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ON GENERALIZED COMPLETE ELLIPTIC INTEGRALS AND MODULAR FUNCTIONS

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Abstract This paper deals with generalized elliptic integrals and generalized modular functions. Several new inequalities are given for these and related functions.

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1. Introduction

Since the publication of the landmark paper [**16**], numerous papers have been written about generalized elliptic integrals, modular functions and their inequalities (see, for example, [**2**, **3**, **10**–**15**, **19**, **20**, **25**, **29**–**32**]). Modular equations have a long history, which goes back to the works of Legendre, Gauss, Jacobi and Ramanujan on number theory. Modular equations also occur in geometric function theory, as shown in [**3**, **21**–**23**, **28**] and in numerical computations of moduli of quadrilaterals [**18**]. For recent surveys of this topic from the point of view of geometric function theory, see [**6**, **7**, **9**, **28**]. The study of these functions is motivated by potential applications to geometric function theory and to number theory. Special functions have an important role in geometric function theory [**4**, **5**, **21**, **22**, **27**].

Given complex numbers a, b and c with $c \neq 0, -1, -2, \dots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \quad |z| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{Z}_+$.

For later use we define the classical *gamma function* $\Gamma(x)$ and *beta function* $B(x,y)$. For $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$, these functions are defined by

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1}\,\mathrm{d}t, \qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively. By [**1**, 6.1.8] we see that $B(\frac{1}{2},\frac{1}{2}) = \pi$.

For the formulation of our main results and for later use we introduce some basic notation. The decreasing homeomorphism $\mu_a\colon (0,1) \to (0,\infty)$ is defined by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{F(a,1-a;1;r'^2)}{F(a,1-a;1;r^2)} = \frac{\pi}{2\sin(\pi a)} \frac{\mathcal{K}_a(r')}{\mathcal{K}_a(r)}$$

for $r \in (0,1)$ and $r' = \sqrt{1-r^2}$. A generalized modular equation with signature $1/a$ and order (or degree) p is

$$\mu_a(s) = p\mu_a(r), \quad 0 < r < 1. \tag{1.1}$$

We define

$$s = \varphi_K^a(r) \equiv \mu_a^{-1}\left(\frac{\mu_a(r)}{K}\right), \quad K \in (0,\infty), \quad p = \frac{1}{K}, \tag{1.2}$$

which is the solution of (1.1).

For $a \in (0,\frac{1}{2}]$, $K \in (0,\infty)$, $r \in (0,1)$, we have, by [**3**, Lemma 6.1] and [**8**, Theorem 10.5],

$$\varphi_K^a(r)^2 + \varphi_{1/K}^a(r')^2 = 1. \tag{1.3}$$

For $a \in (0,\frac{1}{2}]$, $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, the generalized elliptic integrals are defined by

$$\begin{aligned}
\mathcal{K}_a(r) &= \tfrac{1}{2}\pi F(a,1-a;1;r^2), & \mathcal{E}_a(r) &= \tfrac{1}{2}\pi F(a-1,1-a;1;r^2),\\
\mathcal{K}'_a(r) &= \mathcal{K}_a(r'), & \mathcal{E}'_a(r) &= \mathcal{E}_a(r'),\\
\mathcal{K}_a(0) &= \tfrac{1}{2}\pi, & \mathcal{E}_a(0) &= \tfrac{1}{2}\pi,\\
\mathcal{K}_a(1) &= \infty, & \mathcal{E}_a(1) &= \frac{\sin(\pi a)}{2(1-a)}.
\end{aligned}$$

In this paper we study the modular function $\varphi_K^a(r)$ for general $a \in (0,\frac{1}{2}]$, as well as related functions μ_a, $\mathcal{K}_a$, η_K^a, λ_a and their dependency on r and K, where

$$\eta_K^a(x) = \left(\frac{s}{s'}\right)^2, \quad s = \varphi_K^a(r), \quad r = \sqrt{\frac{x}{1+x}} \quad \text{for } x, K \in (0,\infty),$$

and

$$\lambda_a(K) = \left(\frac{\varphi_K^a(\frac{1}{\sqrt{2}})}{\varphi_{1/K}^a(\frac{1}{\sqrt{2}})}\right)^2 = \left(\frac{\mu_a^{-1}(\pi/(2K\sin(\pi a)))}{\mu_a^{-1}(\pi K/(2\sin(\pi a)))}\right)^2 = \eta_K^a(1). \tag{1.4}$$

Motivated by [**17**, **24**], we define, for $p > 1$ and $r \in (0,1)$,

$$\operatorname{artanh}_p(x) = \int_0^x (1-t^p)^{-1}\,\mathrm{d}t = xF\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right).$$

Then $\operatorname{artanh}_2(x)$ is the usual inverse hyperbolic tangent (artanh) function.

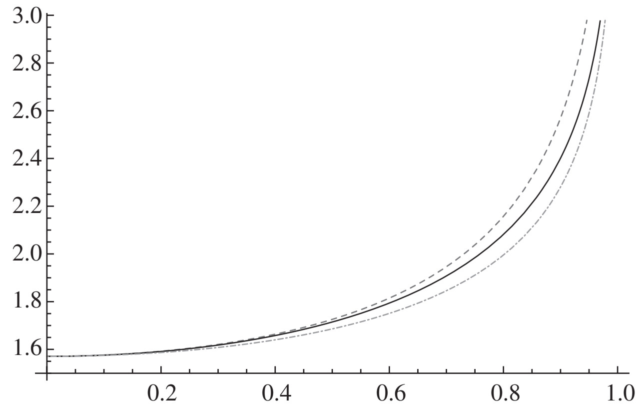

Figure 1. Comparison of upper bounds given in Theorem 1.2 (black line) and in (1.5) (dark grey dashed line) for $\mathcal{K}(r)$ (light grey dot-dashed line).

We give some of the main results of this paper next.

Theorem 1.1. *For $a,b,c > 0$ and $r \in (0,1)$, the function $g(p) = F(a,b;c;r^p)^{1/p}$ is decreasing in $p \in (0,\infty)$. In particular, for $p \geqslant 1$,*

(i) $F(a,b;c;r^p)^{1/p} \leqslant F(a,b;c;r) \leqslant F(a,b;c;r^{1/p})^p$,

(ii) $(\frac{1}{2}\pi)^{1-1/p}\mathcal{K}_a(r^p)^{1/p} \leqslant \mathcal{K}_a(r) \leqslant (\frac{1}{2}\pi)^{1-p}\mathcal{K}_a(r^{1/p})^p$,

(iii) $(\frac{1}{2}\pi)^{1-p}\mathcal{E}_a(r^{1/p})^p \leqslant \mathcal{E}_a(r) \leqslant (\frac{1}{2}\pi)^{1-1/p}\mathcal{E}_a(r^p)^{1/p}$.

Alzer and Qiu gave the following bounds for $\mathcal{K} = \mathcal{K}_{1/2}$ in [**2**, Theorem 18]:

$$\frac{\pi}{2}\left(\frac{\operatorname{artanh}(r)}{r}\right)^{3/4} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{artanh}(r)}{r}\right). \tag{1.5}$$

In the following theorem we generalize their result to the case of $\mathcal{K}_a$, and for the particular case $a = \frac{1}{2}$ our upper bound is better than their bound in (1.5). For a graphical comparison of the bounds see Figure 1.

Theorem 1.2. *For $p \geqslant 2$ and $r \in (0,1)$, we have*

$$\begin{aligned}\frac{\pi}{2}\left(\frac{\operatorname{artanh}_p(r)}{r}\right)^{1/2} &< \frac{\pi}{2}\left(1 - \frac{p-1}{p^2}\log(1-r^2)\right)\\ &< \mathcal{K}_a(r)\\ &< \frac{\pi}{2}\left(1 - \frac{2}{p\pi_p}\log(1-r^2)\right),\end{aligned}$$

where $a = 1/p$ and $\pi_p = 2\pi/(p\sin(\pi/p))$.

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In [**3**, Theorem 5.6] (see also [**10**, Theorem 1.5, 1.8]) it was proved that for $a \in (0, \frac{1}{2}]$ we have

$$\mu_a\left(\frac{rs}{1+r's'}\right) \leqslant \mu_a(r) + \mu_a(s) \leqslant 2\mu_a\left(\frac{\sqrt{2rs}}{\sqrt{1+rs+r's'}}\right)$$

for all $r, s \in (0,1)$. This inequality will be generalized in Theorem 4.3. In the next theorem we give a similar result for the function $\mathcal{K}_a$.

Theorem 1.3. *The function $f(x) = 1/\mathcal{K}_a(1/\cosh(x))$ is increasing and concave from $(0,\infty)$ onto $(0, 2/\pi)$. In particular,*

$$\frac{\mathcal{K}_a(r)\mathcal{K}_a(s)}{\mathcal{K}_a(rs/(1+r's'))} \leqslant \mathcal{K}_a(r) + \mathcal{K}_a(s) \leqslant \frac{2\mathcal{K}_a(r)\mathcal{K}_a(s)}{\mathcal{K}_a(\sqrt{rs/(1+rs+r's')})} \leqslant \frac{2\mathcal{K}_a(r)\mathcal{K}_a(s)}{\mathcal{K}_a(rs)},$$

for all $r, s \in (0,1)$, with equality in the third inequality if and only if $r = s$.

There are several bounds for the function $\mu_a(r)$ when $a = \frac{1}{2}$ in [**8**, Chapter 5]. In the next theorem we give a two-sided bound for $\mu_a(r)$.

Theorem 1.4. *For $p \geqslant 2$ and $r \in (0,1)$, let*

$$l_p(r) = \left(\frac{\pi_p}{2}\right)^2 \left(\frac{p^2 - (p-1)\log r^2}{p\pi_p - 2\log r'^2}\right) \quad \text{and} \quad u_p(r) = \left(\frac{p}{2}\right)^2 \left(\frac{p\pi_p - 2\log r^2}{p^2 - (p-1)\log r'^2}\right).$$

(i) *The following inequalities hold:*

$$l_p(r) < \mu_a(r) < u_p(r),$$

where $a = 1/p$.

(ii) *For $p = 2$ we have*

$$u_2(r) < \frac{4}{\pi} l_2(r).$$

2. Proofs of Theorems 1.1–1.4

For easy reference, we record the next two lemmas from [**8**], which have found many applications. Some of the applications are reviewed in [**7**]. The first result is sometimes called the *monotone l'Hôpital rule*.

Lemma 2.1 (Anderson *et al.* [8, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (Anderson *et al.* [8, Lemma 1.24]). *For $p \in (0,\infty]$, let $I = [0,p)$ and suppose that $f, g : I \to [0,\infty)$ are functions such that $f(x)/g(x)$ is decreasing on $I \setminus \{0\}$ and $g(0) = 0$ and $g(x) > 0$ for $x > 0$. Then*

$$f(x+y)(g(x)+g(y)) \leqslant g(x+y)(f(x)+f(y))$$

for $x, y, x+y \in I$. Moreover, if the monotonicity of $f(x)/g(x)$ is strict, then the above inequality is also strict on $I \setminus \{0\}$.

For easy reference we recall the following lemmas from [**3**].

Lemma 2.3. *For $a \in (0,\frac{1}{2}]$, $K \in (1,\infty)$, $r \in (0,1)$ and $s = \varphi_K^a(r)$, we have the following.*

(i) *$f(r) = s'\mathcal{K}_a(s)^2/(r'\mathcal{K}_a(r)^2)$ is decreasing from $(0,1)$ onto $(0,1)$.*

(ii) *$g(r) = s\mathcal{K}'_a(s)^2/(r\mathcal{K}'_a(r)^2)$ is decreasing from $(0,1)$ onto $(1,\infty)$.*

(iii) *The function $r'^c\mathcal{K}_a(r)$ is decreasing if and only if $c \geqslant 2a(1-a)$, in which case $r'^c\mathcal{K}_a(r)$ is decreasing from $(0,1)$ onto $(0,\frac{1}{2}\pi)$. Moreover, $\sqrt{r'}\mathcal{K}_a(r)$ is decreasing for all $a \in (0,\frac{1}{2}]$.*

Lemma 2.4. *The following formulae hold for $a \in (0,\frac{1}{2}]$, $r \in (0,1)$ and $x, y, K \in (0,\infty)$:*

$$\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{lm}{n}F(1+l,1+m;1+n;r), \quad F = F(l,m;n;r), \tag{2.1}$$

$$\frac{\mathrm{d}\mathcal{K}_a(r)}{\mathrm{d}r} = \frac{2(1-a)(\mathcal{E}_a(r) - r'^2\mathcal{K}_a(r))}{rr'^2}, \tag{2.2}$$

$$\frac{\mathrm{d}\mathcal{E}_a(r)}{\mathrm{d}r} = \frac{2(a-1)(\mathcal{K}_a(r) - \mathcal{E}_a(r))}{r}, \tag{2.3}$$

$$\frac{\mathrm{d}\mu_a(r)}{\mathrm{d}r} = \frac{-\pi^2}{4rr'^2\mathcal{K}_a(r)^2}, \tag{2.4}$$

$$\frac{\mathrm{d}\varphi_K^a(r)}{\mathrm{d}r} = \frac{ss'^2\mathcal{K}_a(s)^2}{Krr'^2\mathcal{K}_a(r)^2} = \frac{ss'^2\mathcal{K}_a(s)\mathcal{K}'_a(s)}{rr'^2\mathcal{K}_a(r)\mathcal{K}'_a(r)} = K\frac{ss'^2\mathcal{K}'_a(s)^2}{rr'^2\mathcal{K}'_a(r)^2}, \tag{2.5}$$

$$\frac{\mathrm{d}\varphi_K^a(r)}{\mathrm{d}K} = \frac{4ss'^2\mathcal{K}_a(s)^2\mu_a(r)}{\pi^2K^2}, \quad \text{where } s = \varphi_K^a(r), \tag{2.6}$$

$$\frac{\mathrm{d}\eta_K^a(x)}{\mathrm{d}x} = \frac{1}{K}\left(\frac{r's\mathcal{K}_a(s)}{rs'\mathcal{K}_a(r)}\right)^2 = K\left(\frac{r's\mathcal{K}'_a(s)}{rs'\mathcal{K}'_a(r)}\right)^2 = \left(\frac{r's}{rs'}\right)^2\frac{\mathcal{K}_a(s)\mathcal{K}'_a(s)}{\mathcal{K}_a(r)\mathcal{K}'_a(r)}, \tag{2.7}$$

$$\frac{\mathrm{d}\eta_K^a(x)}{\mathrm{d}K} = \frac{8\eta_K^a(x)\mu_a(r)K_a(s)^2}{\pi^2K^2}. \tag{2.8}$$

In (2.7), (2.8), $r = \sqrt{x/(1+x)}$ and $s = \varphi_K^a(r)$.

Lemma 2.5 (Anderson *et al.* [8, Theorem 1.52(1)]). *For $a, b > 0$, the function*

$$f(x) = \frac{F(a,b;a+b;x) - 1}{\log(1/(1-x))}$$

is strictly increasing from $(0,1)$ onto $(ab/(a+b), 1/B(a,b))$.

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Proof of Theorem 1.1. With $G(r) = F(a, b; c; r^p)$ and g as in Theorem 1.1 we get, by (2.1),

$$g'(p) = -\frac{(G(r))^{1/p-1}}{cp^2}\left(cG(r)\log(G(r)) + abpr^pF(a+1, b+1; c+1; r^p)\log\left(\frac{1}{r}\right)\right),$$

which is negative. Hence, this implies (i), and (ii) follows from (i). For (iii), write $F(r) = F(-a, b; c; r^p)$. We define $h(p) = F(r)^{1/p}$ and obtain

$$h'(p) = \frac{(F(r))^{1/p-1}}{cp^2}\left(cF(r)\log(1/F(r)) + abpr^pF(a+1, b+1; c+1; r^p)\log\left(\frac{1}{r}\right)\right),$$

which is positive because $F(r) \in (0, 1)$. Hence, h is increasing in p, and (iii) follows easily. □

Proof of Theorem 1.2. By the definition of artanh_p, Lemma 2.5 and the Bernoulli inequality, we obtain

$$\begin{aligned}
\left(\frac{\operatorname{artanh}_p(r)}{r}\right)^{1/2} &= \left(F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; r^p\right)\right)^{1/2} \\
&< \left(1 - \frac{1}{p}\log(1 - r^p)\right)^{1/2} \\
&\leqslant 1 + \frac{1}{2p}\log\left(\frac{1}{1 - r^p}\right) \\
&\leqslant 1 + \frac{p-1}{p^2}\log\left(\frac{1}{1 - r^p}\right) \\
&\leqslant 1 - \frac{p-1}{p^2}\log(1 - r^2) \\
&= \xi.
\end{aligned}$$

Again, by Lemma 2.5 and [**1**, 6.1.17] we obtain

$$\begin{aligned}
\xi &< F\left(\frac{1}{p}, 1 - \frac{1}{p}; 1; r^2\right) \\
&= \frac{2}{\pi}\mathcal{K}_{1/p}(r) \\
&< 1 - \frac{1}{B(1/p, 1 - 1/p)}\log(1 - r^2) \\
&= 1 - \frac{2}{p\pi_p}\log(1 - r^2),
\end{aligned}$$

and this completes the proof. □

Proof of Theorem 1.3. Setting $r = 1/\cosh(x)$, we have

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -\frac{\sinh x}{\cosh^2 x} = -rr'$$

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and

$$f'(x) = -\frac{\mathcal{K}_a'(r)}{\mathcal{K}_a^2(r)}\frac{\mathrm{d}r}{\mathrm{d}x}$$
$$= -\frac{2(1-a)}{\mathcal{K}_a^2(r)}\frac{\mathcal{E}_a(r) - r'^2\mathcal{K}_a(r)}{rr'^2}(-rr')$$
$$= 2(1-a)\frac{\mathcal{E}_a(r) - r'^2\mathcal{K}_a(r)}{r'\mathcal{K}_a(r)^2},$$

which is positive and increasing in r by Lemma 2.3 (iii) and therefore $f'(x)$ is decreasing in x and f is concave. Hence,

$$\tfrac{1}{2}(f(x)+f(y)) \leqslant f\left(\frac{x+y}{2}\right)$$
$$\iff \frac{1}{2}\left(\frac{1}{\mathcal{K}_a(1/\cosh(x))} + \frac{1}{\mathcal{K}_a(1/\cosh(y))}\right) \leqslant \frac{1}{\mathcal{K}_a(1/\cosh(\frac{1}{2}(x+y)))}$$
$$\iff \mathcal{K}_a(r) + \mathcal{K}_a(s) \leqslant \frac{2\mathcal{K}_a(r)\mathcal{K}_a(s)}{\mathcal{K}(\sqrt{rs/(1+rs+r's')})},$$

using $\cosh^2(\frac{1}{2}(x+y)) = (1+rs+r's')/rs$ and setting $s = 1/\cosh(y)$. Clearly,

$$(r-s)^2 \geqslant 0 \iff 1 - 2rs + r^2s^2 \geqslant 1 - r^2 - s^2 + r^2s^2$$
$$\iff 1 - rs \geqslant r's'$$
$$\iff 2 \geqslant 1 + rs + r's'$$
$$\iff \frac{2rs}{1+rs+r's'} \geqslant rs,$$

and the third inequality follows. Obviously, $f(0+) = 0$, and $f'(x)$ is decreasing in x. Then $f(x)/x$ is decreasing and $f(x+y) \leqslant f(x) + f(y)$ by Lemmas 2.1 and 2.2, respectively. This implies the first inequality. □

Proof of Theorem 1.4. By Lemma 2.5 we obtain

(a) $1 - \dfrac{p-1}{p^2}\log r^2 < F\left(\dfrac{1}{p}, 1-\dfrac{1}{p}; 1; 1-r^2\right) < 1 - \dfrac{2}{p\pi_p}\log r^2,$

(b) $1 - \dfrac{p-1}{p^2}\log(1-r^2) < F\left(\dfrac{1}{p}, 1-\dfrac{1}{p}; 1; r^2\right) < 1 - \dfrac{2}{p\pi_p}\log(1-r^2).$

By using (a), (b) and the definition of μ_a, we get (i). The claim (ii) is equivalent to

$$\frac{2(\pi - \log(r^2))}{4 - \log(1-r^2)} < \frac{4}{\pi}\left(\frac{\pi}{2}\right)^2\frac{4-\log(r^2)}{\pi - \log(1-r^2)}$$
$$\iff 4(\pi - \log(r^2))(\pi - \log(1-r^2)) - (4 - \log(r^2))(4 - \log(1-r^2)) < 0$$
$$\iff (\pi - 4)(4\pi - \log(r^2)\log(1-r^2)) < (\pi-4)(4\pi - (\log(2))^2) < 0.$$

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For the penultimate inequality we define $w(x) = \log(x)\log(1-x)$ and we get

$$w'(x) = \frac{(1-x)\log(1-x) - x\log(x)}{x(1-x)} = \frac{-g(x)}{x(1-x)}.$$

We also see that $g(x) = x\log(x) - (1-x)\log(1-x)$ is convex on $(0, \frac{1}{2})$ and concave on $(\frac{1}{2}, 1)$. This implies that $g(x) < 0$ for $x \in (0, \frac{1}{2})$ and $g(x) > 0$ for $x \in (\frac{1}{2}, 1)$. Therefore, w is increasing in $(0, \frac{1}{2})$ and decreasing in $(\frac{1}{2}, 1)$. Hence, the function w has a global maximum at $x = \frac{1}{2}$ and this completes the proof. □

One can obtain the following inequalities by using the proof of Theorem 1.4:

$$\frac{p\pi_p}{2\pi}\frac{\mathcal{K}_a(r)}{(1-(2/(p\pi_p))\log r^2)} \leqslant \mu_a(r') \leqslant \frac{p\pi_p}{2\pi}\frac{\mathcal{K}_a(r)}{(1-((p-1)/p)\log r^2)},$$

with $a = 1/p$ and $p \geqslant 2$.

Lemma 2.6. *The following inequalities hold for all* $r, s \in (0, 1)$ *and* $a \in (0, \frac{1}{2}]$*:*

(i) $\mathcal{K}_a(rs) \leqslant \sqrt{\mathcal{K}_a(r^2)\mathcal{K}_a(s^2)} \leqslant \frac{2}{\pi}\mathcal{K}_a(r)\mathcal{K}_a(s)$,

(ii) $\dfrac{2}{\pi}\mathcal{E}_a(r)\mathcal{E}_a(s) \leqslant \sqrt{\mathcal{E}_a(r^2)\mathcal{E}_a(s^2)} \leqslant \mathcal{E}_a(rs)$.

Proof. Define $f(x) = \log(\mathcal{K}_a(\mathrm{e}^{-x}))$, $x > 0$. We get, by (2.2),

$$f'(x) = -2(1-a)\frac{\mathcal{E}_a(r) - r'^2\mathcal{K}_a(r)}{r'^2\mathcal{K}_a(r)}, \quad r = \mathrm{e}^{-x},$$

and this is negative by the fact that $h(r) = \mathcal{E}_a(r) - r'^2\mathcal{K}_a(r) > 0$ and decreasing in r by [**3**, Lemma 5.4 (1)] and the fact that h is increasing ($h'(r) = 2ar\mathcal{K}_a(r) > 0$). Therefore, $f'(x)$ is increasing in x; hence, f is convex, and this implies the first inequality of part (i). The second inequality follows from Theorem 1.1 (ii).

The first inequality of (ii) follows from Theorem 1.1 (iii); for the second inequality we define $g(x) = \log(\mathcal{E}_a(z))$, $z = \mathrm{e}^{-x}$, $x > 0$, and get, by (2.3),

$$g'(x) = 2(1-a)\frac{\mathcal{K}_a(z) - \mathcal{E}_a(z)}{\mathcal{E}_a(z)},$$

which is positive and increasing in z by [**3**, Theorem 4.1 (3), Lemma 5.2 (3)]; hence, $g'(x)$ is decreasing in x, and therefore g is increasing and concave. This implies that

$$\log(\mathcal{E}_a(\mathrm{e}^{-(x+y)/2})) \geqslant \tfrac{1}{2}(\log(\mathcal{E}_a(\mathrm{e}^{-x})) + \log(\mathcal{E}_a(\mathrm{e}^{-y}))),$$

and the second inequality follows if we set $r = \mathrm{e}^{-x/2}$ and $s = \mathrm{e}^{-y/2}$. □

3. A few remarks on special functions

In this section we generalize some results from [**8**, Chapter 10].

Theorem 3.1. *The function $\mu_a^{-1}(y)$ has exactly one inflection point and it is log-concave from $(0,\infty)$ onto $(0,1)$. In particular,*

$$(\mu_a^{-1}(x))^p(\mu_a^{-1}(y))^q \leqslant \mu_a^{-1}(px+qy)$$

for $p,q,x,y>0$ with $p+q=1$.

Proof. Letting $s=\mu_a^{-1}(y)$ we see that $\mu_a(s)=y$. By (2.4), we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}y} = -\frac{4}{\pi^2}ss'^2\mathcal{K}_a(s)^2,$$

and

$$\begin{aligned}\frac{\mathrm{d}^2s}{\mathrm{d}y^2} &= -\frac{\mathrm{d}s}{\mathrm{d}y}\frac{4}{\pi^2}(s'^2\mathcal{K}_a(s)^2 - 2s^2\mathcal{K}_a(s)^2 + 2\mathcal{K}_a(s)^2(\mathcal{E}_a(s) - s'^2\mathcal{K}_a(s))) \\ &= \frac{16}{\pi^4}ss'^2\mathcal{K}_a(s)^3(2\mathcal{E}_a(s)-(1+s^2)\mathcal{K}_a(s)).\end{aligned}$$

We see that $2\mathcal{E}_a(s)-(1+s^2)\mathcal{K}_a(s)$ is increasing from $(0,\infty)$ onto $(-\infty,\frac{1}{2}\pi)$ as a function of y. Hence, $\mathrm{d}(\mu_a^{-1}(y_0))/\mathrm{d}y^2=0$ for $y_0\in(0,\infty)$, and μ_a^{-1} has exactly one inflection point. Let $f(y)=\log(\mu_a^{-1}(y))=\log s$. Then

$$f'(y) = -\frac{4}{\pi^2}s'^2\mathcal{K}_a(s)^2,$$

which is decreasing as a function of y, by Lemma 2.3 (iii); hence μ_a^{-1} is log-concave. This completes the proof. □

Corollary 3.2.

(i) *For $K\geqslant 1$, the function $f(r)=(\log\varphi_K^a(r))/\log r$ is strictly decreasing from $(0,1)$ onto $(0,1/K)$.*

(ii) *For $K\geqslant 1, r\in(0,1)$, the function $g(p)=\varphi_K^a(r^p)^{1/p}$ is decreasing from $(0,\infty)$ onto $(r^{1/K},1)$. In particular,*

$$r^{p/K}\leqslant\varphi_K^a(r^p)\leqslant\varphi_K^a(r)^p, \quad p\geqslant 1,$$

and

$$\varphi_K^a(r^p)\geqslant\varphi_K^a(r)^p, \quad 0<p\leqslant 1.$$

Proof. Let $s=\varphi_K^a(r)$. By (2.5) we get

$$f'(r) = \frac{rss'^2}{srr'^2}\frac{\mathcal{K}_a(s)\mathcal{K}_a'(s)}{\mathcal{K}_a(r)\mathcal{K}_a'(r)}\log r - \log s,$$

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and this is equivalent to

$$r(\log r)^2 f'(r) = s'^2\mathcal{K}_a(s)\mathcal{K}'_a(s)\left(\frac{\log r}{r'^2\mathcal{K}_a(r)\mathcal{K}'_a(r)} - \frac{\log s}{s'^2\mathcal{K}_a(s)\mathcal{K}'_a(s)}\right),$$

which is negative by Lemma 2.3 (iii). The limiting values follow from l'Hôpital's rule and Lemma 2.3 (i). We observe that

$$\log g(p) = \left(\frac{\log \varphi_K^a(r^p)}{\log(r^p)}\right)\log r,$$

and (ii) follows from (i). □

Lemma 3.3. *For $0 < a \leqslant \frac{1}{2}$, $K, p \geqslant 1$ and $r, s \in (0,1)$, the following inequalities hold:*

$$\frac{\sqrt[p]{\varphi_K^a(r^p)} + \sqrt[p]{\varphi_K^a(s^p)}}{1 + \sqrt[p]{\varphi_K^a(r^p)\varphi_K^a(s^p)}} \leqslant \frac{\varphi_K^a(r) + \varphi_K^a(s)}{1 + \varphi_K^a(r)\varphi_K^a(s)} \leqslant \frac{\varphi_K^a(\sqrt[p]{r})^p + \varphi_K^a(\sqrt[p]{s})^p}{1 + (\varphi_K^a(\sqrt[p]{r})\varphi_K^a(\sqrt[p]{s}))^p}.$$

Proof. It follows from Corollary 3.2 (ii) that

$$\varphi_K^a(r^p)^{1/p} \leqslant \varphi_K^a(r).$$

From the fact that artanh is increasing, we conclude that

$$\operatorname{artanh}(\varphi_K^a(r^p)^{1/p}) + \operatorname{artanh}(\varphi_K^a(s^p)^{1/p}) \leqslant \operatorname{artanh}(\varphi_K^a(r)) + \operatorname{artanh}(\varphi_K^a(s)).$$

This is equivalent to

$$\operatorname{artanh}\left(\frac{\varphi_K^a(r^p)^{1/p} + \varphi_K^a(s^p)^{1/p}}{1 + (\varphi_K^a(r^p) + \varphi_K^a(s^p))^{1/p}}\right) \leqslant \operatorname{artanh}\left(\frac{\varphi_K^a(r) + \varphi_K^a(s)}{1 + (\varphi_K^a(r) + \varphi_K^a(s))}\right),$$

and the first inequality holds. Similarly, the second inequality follows from $\varphi_K^a(r) \leqslant \varphi_K^a(r^{1/p})^p$. □

For $0 < a \leqslant 1$, $K \geqslant 1$ and $r, s \in (0,1)$, the inequality

$$\varphi_K^a\left(\frac{r+s}{1+rs}\right) \leqslant \frac{\varphi_K^a(r) + \varphi_K^a(s)}{1 + \varphi_K^a(r)\varphi_K^a(s)} \tag{3.1}$$

is given in [**3**, Remark 6.17]. For a graphical comparison of (3.1) and the first inequality of Lemma 3.3, see Figure 2.

Theorem 3.4. *For $r, s \in (0,1)$, we have*

$$|\varphi_K^a(r) - \varphi_K^a(s)| \leqslant \varphi_K^a(|r-s|) \leqslant \mathrm{e}^{(1-1/K)R(a)/2}|r-s|^{1/K}, \quad K \geqslant 1. \tag{3.2}$$

*Here $R(a)$ is as in [**3**, Theorem 6.7] and*

$$|\varphi_K^a(r) - \varphi_K^a(s)| \geqslant \varphi_K^a(|r-s|) \geqslant \mathrm{e}^{(1-1/K)R(a)/2}|r-s|^{1/K}, \quad 0 < K \leqslant 1. \tag{3.3}$$

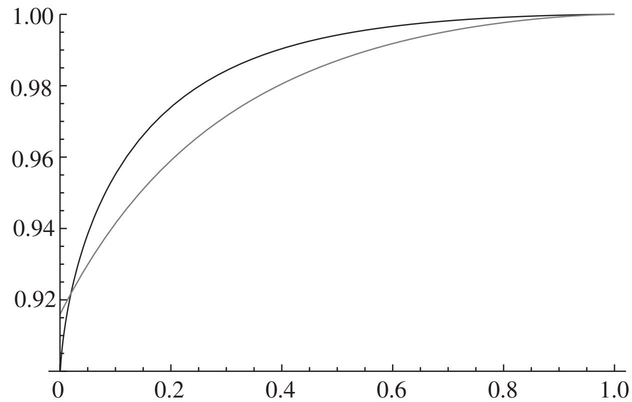

Figure 2. Let $g(a,K,p,r,s) = ((\varphi_K^a(r^p))^{1/p} + (\varphi_K^a(s^p))^{1/p})/(1 + (\varphi_K^a(r^p)\varphi_K^a(s^p))^{1/p})$ and $h(a,K,r,s) = \varphi_K^a((r+s)/(1+rs))$ be the lower bounds in Lemma 3.3 (black line) and in (3.1) (grey line), respectively. For $a = 0.2$, $K = 1.5$, $p = 1.3$ and $s = 0.5$ the functions g and h are plotted. We see that for $r \in (0.2, 1)$ the first lower bound is better.

Proof. It follows from [**3**, Theorem 6.7] that $r^{-1}\varphi_K^a(r)$ is decreasing on $(0,1)$ if $K > 1$, and by Lemma 2.2 we obtain

$$\varphi_K^a(x+y) \leqslant \varphi_K^a(x) + \varphi_K^a(y), \quad x, y \in (0,1).$$

Now the first inequality in (3.2) follows if we take $r = x + y$ and $s = y$; the second one follows from [**3**, Theorem 6.7]. Next, (3.3) follows from (3.2) and the fact that

$$\varphi_{AB}^a(r) = \varphi_A^a(\varphi_B^a(r)), \quad A, B > 0,\ r \in (0,1),$$

when we replace K, r and s by $1/K$, $\varphi_{1/K}^a(r)$ and $\varphi_{1/K}^a(s)$, respectively. □

Theorem 3.5. *For $a \in (0, \frac{1}{2}]$, $c, r \in (0,1)$ and $K, L \in (0,\infty)$ we have the following.*

(i) $f(K) = \log(\varphi_K^a(r))$ *is increasing and concave from $(0,\infty)$ onto $(-\infty, 0)$.*

(ii) $g(K) = \operatorname{artanh}(\varphi_K^a(r))$ *is increasing and convex from $(0,\infty)$ onto $(0,\infty)$.*

(iii) *We have*

$$\varphi_K^a(r)^c \varphi_L^a(r)^{1-c} \leqslant \varphi_{cK+(1-c)L}^a(r) \leqslant \tanh(c \operatorname{artanh}(\varphi_K^a(r)) + (1-c)\operatorname{artanh}(\varphi_L^a(r))).$$

(iv) *We have*

$$\sqrt{\varphi_K^a(r)\varphi_L^a(r)} \leqslant \varphi_{(K+L)/2}^a(r) \leqslant \frac{\varphi_K^a(r) + \varphi_L^a(r)}{1 + \varphi_K^a(r)\varphi_L^a(r) + \varphi_{1/K}^a(r')\varphi_{1/L}^a(r')}.$$

Proof. For (i), by (2.6) we get

$$f'(K) = \frac{4s'^2 \mathcal{K}_a(s)^2 \mu_a(r)}{\pi^2 K},$$

which is positive and decreasing by Lemma 2.3 (iii). For (ii), we get

$$f'(K) = \frac{4s\mathcal{K}_a(s)^2\mu_a(r)}{\pi^2 K^2} = \frac{s\mathcal{K}'_a(s)^2}{\mu_a(r)}$$

by (2.6), which is positive and increasing by Lemma 2.3 (iii). By (i) and (ii) we get

$$c\log(\varphi^a_K(r)) + (1-c)\log(\varphi^a_L(r)) \leqslant \log(\varphi^a_{cK+(1-a)L}(r))$$

and

$$\operatorname{artanh}(\varphi^a_{cK+(1-c)K}(r)) \leqslant a\operatorname{artanh}(\varphi^a_K(r)) + (1-c)\operatorname{artanh}(\varphi^a_L(r)),$$

respectively, and (iii) follows. Also

$$\tfrac{1}{2}(\log(\varphi^a_K(r)) + \log(\varphi^a_L(r))) \leqslant \log(\varphi^a_{(K+L)/2}(r))$$

and

$$\operatorname{artanh}(\varphi^a_{(K+L)/2}(r)) \leqslant \tfrac{1}{2}(\operatorname{artanh}(\varphi^a_K(r)) + \operatorname{artanh}(\varphi^a_L))$$

follow from (i) and (ii), and hence (iv) holds. □

Theorem 3.6. *For $K \geqslant 1$ and $0 < m < n$, the following inequalities hold:*

$$\eta^a_K(mn) \leqslant \sqrt{\eta^a_K(m^2)\eta^a_K(n^2)}, \tag{3.4}$$

$$\left(\frac{n}{m}\right)^{1/K} < \frac{\eta^a_K(n)}{\eta^a_K(m)} < \left(\frac{n}{m}\right)^K, \tag{3.5}$$

$$\eta^a_K(m)\eta^a_K(n) < \left(\eta^a_K\left(\frac{m+n}{2}\right)\right)^2, \tag{3.6}$$

$$2\frac{\eta^a_K(m)\eta^a_K(n)}{\eta^a_K(m)+\eta^a_K(n)} < \eta^a_K(\sqrt{mn}) < \sqrt{\eta^a_K(m)\eta^a_K(n)}. \tag{3.7}$$

Proof. We define a function $g(x) = \log\eta^a_K(\mathrm{e}^x)$ on $\mathbb{R}$. By [**3**, Theorem 1.16], g is increasing, convex and satisfies $1/K \leqslant g'(x) \leqslant K$. Then

$$\begin{aligned}\log\eta^a_K(\mathrm{e}^{(x+y)/2}) = g\left(\frac{x+y}{2}\right) &\leqslant \frac{g(x)+g(y)}{2}\\ &= \tfrac{1}{2}\log(\eta^a_K(\mathrm{e}^x)) + \tfrac{1}{2}\log(\eta^a_K(\mathrm{e}^y)),\end{aligned}$$

and this is equivalent to

$$\log\eta^a_K(\mathrm{e}^{x/2}\mathrm{e}^{y/2}) \leqslant \log(\eta^a_K(\mathrm{e}^{x/2})\eta^a_K(\mathrm{e}^{y/2})).$$

Hence, (3.4) follows if we set $\mathrm{e}^{x/2} = m$ and $\mathrm{e}^{y/2} = n$. For (3.5), let $x > y$. Then, by the inequality $1/K \leqslant g'(x) \leqslant K$ and the Mean-Value Theorem, we get

$$\frac{x-y}{K} \leqslant g(x) - g(y) \leqslant K(x-y),$$

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and this is equivalent to

$$\frac{\log(\mathrm{e}^x)-\log(\mathrm{e}^y)}{K}\leqslant \log(\eta_K^a(\mathrm{e}^x))-\log(\eta_K^a(\mathrm{e}^y))\leqslant K(\log(\mathrm{e}^x)-\log(\mathrm{e}^y)).$$

By setting $\mathrm{e}^{x/2}=m$ and $\mathrm{e}^{y/2}=n$, we get the desired inequality. For (3.6), let $f(x)=\log(\eta_K^a(x))$, $r=\sqrt{x/(1+x)}$ and $s=\varphi_K^a(r)$. Then by (2.7) we get

$$\begin{aligned} f'(x) &= \frac{1}{K}\left(\frac{s'}{s}\right)^2\left(\frac{sr'\mathcal{K}_a(s)}{rs'\mathcal{K}_a(r)}\right)^2=\frac{1}{K}\left(\frac{r'}{r}\right)^2\left(\frac{\mathcal{K}_a(s)}{\mathcal{K}_a(r)}\right)^2 \\ &= \frac{1}{K}\left(\frac{r'}{s}\right)^2\left(\frac{s\mathcal{K}_a(s)}{r\mathcal{K}_a(r)}\right)^2, \end{aligned}$$

which is positive and decreasing by Lemma 2.3 (ii). Hence, $\frac{1}{2}(f(x)+f(y))\leqslant f(\frac{1}{2}(x+y))$, and the inequality follows.

For (3.7), letting $h(x)=1/\eta_K^a(\mathrm{e}^x)$, we see that this is log-concave by (3.4), and we get

$$\log\left(\frac{1}{\eta_K^a(\mathrm{e}^x)}\right)+\log\left(\frac{1}{\eta_K^a(\mathrm{e}^y)}\right)<2\log\left(\frac{1}{\eta_K^a(\mathrm{e}^{(x+y)/2})}\right),$$

Setting $\mathrm{e}^x=m$ and $\mathrm{e}^y=n$, we get the second inequality. We observe that $h(x)=(s'/s)$, $s=\varphi_K^a(r)$, $r=\sqrt{\mathrm{e}^x/(\mathrm{e}^x+1)}$. We get

$$-f'(x)=\frac{1}{K}\left(\frac{r'}{s}\right)\left(\frac{s'\mathcal{K}_a(s)}{r'\mathcal{K}_a(r)}\right)^2,$$

which is positive and decreasing by Lemma 2.3 (i); hence h is convex, and the first inequality follows easily. □

Theorem 3.7. *For $x\in(0,\infty)$, the function $f\colon(0,\infty)\to(0,\infty)$ defined by $f(K)=\eta_K^a(x)$ is increasing, convex and log-concave. In particular,*

$$\eta_K^a(x)^c\eta_L^a(x)^{1-c}\leqslant \eta_{cK+(1-c)L}^a(x)\leqslant c\eta_K^a(x)+(1-c)\eta_L^a(x)$$

for $K,L,x\in(0,\infty)$ and $c\in(0,1)$, with equality if and only if $K=L$.

Proof. We observe that $f(K)=(s/s')^2$, where $s=\varphi_K^a(r)$ and $r=\sqrt{x/(x+1)}$. We get by (2.8)

$$f'(K)=\frac{8s^2\mathcal{K}_a(s)^2}{\pi^2s'^2K^2}\mu_a(r)=\frac{4}{\pi\sin(\pi a)}\frac{\mathcal{K}_a(r)}{\mathcal{K}_a'(r)}\left(\frac{s\mathcal{K}_a'(s)}{s'}\right)^2,$$

which is positive and increasing by Lemma 2.3 (iii); hence, f is increasing and convex. For log-concavity, let $g(K)=\log(\eta_K^a(x))$. By (2.8), we get

$$g'(K)=\frac{8\mathcal{K}_a(s)^2}{\pi^2K^2}\mu_a(r)=\frac{4}{\pi\sin(\pi a)}\frac{\mathcal{K}_a(r)}{\mathcal{K}_a'(r)}\mathcal{K}_a'(s)^2,$$

which is decreasing; hence, f is log-concave. □

Theorem 3.8. *The function*

$$f(K) = \frac{\log \eta_K^a(x) - \log(x)}{K-1}$$

is decreasing from $(1,\infty)$ *onto*

$$\left(\frac{\pi \mathcal{K}_a(r)}{\sin(\pi a)\mathcal{K}'_a(r)}, \frac{4\mathcal{K}_a(r)\mathcal{K}'_a(r)}{\pi \sin(\pi a)}\right),$$

and the function

$$g(K) = \frac{\eta_K^a(x) - (x)}{K-1}$$

is increasing from $(1,\infty)$ *onto*

$$(4r^2 \sin(\pi a)\mathcal{K}_a(r)\mathcal{K}'_a(r)/(\pi r'^2), \infty),$$

where $r = \sqrt{x/(x+1)}$.

Proof. It follows from Theorem 3.7 and Lemma 2.1 that f is monotone. Let $s = \varphi_K^a(r)$; by (2.6), l'Hôpital's rule and definition of μ_a, we get

$$\begin{aligned}\lim_{K\to 1} f(K) &= \lim_{K\to 1} \frac{2}{K-1} \log\left(\frac{sr'}{s'r}\right)\\ &= \lim_{K\to 1} \frac{8\mathcal{K}_a(s)^2\mu_a(r)}{K^2\pi^2}\\ &= \frac{8}{\pi^2}\mathcal{K}_a(r)^2\mu_a(r)\\ &= \frac{4\mathcal{K}_a(r)\mathcal{K}'_a(r)}{\pi\sin(\pi a)}.\end{aligned}$$

By using the fact that $K = \mu_a(r)/\mu_a(s)$ and l'Hôpital's rule, we get

$$\begin{aligned}\lim_{K\to\infty} f(K) &= \lim_{K\to\infty} \frac{8\mu_a(s)^2\mathcal{K}_a(s)^2}{\pi^2\mu_a(r)}\\ &= \lim_{K\to\infty} \frac{2\mathcal{K}'_a(s)^2}{\sin^2(\pi a)\mu_a(r)}\\ &= \frac{2\mathcal{K}_a(0)^2}{\sin^2(\pi a)\mu_a(r)}\\ &= \frac{\pi\mathcal{K}_a(r)}{\sin(\pi a)\mathcal{K}'_a(r)}.\end{aligned}$$

Next, let $g(K) = G(K)/H(K)$, where $G(K) = (s/s')^2 - (r/r')^2$ and $H(K) = K-1$. We see that $G(1) = H(1) = 0$ and $G(\infty) = H(\infty) = \infty$. We see that

$$\frac{G'(K)}{H'(K)} = \frac{2(s\mathcal{K}'_a(s))^2}{s'^2\mu_a(r)},$$

and it follows from Lemmas 2.3 (iii) and 2.1 that $g(K)$ is increasing and the required limiting values follow from $\varphi_K^a(r) = \mu_a^{-1}(\mu_a(r)/K)$. □

Remark 3.9. If we take $x = 1$ in Theorem 3.8, then with $t = 4\mathcal{K}_a(\frac{1}{\sqrt{2}})^2/(\pi\sin(\pi a))$ we have the following:

1. $\log(\lambda_a(K))/(K-1)$ is strictly decreasing from $(1,\infty)$ onto $(\pi/\sin(\pi a), t)$;
2. $(\lambda_a(K)-1)/(K-1)$ is increasing from $(1,\infty)$ onto $(t\sin^2(\pi a), \infty)$.

In particular,

$$\exp\left(\frac{\pi(K-1)}{\sin(\pi a)}\right) < \lambda_a(K) < \exp(t(K-1))$$

and

$$1 + t(K-1)\sin^2(\pi a) < \lambda_a(K) < \infty,$$

respectively, and we get

$$\max\left\{\exp\left(\frac{\pi(K-1)}{\sin(\pi a)}\right), 1 + t(K-1)\sin^2(\pi a)\right\} < \lambda_a(K) < \mathrm{e}^{t(K-1)}.$$

Lemma 3.10. *For $c \in [-3, 0)$, the function $f(r) = \mathcal{K}_a(r)^c + \mathcal{K}'_a(r)^c$ is strictly increasing from $(0, \frac{1}{\sqrt{2}})$ onto $((\frac{1}{2}\pi)^c, 2\mathcal{K}_a(\frac{1}{\sqrt{2}})^c)$.*

Proof. By (2.2), we get

$$\begin{aligned} f'(r) &= \frac{2(1-a)c\mathcal{K}_a(r)^{c-1}(\mathcal{E}_a(r) - r'^2\mathcal{K}_a(r))}{rr'} - \frac{2(1-a)c\mathcal{K}'_a(r)^{c-1}(\mathcal{E}'_a(r) - r^2\mathcal{K}'_a(r))}{rr'} \\ &= \frac{2(1-a)c(\mathcal{K}_a(r)\mathcal{K}'_a(r))^{c-1}}{rr'}(h(r) - h(r')), \end{aligned}$$

and here

$$h(r) = \frac{r^2\mathcal{K}'_a(r)^{1-c}}{r^2}(\mathcal{E}_a(r) - r'^2\mathcal{K}_a(r)),$$

which is increasing on $(0,1)$ by [**8**, Theorem 3.21(1)] and Lemma 2.3 (iii). Hence, $f'(r) < 0$ on $(0, \frac{1}{\sqrt{2}})$, and the limiting values are clear. □

Theorem 3.11.

(i) *For $K > 1$, the function $\log(\lambda_a(K))/(K - 1/K)$ is strictly increasing from $(1,\infty)$ onto $(2\mathcal{K}_a(\frac{1}{\sqrt{2}})/(\pi\sin(\pi a)), \pi/\sin(\pi a))$.*

(ii) *The function $\log(\lambda_a(K)+1)$ is convex on $(0,\infty)$, and $\log(\lambda_a(K))$ is concave.*

(iii) *The function $g(K) = (\log(\lambda_a(K)))/\log K$ is strictly increasing on $(1,\infty)$. In particular, for $c \in (0,1)$,*

$$\lambda_a(K^c) < (\lambda_a(K))^c.$$

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Proof. For (i), let

$$r=\mu_a^{-1}\left(\frac{\pi K}{2\sin(\pi a)}\right),\quad 0\leqslant r\leqslant\frac{1}{\sqrt{2}}.$$

Then, by (1.3),

$$\begin{aligned}r'&=\sqrt{1-\left(\mu_a^{-1}\left(\frac{\pi K}{2\sin(\pi a)}\right)\right)^2}\\&=\sqrt{1-\left(\mu_a^{-1}\left(K\mu_a\left(\frac{1}{\sqrt{2}}\right)\right)\right)^2}\\&=\mu_a^{-1}\left(\frac{\pi}{2K\sin(\pi a)}\right),\end{aligned}$$

we also observe that $K=\mathcal{K}'_a(r)/\mathcal{K}_a(r)$. Now it is enough to prove that the function

$$f(r)=\frac{2\log(r'/r)}{\mathcal{K}'_a(r)/\mathcal{K}_a(r)-\mathcal{K}_a(r)\mathcal{K}'_a(r)}=\frac{\pi\log(r'/r)}{\sin(\pi a)(\mu_a(r)+\mu_a(r'))},$$

is strictly decreasing on $(0,\frac{1}{\sqrt{2}})$. Set $f(r)=G(r)/H(r)$. Clearly, $G(\frac{1}{\sqrt{2}})=H(\frac{1}{\sqrt{2}})=0$. By (2.4), we get

$$\frac{G'(K)}{H'(K)}=\frac{4}{\pi\sin(\pi a)(\mathcal{K}_a(r)^{-2}-\mathcal{K}_a(r')^{-2})},$$

which is strictly decreasing from $(0,\frac{1}{\sqrt{2}})$ onto $(2\mathcal{K}_a(\frac{1}{\sqrt{2}})/(\pi\sin(\pi a)),\pi/\sin(\pi a))$ by Lemma 3.10. Now the proof of (i) follows from Lemma 2.1.

For (ii), it follows from Theorem 3.7 that $\log(\lambda_a(K))$ is concave. Letting $f(K)=\lambda_a(K)+1$, we have

$$f(K)=\left(\mu_a^{-1}\left(\frac{\pi K}{2\sin(\pi a)}\right)\right)^{-2},$$

by (1.4) and (1.3). Now we have $\log f(K)=-2\log y$, where $\mu_a(y)=\pi K/(2\sin(\pi a))$. By (2.4) we get

$$\frac{f'(K)}{f(K)}=-\frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}K}=\frac{4}{\pi}(y'\mathcal{K}_a(y)),$$

which is decreasing in y by Lemma 2.3 (iii), and increasing in K. Hence, $\log f(K)$ is convex.

For (iii), $K>1$, let $h(K)=(K-1/K)/\log K$. We get

$$h'(K)=\frac{(1+K^2)\log K-(K^2-1)}{(K\log K)^2},$$

which is positive because

$$\log K>\frac{2(K-1)}{K+1}>\frac{K^2-1}{K^2+1}$$

by [**8**, § 1.58(4)a]; hence, h is strictly increasing. Also

$$g(K) = h(K)\frac{\log(\lambda_a(K))}{K - 1/K} = \frac{\log(\lambda_a(K))}{\log K}$$

is strictly increasing by (i). This implies that

$$\frac{\log(\lambda_a(K^c))}{c \log K} < \frac{\log(\lambda_a(K))}{\log K},$$

and hence (iii) follows. □

Corollary 3.12. *For* $0 < r < \frac{1}{\sqrt{2}}$ *and* $t = \pi^2/(2\mathcal{K}_a(\frac{1}{\sqrt{2}})^2)$, *we have the following.*

(i) *The function*

$$f(r) = \frac{\mu_a(r) - \mu_a(r')}{\log(r'/r)}$$

is increasing from $(0, \frac{1}{\sqrt{2}})$ *onto* $(1, t)$. *In particular,*

$$\log(r'/r) < \mu_a(r) - \mu_a(r') < \frac{\pi^2}{2\mathcal{K}_a(\frac{1}{\sqrt{2}})^2} \log(r'/r).$$

(ii) *For* $g(r) = \log(r'/r)$,

$$g(r) + \sqrt{(\pi/\sin(\pi a))^2 + g(r)^2} < 2\mu_a(r) < tg(r) + \sqrt{(\pi/\sin(\pi a))^2 + t^2 g(r)^2}.$$

Proof. It follows from the proof of Theorem 3.11 (i) that $f(r)$ is increasing, and limiting values follow easily by l'Hôpital's rule. For (ii), from the definition of μ_a we get $\mu_a(r') = \pi^2/(2\sin(\pi a))^2 \mu_a(r)$; substituting this into (i), we obtain

$$1 < \frac{\mu_a(r)^2 - \pi^2/(2\sin(\pi a))^2}{\mu_a(r)\log(r'/r)} < t = \frac{\pi^2}{2\mathcal{K}_a(1\sqrt{2})^2}.$$

This implies that

$$\mu_a(r)^2 - \mu_a(r)\log(r'/r) > \frac{\pi^2}{(2\sin(\pi a))^2} \tag{3.8}$$

and

$$\mu_a(r)^2 - t\mu_a(r)\log(r'/r) < \frac{\pi^2}{(2\sin(\pi a))^2}. \tag{3.9}$$

We get the left and right inequalities in (ii) by solving (3.8) and (3.9) for $\mu_a(r)$, respectively. □

4. Three-parameter complete elliptic integrals

The results in this section have counterparts in [**3**]. For $a, b, c > 0$, $a+b \geqslant c$, the decreasing homeomorphism $\mu_{a,b,c}\colon (0,1) \to (0,\infty)$ is defined by

$$\mu_{a,b,c}(r) = \frac{B(a,b)}{2} \frac{F(a,b;c;r'^2)}{F(a,b;c;r^2)}, \quad r \in (0,1),$$

where B is the beta function. The (a,b,c)-modular function is defined by

$$\varphi_K^{a,b,c}(r) = \mu_{a,b,c}^{-1}\left(\frac{\mu_{a,b,c}(r)}{K}\right).$$

We define, in the case $a < c$,

$$\mu_{a,c}(r) = \mu_{a,c-a,c}(r) \quad \text{and} \quad \varphi_K^{a,c}(r) = \varphi_K^{a,c-a,c}(r).$$

We define the three-parameter complete elliptic integrals of the first and second kinds for $0 < a < \min\{c, 1\}$ and $0 < b < c \leqslant a+b$ by

$$\mathcal{K}_{a,b,c}(r) = \tfrac{1}{2}B(a,b)F(a,b;c;r^2),$$
$$\mathcal{E}_{a,b,c}(r) = \tfrac{1}{2}B(a,b)F(a-1,b;c;r^2),$$

and set

$$\mathcal{K}_{a,c}(r) = \mathcal{K}_{a,c-a,c}(r) \quad \text{and} \quad \mathcal{E}_{a,c}(r) = \mathcal{E}_{a,c-a,c}(r).$$

Lemma 4.1 (Heikkala *et al.* [19, Theorem 3.6]). *For $0 < a < c \leqslant 1$, the function $f(r) = \mu_{a,c}(r)\,\operatorname{artanh} r$ is strictly increasing from $(0,1)$ onto $(0, (\frac{1}{2}B)^2)$.*

Lemma 4.2 (Heikkala *et al.* [19, Lemma 4.1]). *Let $a < c \leqslant 1, K \in (1,\infty), r \in (0,1)$, and let $s = \varphi_K^{a,c}(r)$ and $t = \varphi_{1/K}^{a,c}(r)$. Then*

(i) *$f_1(r) = \mathcal{K}_{a,c}(s)/\mathcal{K}_{a,c}(r)$ is increasing from $(0,1)$ onto $(1,K)$,*

(ii) *$f_2(r) = s'\mathcal{K}_{a,c}(s)^2/(r'\mathcal{K}_{a,c}(r)^2)$ is decreasing from $(0,1)$ onto $(0,1)$,*

(iii) *$f_3(r) = s\mathcal{K}'_{a,c}(s)^2/(r\mathcal{K}'_{a,c}(r)^2)$ is decreasing from $(0,1)$ onto $(1,\infty)$,*

(iv) *$g_1(r) = \mathcal{K}_{a,c}(t)/\mathcal{K}_{a,c}(r)$ is decreasing from $(0,1)$ onto $(1/K,1)$,*

(v) *$g_2(r) = t'\mathcal{K}_{a,c}(t)^2/(r'\mathcal{K}_{a,c}(r)^2)$ is increasing from $(0,1)$ onto $(1,\infty)$,*

(vi) *$g_3(r) = t\mathcal{K}'_{a,c}(t)^2/(r\mathcal{K}'_{a,c}(r)^2)$ is increasing from $(0,1)$ onto $(0,1)$,*

(vii) *$g_4(r) = s/r$ is decreasing from $(0,1)$ onto $(1,\infty)$,*

(viii) *$g_5(r) = t/r$ is increasing from $(0,1)$ onto $(0,1)$.*

Theorem 4.3. *For $0 < a < c \leqslant 1$, the function $f(x) = \mu_{a,c}(1/\cosh(x))$ is increasing and concave from $(0, \infty)$ onto $(0, \infty)$. In particular,*

$$\mu_{a,c}\left(\frac{rs}{1 + r's'}\right) \leqslant \mu_{a,c}(r) + \mu_{a,c}(s) \leqslant 2\mu_{a,c}\left(\sqrt{\frac{2rs}{1 + rs + r's'}}\right)$$

for all $r, s \in (0, 1)$. The second inequality becomes an equality if and only if $r = s$.

Proof. Let $r = 1/\cosh(x)$ and (see [**19**])

$$M(r^2) = \left(\frac{2}{B(a, b)}\right)^2 b(\mathcal{K}_{a,c}(r)\mathcal{E}'_{a,c}(r) + \mathcal{K}'_{a,c}(r)\mathcal{E}_{a,c}(r) - \mathcal{K}_{a,c}(r)\mathcal{K}'_{a,c}(r)).$$

We get

$$f'(x) = \frac{B(a, b)}{2} \frac{M(r^2)}{r'^2 \mathcal{K}(r)^2},$$

which is positive and increasing in r by [**19**, Lemma 3.4 (1), Theorem 3.12 (2)], and f is decreasing in x. Hence, f is concave. This implies that

$$\frac{1}{2}\left(\mu_{a,c}\left(\frac{1}{\cosh(x)}\right) + \mu_{a,c}\left(\frac{1}{\cosh(y)}\right)\right) \leqslant \mu_{a,c}\left(\frac{1}{\cosh(\frac{1}{2}(x + y))}\right),$$

and we get the second inequality by using the formula

$$\left(\cosh\left(\frac{x + y}{2}\right)\right)^2 = \frac{1 + rs + r's'}{2rs}$$

and setting $s = 1/\cosh(y)$. Next, $f'(x)$ is decreasing in x, and $f(0) = 0$. Then $f(x)/x$ is decreasing on $(0, \infty)$ and $f(x + y) \leqslant f(x) + f(y)$ by Lemmas 2.1 and 2.2, respectively. Hence, the first inequality follows. □

Lemma 4.4. *For $0 < a < c \leqslant 1$, we have*

$$\mu_{a,c}(r) + \mu_{a,c}(s) \leqslant 2\mu_{a,c}(\sqrt{rs}),$$

for all $r, s \in (0, 1)$, with equality if and only if $r = s$.

Proof. Clearly,

$$\begin{aligned}
(r - s)^2 \geqslant 0 &\iff 1 + r^2 s^2 \geqslant 1 - (r - s)^2 + r^2 s^2 \\
&\iff (1 - rs)^2 \geqslant 1 - r^2 - s^2 + r^2 s^2 \\
&\iff 1 - rs \geqslant r's' \\
&\iff 2 \geqslant 1 + rs + r's' \\
&\iff 1/(rs) \geqslant (1 + rs + r's')/(2rs).
\end{aligned}$$

By using the fact that $\mu_{a,c}$ is decreasing, we get

$$\mu_{a,c}\left(\sqrt{\frac{2rs}{1 + rs + r's'}}\right) \leqslant \mu_{a,c}(\sqrt{rs}),$$

and the result follows from Theorem 4.3. □

Theorem 4.5. *For $K > 1$, $0 < a < c$ and $r, s \in (0,1)$,*

$$\tanh(K \operatorname{artanh} r) < \varphi_K^{a,c}(r).$$

The inequality is reversed if we replace K by $1/K$.

Proof. Let $s = \varphi_K^{a,c}(r)$. Then $s > r$, and by the equality $\varphi_K^{a,c}(r) = \mu_{a,c}^{-1}(\mu_{a,c}(r)/K)$ and Lemma 4.1 we get

$$\frac{1}{K}\mu_{a,c}(r) \operatorname{artanh} s = \mu_{a,c}(s) \operatorname{artanh} s > \mu_{a,c}(r) \operatorname{artanh} r,$$

which is equivalent to the required inequality. For the case $1/K$ let $x = \varphi_{1/K}^{a,c}(r)$. Then $x < r$, and similarly we get

$$K\mu_{a,c}(r) \operatorname{artanh} x = \mu_{a,c}(x) \operatorname{artanh} x < \mu_{a,c}(r) \operatorname{artanh} r,$$

and this is equivalent to $\tanh((\operatorname{artanh} r)/K) > \varphi_{1/K}^{a,c}(r)$. □

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