groups are equal to $G$. By 3.7, $p$ has property $(L)$. By 3.8.i, ii, $X(+\underline{\epsilon})$ and $X(-\underline{\epsilon})$ consist of one element each. By 3.8.iv, $X(-\underline{\epsilon}) = k(E(G)) = X(+\underline{\epsilon})$.

Sometimes the end structure of $X$ is understood and allows to infer the end structure of $B$, as in the following example.

Example 4.2. Let $Y$ be a connected compact $CW$-complex. Let $\zeta = (B, \pi, Y)$ be a real line bundle over $Y$. Then

$$\begin{array}{lcl} B \text{ has two ends} & \Leftrightarrow & w_1(\zeta) = 0 \quad \text{(1st Stiefel Whitney class)} \\ B \text{ has one end} & \Leftrightarrow & w_1(\zeta) = 1. \end{array}$$

Proof. If $w_1(\zeta) = 0$, then $B = Y \times \mathbb{R}$ has two ends. Since $w_1(\zeta) \in \mathbb{Z}/2$, it now suffices to show "$w_1(\zeta) = 1$ implies that $B$ has one end." This follows by applying 3.8

to the connected double cover $p : X := Y' \times \mathbb{R} \longrightarrow B$, where $Y' \longrightarrow Y$ is the connected principal $\mathbb{Z}/2$-bundle associated with $\zeta$. □

(4.2) can be useful when dealing with the normal bundle of a closed connected submanifold $C$ of codimension 1 in a connected $C^\infty$-manifold $M$.

PROPOSITION 4.3. (i) *$M - C$ has at most two connected components.*

(ii) *If $M - C$ has two connected components then $\nu$ is trivial.*

PROOF. (i) Let $D$ denote a normal unit disc bundle of $C$ in $M$ with respect to some Riemannian metric on $\nu$. Since the inclusion $(M - D) \longrightarrow (M - C)$ is a deformation retract, we get a bijection Comp$(M - D) \longrightarrow$ Comp$(M - C)$. Further, the inclusion $(D - C) \longrightarrow (M - C)$ induces an onto map Comp$(D - C) \longrightarrow$ Comp$(M - C)$. By 4.2, Comp$(D - C)$ is in bijective correspondence with $E(\nu)$, which has at most two elements.

(ii) The proof of (i) shows that if $(M - C)$ has two components, then $\nu$ has two ends. By 4.2, $\nu$ is trivial. □

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