Log p-divisible groups associated with log 1-motives

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Abstract. We first provide a detailed proof of Kato's classification theorem of log p-divisible groups over a Noetherian Henselian local ring. Exploring Kato's idea further, we then define the notion of a standard extension of a classical finite étale group scheme (resp. classical étale p-divisible group) by a classical finite flat group scheme (resp. classical p-divisible group) in the category of finite Kummer flat group log schemes (resp. log p-divisible groups), with respect to a given chart on the base. These results are then used to prove that log p-divisible groups are formally log smooth. We then study the finite Kummer flat group log schemes $T_n(\mathbf{M}) := H^{-1}(\mathbf{M} \otimes_{\mathbb{Z}}^{L} \mathbb{Z}/n\mathbb{Z})$ (resp. the log p-divisible group $\mathbf{M}[p^{\infty}]$) of a log 1-motive $\mathbf{M}$ over an fs log scheme and show that they are étale locally standard extensions. Lastly, we give a proof of the Serre–Tate theorem for log abelian varieties with constant degeneration.

1 Introduction

When studying degenerations of abelian varieties, one is led to Kato's theory of finite Kummer flat group log schemes and log p-divisible groups (see [12] and [26, Appendix][1] for a brief account of finite Kummer flat group log schemes). For instance, by a theorem of Kato (see [28, Theorem 1.3]), the p-divisible group of a semistable abelian variety over a complete discrete valuation field extends to a log p-divisible group over the corresponding discrete valuation ring. In this article, we will expand on some ideas developed in [12] and connect them with the theory of log 1-motives and log abelian varieties.

Let S be an fs log scheme endowed with a suitable global chart, whose underlying scheme is the spectrum of a Noetherian Henselian local ring with positive residue characteristic p. When studying finite Kummer flat group log schemes over S, one key point is to understand extensions of classical finite étale group schemes by classical

[1]The terminology "log finite flat group scheme" in [26, Appendix] is replaced by "finite Kummer flat group log schemes" here, which should be more suitable.

finite flat group schemes over S in a logarithmic category (see [26, the paragraph before Theorem A.3] for an explanation of this). In [12, Theorem 3.3] (see also Theorem 3.8), Kato gives a description of such an extension in terms of a classical extension together with a certain monodromy datum. We call such extensions *standard* (see Definition 3.1).

As a corollary of Kato's theorem (Theorem 3.8), one then also gets a description of the extensions of a classical étale p-divisible group by a classical p-divisible group over S (see Theorem 3.13). After presenting the basic objects in Section 2, we then in Section 3 present a complete proof to Kato's theorem following Kato's approach. In the procedure, we explore Kato's idea further and define the notion of a standard extension of a classical finite étale group scheme (resp. classical étale p-divisible group) by a classical finite flat group scheme (resp. classical p-divisible group) (see Definition 3.1 [resp. Definition 3.12]). Then Kato's results amount to saying that, over a Noetherian Henselian local fs log scheme admitting a global chart, any extension of a classical finite étale group scheme (resp. classical étale p-divisible group) by a classical finite flat group scheme (resp. classical p-divisible group) is always standard. Over a general base, we get the following.

Theorem (See Theorem 3.11) *Let S be a locally Noetherian fs log scheme, and let G be a finite Kummer flat group log scheme over S, which is an extension of a classical finite étale group scheme by a classical finite flat group scheme. Then G is a standard extension étale locally on S.*

Moreover, as an application of Kato's results, we can prove the following theorem for objects in the category $(p\text{-div}/S)_{\mathrm{d}}^{\log}$, which consists of log p-divisible groups whose dual is also a log p-divisible group. This is analogous to the formal smoothness of classical p-divisible groups (see [16, Chapter II 3.3.13]).

Theorem (See Theorem 3.16) *Let S be a locally Noetherian fs log scheme on which p is locally nilpotent, and let H be a log p-divisible group over S, which lies in $(p\text{-div}/S)_{\mathrm{d}}^{\log}$. Then H is formally log smooth, i.e., for any strict closed square-zero thickening $T_0 \hookrightarrow T$ in (fs/S), any element of $H(T_0)$ can be lifted to an element of $H(T)$ étale locally on T.*

In Section 4, we study the finite Kummer flat group log scheme

$$T_n(\mathbf{M}) := H^{-1}(\mathbf{M} \otimes_{\mathbb{Z}}^{L} \mathbb{Z}/n\mathbb{Z})$$

for a log 1-motive $\mathbf{M} = [Y \xrightarrow{u} G_{\log}]$ over a locally Noetherian fs log scheme S, as well as the log p-divisible group $\mathbf{M}[p^{\infty}] := \varinjlim_n T_{p^n}(\mathbf{M})$ of $\mathbf{M}$. Let T be the torus part of G, and let X be the character group of T. The composition

$$Y \xrightarrow{u} G_{\log} \to G_{\log}/G \cong T_{\log}/T = \mathcal{H}om(X, \mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})$$

gives rise to a pairing $\langle -, - \rangle : X \times Y \to \mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}}$ (see [8, Section 2.3]). We call this pairing the *monodromy pairing of* $\mathbf{M}$. In Proposition 4.7, we show that étale locally the monodromy pairing of $\mathbf{M}$ gives rise to a canonical pseudo-monodromy of $T_n(\mathbf{M})$, as well as of $\mathbf{M}[p^{\infty}]$.

In Section 5, we turn to the Serre–Tate theorem for log abelian varieties with constant degeneration.

Theorem (see Theorem 5.14) *Let S_0 be a locally Noetherian fs log scheme on which p is locally nilpotent, and let $S_0 \subset S$ be a strict infinitesimal thickening of fs log schemes. Let A_0 be a log abelian variety with constant degeneration over S_0. Let $\mathrm{Def}_{A_0}(S)$ be the category of pairs (A, ϕ) where A is log abelian variety over S, and $\phi : A_{S_0} \xrightarrow{\cong} A_0$ is an isomorphism. Similarly, let $\mathrm{Def}_{A_0[p^\infty]}(S)$ be the category of pairs (H, ψ), where H is a log p-divisible group over S, with an identification $\psi : H_{S_0} \xrightarrow{\cong} A_0[p^\infty]$.*

Then the functor $A \mapsto A[p^\infty]$, taking a log abelian variety over S to its associated log p-divisible group, induces an equivalence of categories

$$\mathrm{Def}_{A_0}(S) \leftrightarrow \mathrm{Def}_{A_0[p^\infty]}(S).$$

For this, we follow Drinfeld's approach for abelian varieties as in [15, Section 1]. The key point here is to verify the hypothesis about formal (log) smoothness from [15, Lemma 1.1.3] for log p-divisible groups in $(p\text{-div}/S)_{\mathrm{d}}^{\log}$ and log abelian varieties with constant degeneration.

The above theorem can also be regarded as a Serre–Tate theorem for pointwise polarizable log 1-motives. There is also a Serre–Tate theorem for classical 1-motives over local Artin rings, which is the main theorem of [4].

2 Finite Kummer flat group log schemes and log *p*-divisible groups

2.1 Kummer log topologies

Unless otherwise stated, we always denote by S a locally Noetherian fs log scheme. By log structure, we always mean log structure on the classical étale site. We denote by (fs/S) the category of fs log schemes over S. We recall the Kummer log flat topology and the Kummer log étale topology on (fs/S) (see [13, Definition 2.3] or [20, Definition 2.13]).

Definition 2.1   For $U \in (\mathrm{fs}/S)$, a family of morphisms $\{U_i \xrightarrow{f_i} U\}$ is called a *Kummer log flat cover* (resp. *Kummer log étale cover*), if the following are satisfied.
 (1) Each f_i is log flat (resp. log étale) and of Kummer type, and its underlying map of schemes is locally of finite presentation.
 (2) The family is set-theoretically surjective, i.e., $U = \bigcup_i f_i(U_i)$.
The *Kummer log flat topology* (resp. *Kummer log étale topology*) on (fs/S) is the Grothendieck topology given by the Kummer log flat cover (resp. Kummer log étale cover) on (fs/S). We will sometimes call the Kummer log flat (resp. Kummer log étale) topology simply the Kummer flat (resp. Kummer étale) topology, and denote the resulting site by $(\mathrm{fs}/S)_{\mathrm{kfl}}$ (resp. $(\mathrm{fs}/S)_{\mathrm{két}}$).

To see that these are indeed Grothendieck topology, we refer to [13, Section 2] and [20, Section 2].

By taking the strict flat covers, i.e., families of strict morphisms whose underlying maps of schemes form flat covers of schemes, one gets the classical flat site on (fs/S), denoted by $(\mathrm{fs}/S)_{\mathrm{fl}}$. Similarly, one also gets the classical étale site on (fs/S), denoted by $(\mathrm{fs}/S)_{\mathrm{ét}}$. We have a natural "forgetful" map of sites

$$\varepsilon : (\mathrm{fs}/S)_{\mathrm{kfl}} \to (\mathrm{fs}/S)_{\mathrm{fl}}.$$

There is of course the étale version of the above forgetful map of sites, but we do not need it in this article. In order to shorten formulas, we will mostly abbreviate $(\mathrm{fs}/S)_{\text{ét}}$ (resp. $(\mathrm{fs}/S)_{\text{két}}$, resp. $(\mathrm{fs}/S)_{\text{fl}}$, resp. $(\mathrm{fs}/S)_{\text{kfl}}$) as $S_{\text{ét}}$ (resp. $S_{\text{két}}$, resp. S_{fl}, resp. S_{kfl}).

Definition 2.2 Kato's multiplicative group (or the log multiplicative group) $\mathbb{G}_{\mathrm{m,log}}$ is the sheaf on $S_{\text{ét}}$ defined by $\mathbb{G}_{\mathrm{m,log}}(U) = \Gamma(U, M_U^{\mathrm{gp}})$ for any $U \in (\mathrm{fs}/S)$, where M_U denotes the log structure of U and M_U^{gp} denotes the group envelope of M_U.

The étale sheaf $\mathbb{G}_{\mathrm{m,log}}$ is also a sheaf on S_{kfl} (see [13, Theorem 3.2] or [20, Corollary 2.22]).

By convention, for any sheaf of abelian groups F on S_{kfl} and a subgroup sheaf G of F on S_{kfl}, we denote by $(F/G)_{S_{\text{ét}}}$ (resp. $(F/G)_{S_{\text{fl}}}$, resp. $(F/G)_{S_{\text{két}}}$) the quotient sheaf on $S_{\text{ét}}$ (resp. S_{fl}, resp. $S_{\text{két}}$), while F/G denotes the quotient sheaf on S_{kfl}.

2.2 Finite Kummer flat group log schemes and log *p*-divisible groups

Definition 2.3 The category $(\mathrm{fin}/S)_{\mathrm{c}}$ is the full subcategory of the category of sheaves of abelian groups over $(\mathrm{fs}/S)_{\text{kfl}}$ consisting of objects which are representable by a classical finite flat group scheme over S. Here, classical means that the log structure of the representing log scheme is induced from S.

The category $(\mathrm{fin}/S)_{\mathrm{f}}$ is the full subcategory of the category of sheaves of abelian groups over $(\mathrm{fs}/S)_{\text{kfl}}$ consisting of objects which are representable by a classical finite flat group scheme over a Kummer log flat cover of S. For $F \in (\mathrm{fin}/S)_{\mathrm{f}}$, let $U \to S$ be a Kummer log flat cover of S such that $F_U := F \times_S U \in (\mathrm{fin}/U)_{\mathrm{c}}$. Then the rank of F is defined to be the rank of F_U over U.

The category $(\mathrm{fin}/S)_{\mathrm{r}}$ is the full subcategory of $(\mathrm{fin}/S)_{\mathrm{f}}$ consisting of objects which are representable by a log scheme over S.

Let $F \in (\mathrm{fin}/S)_{\mathrm{f}}$, and the Cartier dual of F is the sheaf $F^* := \mathcal{H}om_{S_{\text{kfl}}}(F, \mathbb{G}_{\mathrm{m}})$. By the definition of $(\mathrm{fin}/S)_{\mathrm{f}}$, it is clear that $F^* \in (\mathrm{fin}/S)_{\mathrm{f}}$.

The category $(\mathrm{fin}/S)_{\mathrm{d}}$ is the full subcategory of $(\mathrm{fin}/S)_{\mathrm{r}}$ consisting of objects whose Cartier dual also lies in $(\mathrm{fin}/S)_{\mathrm{r}}$.

The category $(\mathrm{fin}/S)_{\mathrm{r}}$ also has an alternative description as follows.

Proposition 2.4 *Let G be a sheaf of abelian groups on $(\mathrm{fs}/S)_{\text{kfl}}$. Then we have $G \in (\mathrm{fin}/S)_{\mathrm{r}}$ if and only if G satisfies the following condition.*

(⋆) *G is representable by an fs log scheme such that the structure morphism $G \to S$ is Kummer log flat and its underlying morphism of schemes is finite.*

Proof Assume that G satisfied the condition (⋆). We want to show that $G \in (\mathrm{fin}/S)_{\mathrm{r}}$. As the problem is classically étale local on S, we can assume that S is quasi-compact and that the log structure on S admits a global chart. Moreover, if S is quasi-compact, so is G. Hence, by [13, Theorem 2.7(2)], there is a Kummer log flat cover $S' \to S$ such that $G_{S'} \to S'$ is strict. By [14, Lemma 4.3.1], the morphism $G_{S'} \to S'$ is classically flat. The underlying morphism of schemes of $G_{S'} \to S'$ is also finite by [19, 1.10]. Hence, $G_{S'} \to S'$ is a classical finite flat group scheme over S'. It follows that $G \in (\mathrm{fin}/S)_{\mathrm{r}}$.

Conversely, assume that $G \in (\mathrm{fin}/S)_{\mathrm{r}}$, i.e., there is a Kummer log flat cover $S' \to S$ such that $G_{S'} \to S'$ is a classical finite flat group scheme. The property of being log-flat descends along Kummer log flat covers of the base, by [7, Theorem 0.1]. Hence, G is log

flat over S. Moreover, by [13, Proposition 2.7(1)], $G \to S$ is of Kummer type. To finish the proof, we are left with showing that the underlying map of schemes of $G \to S$ is finite, which follows from Proposition C.2.[2] ∎

Definition 2.5 Due to Proposition 2.4, we call an object of $(\text{fin}/S)_r$ a *finite Kummer log flat group log scheme*, or simply *finite Kummer flat group log scheme*, or even a *finite kfl group log scheme*.

Definition 2.6 Let p be a prime number. A log p-divisible group over S is a sheaf of abelian groups G on $(\text{fs}/S)_{\text{kfl}}$ satisfying:

(1) $G = \varinjlim_n G_n$ with $G_n := \ker(p^n : G \to G)$;
(2) $p : G \to G$ is surjective;
(3) $G_n \in (\text{fin}/S)_r$ for any $n > 0$.

We denote the category of log p-divisible groups over S by $(p\text{-div}/S)_r^{\log}$. The full subcategory of $(p\text{-div}/S)_r^{\log}$ consisting of objects G with $G_n \in (\text{fin}/S)_d$ for $n > 0$ will be denoted by $(p\text{-div}/S)_d^{\log}$. A log p-divisible group G with $G_n \in (\text{fin}/S)_c$ for $n > 0$ is clearly just a classical p-divisible group, and we denote the full subcategory of $(p\text{-div}/S)_d^{\log}$ consisting of classical p-divisible groups by $(p\text{-div}/S)_c^{\log}$.

Remark 2.7. In Definition 2.6, one can replace the condition (3) by simply requiring $G_1 \in (\text{fin}/S)_r$. This follows from the fact that the category $(\text{fin}/S)_r$ is closed under extension in the category of sheaves of abelian groups on $(\text{fs}/S)_{\text{kfl}}$ (see [26, Proposition A.1]). Similarly, a log p-divisible group G lies in $(p\text{-div}/S)_d^{\log}$ if $G_1 \in (\text{fin}/S)_d$; this also follows from [26, Proposition A.1]. Note that the corresponding statement for $(p\text{-div}/S)_c^{\log}$ does not hold.

3 Kato's classification theorem of log p-divisible groups

3.1 Standard extensions of log finite flat group objects

In this subsection, we further assume that S admits a global chart $P \to M_S$ with P an fs monoid. Throughout the article, if Q is a monoid, we also denote by Q the constant sheaf associated with Q.

Let $F'' \in (\text{fin}/S)_c$ be étale, let $F' \in (\text{fin}/S)_c$, and let n be a positive integer which kills both F' and F''. We denote by

$$\mathfrak{Ext}_{S_{\text{kfl}}}(F'', F') \quad (\text{resp. } \mathfrak{Ext}_{S_{\text{fl}}}(F'', F'))$$

the category of extensions of F'' by F' in $(\text{fin}/S)_r$ (resp. $(\text{fin}/S)_c$). Let $F''(1) := F'' \otimes_{\mathbb{Z}/n\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}(1)$, and we denote by

$$\mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\text{gp}}$$

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[2] This is actually contained in [13, Theorem 7.1]. However, the proof for the descent of finiteness there is referred to Tani's thesis [25], which is in Japanese. Therefore, we present a proof in Proposition C.2.

the discrete category associated with the set $\mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$. There is a natural functor

$$(3.1) \qquad \Phi : \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F') \times \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')$$

constructed as follows.

First, we construct a functor

$$(3.2) \qquad \Phi_2 : \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F').$$

For every element $a \in P^{\mathrm{gp}}$, let $\mathbf{M}_a$ be the log 1-motive $[\mathbb{Z} \xrightarrow{1 \mapsto a} \mathbb{G}_{\mathrm{m,log}}]$. Then $E_a := H^{-1}(\mathbf{M}_a \otimes_{\mathbb{Z}}^L \mathbb{Z}/n\mathbb{Z})$ fits into a short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z}(1) \to E_a \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

which splits Kummer flat locally. Hence, we get another short exact sequence

$$0 \to F''(1) \to E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F'' \to F'' \to 0$$

after tensoring with F''. For any $N \in \mathrm{Hom}_S(F''(1), F')$, we define $\Phi_2(N \otimes a)$ as the pushout

$$(3.3) \quad \begin{array}{ccccccccc}
0 & \longrightarrow & F''(1) & \longrightarrow & E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F'' & \longrightarrow & F'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle N} & & \downarrow & & \| & & \\
0 & \longrightarrow & F' & \longrightarrow & N_*(E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F'') & \longrightarrow & F'' & \longrightarrow & 0
\end{array}$$

Now, for any $\beta = \sum_i N_i \otimes a_i \in \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$, we define $\Phi_2(\beta) \in \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')$ as the Baer sum of the extensions $\Phi_2(N_i \otimes a_i)$. Now the functor Φ is defined as

$$\Phi(F^{\mathrm{cl}}, \beta) := F^{\mathrm{cl}} +_{\mathrm{Baer}} \Phi_2(\beta)$$

for $F^{\mathrm{cl}} \in \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F')$ and $\beta \in \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$. Here, the sum $+_{\mathrm{Baer}}$ denotes the Baer sum.

Definition 3.1 An extension F of F'' by F' in the category $(\mathrm{fin}/S)_{\mathrm{r}}$ is called *standard with respect to the given chart*, if it lies in the essential image of the functor Φ. We denote by $\mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}}$ the full subcategory of $\mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')$ consisting of standard extensions. Then the functor Φ induces a functor

$$(3.4) \qquad \Phi_{\mathrm{std}} : \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F') \times \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}}.$$

For any $F = \Phi(F^{\mathrm{cl}}, \beta)$ in $\mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}}$, we call β a *pseudo-monodromy* of the extension F. If the functor above is an equivalence of categories, we call β a *monodromy* of F.

In general, not every extension is standard, and a standard extension F admits more than one pseudo-monodromy. The name pseudo-monodromy comes from the fact that, for arbitrary charts, the decompositions one gets might be strange. In particular, a purely classical group scheme might have nonzero pseudo-monodromy.

Now we are going to investigate the essential image of the functor Φ.

Given an element $0 \to F' \to F \to F'' \to 0$ in $\mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')$, applying the direct image functor ε_*, we get a long exact sequence

$$(3.5) \qquad\qquad 0 \to F' \to F \to F'' \xrightarrow{\delta_F} R^1\varepsilon_* F'.$$

Note that if $F = \Phi(F^{\mathrm{cl}}, \beta)$, we have $\delta_{F^{\mathrm{cl}}} = 0$, so the connecting map δ_F only depends on β. By [20, Theorem 3.12] or [13, Theorem 4.1], we have

$$R^1\varepsilon_* F' = W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$$

with

$$W := \varinjlim_r \mathcal{H}om_S(\mathbb{Z}/r\mathbb{Z}(1), F').$$

Note that $W = \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F')$, since F' is n-torsion. Then the homomorphism δ_F can be rewritten as $\delta_F : F'' \to W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$, which is an element of the group $\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}})$. At the same time, the chart $P \to M_S$ induces a canonical homomorphism $\gamma_P : P^{\mathrm{gp}} \to (\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$, where P^{gp} denotes the group envelope of the monoid P. Therefore, we have a canonical homomorphism

$$(3.6) \qquad \mathrm{id}_W \otimes \gamma_P : W \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}} = R^1\varepsilon_* F'.$$

Remark 3.2. In [12, Section 3.7],

$$R^1\varepsilon_* F' = \varinjlim_r \mathcal{H}om_S(\mathbb{Z}/r\mathbb{Z}(1), F') \otimes (\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$$

is simply replaced by

$$\varinjlim_r \mathcal{H}om_S(\mathbb{Z}/r\mathbb{Z}(1), F') \otimes P^{\mathrm{gp}}$$

without further explanation. From the authors' point of view, this is not completely correct, as $(\mathbb{G}_{\mathrm{m},\log}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$ is not constant, even in the Henselian local case. This is one reason why we present a complete treatment to Kato's Theorem 3.8.

We have a canonical identification

$$(3.7) \quad \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) = \mathrm{Hom}_S(F'', W) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} = \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

Lemma 3.3 Let $\beta \in \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$, and let $F_\beta := \Phi_2(\beta)$. Applying the functor ε_* to the short exact sequence $0 \to F' \to F_\beta \to F'' \to 0$ gives rise to a long exact sequence $0 \to F' \to F_\beta \to F'' \xrightarrow{\delta_{F_\beta}} R^1\varepsilon_* F'$. Let $\bar{\delta}_{F_\beta}$ be the homomorphism corresponding to β under the canonical identification (3.7). Then the homomorphism δ_{F_β} factors as

$$
\begin{array}{ccc}
 & W \otimes_{\mathbb{Z}} P^{\mathrm{gp}} & \\
{\scriptstyle\bar{\delta}_{F_\beta}} \nearrow & \downarrow {\scriptstyle \mathrm{id}_W \otimes \gamma_P} & \\
F'' \xrightarrow[\delta_{F_\beta}]{} & R^1\varepsilon_* F' &
\end{array}
$$

Proof It suffices to consider the case $\beta = N \otimes a \in \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$.

We abbreviate $(\mathbb{G}_{m,\log}/\mathbb{G}_m)_{S_{fl}}$ as $\mathcal{G}$ in order to shorten formulas. Applying the functor ε_* to the short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z}(1) \to E_a \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

we get a map

$$\delta_{E_a} : \mathbb{Z}/n\mathbb{Z} \to R^1\varepsilon_*\mathbb{Z}/n\mathbb{Z}(1) = \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) \otimes_{\mathbb{Z}} \mathcal{G}, \bar{1} \mapsto \bar{1} \otimes a.$$

Applying the functor ε_* to $0 \to F''(1) \to E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F'' \to F'' \to 0$, we get another map

$$\delta_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''} : F'' = \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}/n\mathbb{Z}} F'' \to R^1\varepsilon_*F''(1) = R^1\varepsilon_*\mathbb{Z}/n\mathbb{Z}(1) \otimes_{\mathbb{Z}/n\mathbb{Z}} F''.$$

The map $\delta_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}$ is identical to the map $\delta_{E_a} \otimes 1_{F''}$. Applying the functor ε_* to the commutative diagram (3.3), we get the following commutative diagram:

$$
\begin{array}{ccccc}
F'' & \xrightarrow{\delta_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}} & R^1\varepsilon_*F''(1) & = & \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F''(1)) \otimes_{\mathbb{Z}} \mathcal{G} \\
\| & & \downarrow{\scriptstyle R^1\varepsilon_*N} & & \downarrow{\scriptstyle N_* \otimes \mathrm{id}_{\mathcal{G}}} \\
F'' & \xrightarrow{\delta_{F_\beta}} & R^1\varepsilon_*F' & = & \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F') \otimes_{\mathbb{Z}} \mathcal{G}
\end{array}
$$

It is clear that δ_{E_a} factors through

$$\bar{\delta}_{E_a} : \mathbb{Z}/n\mathbb{Z} \to \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) \otimes_{\mathbb{Z}} P^{gp}, \bar{1} \mapsto \bar{1} \otimes a.$$

Therefore, $\delta_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}$ factors through

$$\bar{\delta}_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''} : F'' = \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}/n\mathbb{Z}} F'' \to \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) \otimes_{\mathbb{Z}} P^{gp} \otimes_{\mathbb{Z}/n\mathbb{Z}} F'',$$

where $\bar{\delta}_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}$ is defined as the map $\bar{\delta}_{E_a} \otimes 1_{F''}$. We can rewrite $\bar{\delta}_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}$ as

$$F'' \to \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F''(1)) \otimes_{\mathbb{Z}} P^{gp}.$$

We have an obvious commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F''(1)) \otimes_{\mathbb{Z}} P^{gp} & \longrightarrow & R^1\varepsilon_*F''(1) \\
\downarrow{\scriptstyle N_* \otimes \mathrm{id}_{P^{gp}}} & & \downarrow{\scriptstyle R^1\varepsilon_*N} \\
\mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F') \otimes_{\mathbb{Z}} P^{gp} & \longrightarrow & R^1\varepsilon_*F'
\end{array}
$$

It follows that δ_{F_β} factors through $\bar{\delta}_{F_\beta} := (N_* \otimes \mathrm{id}_{P^{gp}}) \circ \bar{\delta}_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}$. This finishes the proof. ∎

Remark 3.4. In the proof of Lemma 3.3, the map

$$\bar{\delta}_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''} : F'' = F'' \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F''(1)) \otimes_{\mathbb{Z}} P^{gp}$$

corresponds to

$$\mathrm{id}_{F''(1)} \in \mathrm{Hom}_S(F'', \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F''(1))) = \mathrm{Hom}_S(F''(1), F''(1))$$

and $\mathbb{Z} \to P^{gp}, 1 \mapsto a$. Therefore, $(N_* \otimes \mathrm{id}_{P^{gp}}) \circ \bar{\delta}_{E_a \otimes_{\mathbb{Z}/n\mathbb{Z}} F''}$ is nothing but

$$N \otimes_{\mathbb{Z}} a \in \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{gp}.$$

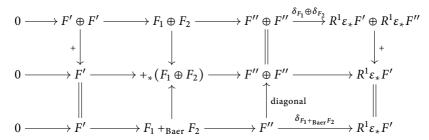**Proposition 3.5**   *Let F be an extension of F'' by F' in the category $(\mathrm{fin}/S)_{\mathrm{r}}$. Then F is a standard extension with respect to the given chart if and only if the homomorphism δ_F from* (3.5) *factors*

$$
\begin{array}{ccc}
 & & W \otimes P^{\mathrm{gp}} \\
 & {\scriptstyle \bar{\delta}_F}\nearrow & \downarrow \\
0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \xrightarrow{\;\delta_F\;} R^1 \varepsilon_* F'
\end{array}
$$

through the canonical homomorphism $W \otimes P^{\mathrm{gp}} \to W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}} = R^1 \varepsilon_* F'$.

Proof   The "only-if" part follows from Lemma 3.3 easily. Now we assume that δ_F has a factorization as above. We want to express F as a standard extension.

First, we digress to investigate the behavior of the map (3.9) with respect to Baer sums. Let F_1 and F_2 be two extensions of F'' by F'. The following commutative diagram

$$
\begin{array}{ccccccccc}
0 \longrightarrow & F' \oplus F' & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F'' \oplus F'' & \xrightarrow{\delta_{F_1} \oplus \delta_{F_2}} & R^1\varepsilon_* F' \oplus R^1\varepsilon_* F'' \\
 & {\scriptstyle +}\downarrow & & \downarrow & & \| & & {\scriptstyle +}\downarrow \\
0 \longrightarrow & F' & \longrightarrow & +_*(F_1 \oplus F_2) & \longrightarrow & F'' \oplus F'' & \longrightarrow & R^1\varepsilon_* F' \\
 & \| & & \uparrow & & {\scriptstyle \text{diagonal}}\uparrow & & \| \\
0 \longrightarrow & F' & \longrightarrow & F_1 +_{\mathrm{Baer}} F_2 & \longrightarrow & F'' & \xrightarrow{\delta_{F_1 +_{\mathrm{Baer}} F_2}} & R^1\varepsilon_* F'
\end{array}
$$

implies that $\delta_{F_1 +_{\mathrm{Baer}} F_2} = \delta_{F_1} + \delta_{F_2}$, i.e., the formation of δ_- is compatible with the Baer sum.

Now we go back to the proof of the proposition. By abuse of notation, we use the same notation $\bar{\delta}_F$ for the element in $\mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$ corresponding to $\bar{\delta}_F$ in $\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}})$. Let $F_{\bar{\delta}_F} := \Phi_2(\bar{\delta}_F)$, and let $F^{\mathrm{cl}} := F -_{\mathrm{Baer}} F_{\bar{\delta}_F}$. By Lemma 3.3, the connection map $F'' \to R^1\varepsilon_* F'$ for F^{cl} is $\delta_F - \delta_F = 0$. Therefore, F^{cl} lies in $\mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F')$, and $F = F^{\mathrm{cl}} +_{\mathrm{Baer}} F_{\bar{\delta}_F} = F^{\mathrm{cl}} +_{\mathrm{Baer}} \Phi_2(\bar{\delta}_F)$ is standard with respect to the given chart. ∎

By Proposition 3.5, it is clear that the homomorphism

$$
(3.8) \qquad \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \xrightarrow{\;\alpha\;} \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}})
$$

induced by $\gamma_P : P^{\mathrm{gp}} \to (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$ is important for understanding standard extensions.

For any $U \in (\mathrm{fs}/S)$, both P^{gp} and $(\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$ are constructible $\mathbb{Z}$-modules on the small étale site of U (see [22, Lemma 3.5(ii)]). By [17, Chapter V, Remark 1.7(e)] and Lemma A.1, the sheaf $W = \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), F')$ is a constructible $\mathbb{Z}$-module for the étale topology. By [24, Tag 095I], the tensor product of W and P^{gp} for the étale topology is a constructible $\mathbb{Z}$-module; hence, it is representable by an algebraic space by [2, Exposé IX, Proposition 2.7], and therefore automatically a sheaf on $(\mathrm{fs}/S)_{\mathrm{fl}}$. It follows that the tensor product $W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$ on $(\mathrm{fs}/S)_{\mathrm{fl}}$ coincides with the corresponding tensor product for the étale topology. So we do not have to distinguish

for which topology we take the tensor product of W and P^{gp}, and just simply write $W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$. We have similar result for $W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$.

Recall that the log structure in this article is defined in the étale topology. Let $M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}$ be the quotient on the small étale site of S. Since F'' is étale locally constant, the homomorphism α is identified with

$$\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \xrightarrow{\alpha} \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})$$

in which the Hom-groups are taken in the small étale site of S. By abuse of notation, we denote this homomorphism also by α.

The chart $P \to M_S$ induces a canonical epimorphism $P^{\mathrm{gp}} \to M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}$. Let K be the kernel of this homomorphism. We then get a short exact sequence

$$0 \to K \to P^{\mathrm{gp}} \to M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times} \to 0$$

of sheaves on the small étale site of S. Since the stalks of K, P^{gp}, and $M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}$ are free abelian groups, the canonical sequence

$$(3.9) \qquad 0 \to W \otimes_{\mathbb{Z}} K \to W \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to W \otimes_{\mathbb{Z}} M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times} \to 0$$

is exact on all stalks at geometric points of S. It follows that the sequence itself is exact.

Now, applying the functor $\mathrm{Hom}_S(F'', -)$ to (3.9), we get another exact sequence

(3.10)
$$0 \to \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} K) \to \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \xrightarrow{\alpha} \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}).$$

Proposition 3.6 *Let the notation and the assumptions be as above.*

(1) *Assume that* $\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} K) = 0$, *i.e., the map* α *from (3.10) is injective. Then the functor (3.4)*

$$\Phi_{\mathrm{std}} : \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F') \times \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}}$$

is an equivalence of categories.

(2) *Assume that* α *is an isomorphism. Then*

$$\mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}} = \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F'),$$

whence the functor Φ *(3.1) is an equivalence of categories.*

Proof We claim that Φ_{std} admits a quasi-inverse, given by

$$(3.11) \qquad \begin{aligned} \Psi_{\mathrm{std}} : \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}} &\to \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F') \times \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}, \\ F &\mapsto (F -_{\mathrm{Baer}} \Phi_2(\beta_F), \beta_F) \end{aligned}$$

where β_F, under the identification

$$\mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \cong \mathrm{Hom}_S(F'', \mathcal{Hom}(\mathbb{Z}/n\mathbb{Z}(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}),$$

corresponds to $\bar{\delta}_F$, which is the unique lift of the connecting map $\delta_F : F'' \to R^1\varepsilon_*F'$ of F. Note that the uniqueness of β_F, which amounts to the uniqueness of $\bar{\delta}_F$, is guaranteed by $\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} K) = 0$.

Since the functor (3.4) is clearly essentially surjective, it suffices to show that $\Psi_{\mathrm{std}} \circ \Phi_{\mathrm{std}} = 1$.

By Lemma 3.3, for $F = \Phi_2(\beta)$ with $\beta \in \mathrm{Hom}_S(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$, we have $\beta_F = \beta$. Hence, for any $F \in \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}}$, we have

$$\beta_{F-\mathrm{Baer}\,\Phi_2(\beta_F)} = \beta_F - \beta_{\Phi_2(\beta_F)} = \beta_F - \beta_F = 0.$$

This implies that the direct image of the short exact sequence

$$0 \to F' \to (F -_{\mathrm{Baer}} \Phi_2(\beta_F)) \to F'' \to 0$$

remains exact; therefore, $F -_{\mathrm{Baer}} \Phi_2(\beta_F)$ is indeed classical. We also have

$$\Psi_{\mathrm{std}}(\Phi_{\mathrm{std}}(F^{\mathrm{cl}}, \beta)) = \Psi_{\mathrm{std}}(F^{\mathrm{cl}} +_{\mathrm{Baer}} \Phi_2(\beta)) = (F^{\mathrm{cl}}, \beta).$$

This proves part (1).

For (2), note that under the assumption that α is an isomorphism, every extension of F'' by F' is standard with respect to the given chart by Proposition 3.5. So part (2) follows from part (1). ∎

Lemma 3.7 *Assume that the underlying scheme of S is* $\mathrm{Spec}\,A$ *with A a Noetherian Henselian local ring, and S admits a global chart $P \to M_S$ such that the induced map $P \to M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}^{\times}$ is an isomorphism for the closed point s of S. Then the canonical homomorphism α (3.8) is an isomorphism, so the group $\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}})$ can canonically be identified with the group*

$$\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) = \mathrm{Hom}_S(F'', W) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

Proof   It suffices to prove that the map α from (3.10) is an isomorphism. Then the result follows from $P^{\mathrm{gp}} \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$.

We first deal with the case that the finite étale group scheme F'' is constant. Then, for any sheaf G on the small étale site of S, the group $\mathrm{Hom}_S(F'', G)$ is determined by the group $\mathrm{Hom}(F''(S), G(S))$. On the other hand, we have $\Gamma(S, G) = \Gamma(s, G|_s)$ by Gabber's theorem (see [24, Tag 09ZI]). This implies that the exact sequence

$$0 \to \Gamma(S, W \otimes_{\mathbb{Z}} K) \to \Gamma(S, W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \xrightarrow{\beta} \Gamma(S, W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}))$$

can be identified with the exact sequence

$$0 \to \Gamma(s, (W \otimes_{\mathbb{Z}} K)|_s) \to \Gamma(s, (W \otimes_{\mathbb{Z}} P^{\mathrm{gp}})|_s) \to \Gamma(s, (W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}))|_s),$$

where $(-)|_s$ denotes the pullback from the small étale site of S to that of s. By the property of the given chart $P \to M_S$, we have $P^{\mathrm{gp}}|_s \cong (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})|_s$. It follows that the map β is an isomorphism. Therefore, the map α is an isomorphism.

In general, choose a finite étale cover $\tilde{S} = \mathrm{Spec}\,\tilde{A} \to \mathrm{Spec}\,A$ with $\tilde{A}$ Henselian local, such that $F''|_{\bar{s}}$ is a constant group scheme. Let

$$f \in \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})).$$

Then $\tilde{f} := f \times_S \tilde{S}$ lifts to a unique $\tilde{g} \in \mathrm{Hom}_{\tilde{S}}(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}})$ by the previous case. Since $\tilde{S} \times_S \tilde{S}$ is a disjoint union of Henselian local schemes which are finite étale over $\tilde{S}$, we have $p_1^* \tilde{g} = p_2^* \tilde{g}$ by the previous case again, where p_1 and p_2 are the projections from $\tilde{S} \times_S \tilde{S}$ to its two factors. It follows that $\tilde{g}$ descends to a unique element of $\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}})$. This finishes the proof. ∎

Theorem 3.8 (Kato) *Let the assumptions be as in Lemma* 3.7*. Then*

$$\mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')_{\mathrm{std}} = \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')$$

and the functor (3.1)

$$\Phi : \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F') \times \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F')$$

is an equivalence of categories with inverse (3.11)

$$\Psi : \mathfrak{Ext}_{S_{\mathrm{kfl}}}(F'', F') \to \mathfrak{Ext}_{S_{\mathrm{fl}}}(F'', F') \times \mathfrak{Hom}(F''(1), F') \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

Proof   This follows from Lemma 3.7 and Proposition 3.6.   ∎

3.2 The connected-étale sequence

Assume that $S = \mathrm{Spec}(R)$ is a local Henselian ring with residue characteristic p. The results in this section are all due to Kato, and we will only sketch the proofs.

For any $G \in (\mathrm{fin}/S)_{\mathrm{f}}$, there is a unique exact sequence

$$0 \to G^{\circ} \to G \to G^{\text{ét}} \to 0$$

which restricts to the usual connected-étale sequence over any finite Kummer log flat cover, over which G becomes a classical flat group scheme. To get this, one can descend the usual connected-étale sequence from some cover over which G is classical (the sequence descends because it is unique and compatible with base change).

One now has the following result.

Proposition 3.9   *Let $G \in (\mathrm{fin}/S)_{\mathrm{d}}$. Assume that the residue characteristic p is positive, and G is killed by a power of p. Then G° and $G^{\text{ét}}$ are classical finite flat group schemes.*

Proof   One has to show that a connected finite kfl log group scheme $f : H \to S$ is classical. For this, one looks at the unit map $e : S \to H$, and sees that for any geometric point $\bar{s}$ of S, one gets that the identity factors as

$$(M_S/\mathcal{O}_S^{\times})_{\bar{s}} \to (M_H/\mathcal{O}_H^{\times})_{e(\bar{s})} \to (M_S/\mathcal{O}_S^{\times})_{\bar{s}}.$$

Using that $H \to S$ is Kummer, one concludes that $f^{-1}(M_S/\mathcal{O}_S^{\times}) \to M_H/\mathcal{O}_H^{\times}$ is an isomorphism.

From this, it then follows that G° is classical. As the dual $(G^{\text{ét}})^D$ of $G^{\text{ét}}$ is representable by assumption and clearly connected, it is classical. Hence, $G^{\text{ét}}$ is classical as well.   ∎

Recall the following definition. Let Z be an fs log scheme, and let z be a point of Z. A chart $P \to M_Z$ of Z is called *neat* at z, if the induced map $P \xrightarrow{\cong} M_{Z,\bar{z}}/\mathcal{O}_{Z,\bar{z}}^{\times}$ is an isomorphism (see [21, Chapter II, Definition 2.3.1]). Neat charts exist étale locally by [21, Chapter II, Proposition 2.3.7].

We also denote by $(\mathrm{fin}/S)_{\mathrm{d}}^p$ the subcategory of $(\mathrm{fin}/S)_{\mathrm{d}}$ whose objects are p-power torsion. Using Theorem 3.8, one then gets the following corollary.

Corollary 3.10   *Assume that $P \to M_S$ is a neat chart and that the residue characteristic p is positive. Then there is an equivalence of categories between $(\mathrm{fin}/S)_{\mathrm{d}}^p$ and*

the category of pairs (G^{cl}, N), where G^{cl} is a classical extension of $G^{\mathrm{ét}}$ by G°, and $N \in \mathrm{Hom}(G^{\mathrm{ét}}(1), G^{\circ}) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$.

3.3 Extensions are always standard étale locally

In the previous section, we studied standard extensions of classical finite flat group schemes after fixing a chart on the base. In this section, we show that any extension of one classical finite étale group scheme by another classical finite flat group scheme is always étale locally standard with respect to a suitable chosen local chart.

Theorem 3.11 *Let S be an fs log scheme whose underlying scheme is locally Noetherian. Let $F', F'' \in (\mathrm{fin}/S)_c$ with F'' étale, and let $F \in (\mathrm{fin}/S)_r$ be an extension of F'' by F'. Then, for any $s \in S$, there exists an étale neighborhood $\tilde{S}$ of s such that the log structure of S admits a chart $P_{\tilde{S}} \to M_{\tilde{S}} = (M_S)|_{\tilde{S}}$ on $\tilde{S}$ with P an fs monoid, and F is standard with respect to this chart.*

Proof   By shrinking S étale locally if necessary, we assume that S admits a chart $P \to M_S$ such that the induced map $P \to M_{S,\bar{s}}/\mathcal{O}^{\times}_{S,\bar{s}}$ is an isomorphism. We adopt the notation from Section 3.1. By Proposition 3.5, we are left with lifting

$$\delta_F : F'' \to R^1\varepsilon_* F' = W \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$$

to a homomorphism $\bar{\delta}_F : F'' \to W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$ étale locally around s.

The short exact sequence (3.10) extends to a longer exact sequence

$$\mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \xrightarrow{\alpha} \mathrm{Hom}_S(F'', W \otimes_{\mathbb{Z}} M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}) \xrightarrow{\gamma} \mathrm{Ext}^1_{S_{\mathrm{ét}}}(F'', W \otimes_{\mathbb{Z}} K).$$

The local-to-global extension spectral sequence gives rise to an exact sequence

$$0 \to H^1(S_{\mathrm{ét}}, \mathcal{H}om_S(F'', W \otimes_{\mathbb{Z}} K)) \to \mathrm{Ext}^1_{S_{\mathrm{ét}}}(F'', W \otimes_{\mathbb{Z}} K) \to$$
$$\Gamma(S, \mathcal{E}xt^1_{S_{\mathrm{ét}}}(F'', W \otimes_{\mathbb{Z}} K)).$$

We claim that $\gamma(\delta_F)$ has trivial image in $\Gamma(S, \mathcal{E}xt^1_{S_{\mathrm{ét}}}(F'', W \otimes_{\mathbb{Z}} K))$. It suffices to show that $\gamma(\delta_F)$ vanishes at all stalks of $\mathcal{E}xt^1_{S_{\mathrm{ét}}}(F'', W \otimes_{\mathbb{Z}} K)$. Let t be any point of S with $\bar{t}$ a geometric point above t, and let $S_{\bar{t}}$ be the strict Henselization of S at $\bar{t}$. Pulling back the short exact sequence (3.9) to $S_{\bar{t}}$ and then taking global section, we get a short exact sequence

$$0 \to \Gamma(S_{\bar{t}}, W \otimes_{\mathbb{Z}} K) \to \Gamma(S_{\bar{t}}, W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \to \Gamma(S_{\bar{t}}, W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})) \to 0.$$

By [24, Tag 09ZH], this exact sequence can be identified with the short exact sequence

$$0 \to \Gamma(\bar{t}, (W \otimes_{\mathbb{Z}} K)|_{\bar{t}}) \to \Gamma(\bar{t}, (W \otimes_{\mathbb{Z}} P^{\mathrm{gp}})|_{\bar{t}}) \to \Gamma(\bar{t}, (W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}))|_{\bar{t}}) \to 0.$$

Now, over the geometric point $\bar{t}$, the epimorphism $P^{\mathrm{gp}} \to M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}$ admits a section. It follows that the epimorphism $\Gamma(S_{\bar{t}}, W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \to \Gamma(S_{\bar{t}}, W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}))$ also admits a section. This forces the homomorphism

$$\mathrm{Hom}_{S_{\bar{t}}}(F'', W \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \xrightarrow{\alpha} \mathrm{Hom}_{S_{\bar{t}}}(F'', W \otimes_{\mathbb{Z}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}))$$

to be surjective. It follows that the stalk of $\gamma(\delta_F)$ at $\bar{t}$ vanishes.

Now we see that $\gamma(\delta_F)$ lies in the subgroup $H^1(S, \mathcal{H}om_S(F'', W \otimes_{\mathbb{Z}} K))$ of $\mathrm{Ext}^1_{S_{\text{ét}}}(F'', W \otimes_{\mathbb{Z}} K)$. Hence, passing to an étale cover of S, $\gamma(\delta_F)$ vanishes; in other words, δ_F lifts to a homomorphism $\bar\delta_F : F'' \to W \otimes_{\mathbb{Z}} P^{\text{gp}}$ after passing to a suitable étale cover of S. This finishes the proof. ∎

3.4 Standard extensions of log *p*-divisible groups

First, we assume that S admits a global chart $P \to M_S$ with P an fs monoid.

Let $H' = \varinjlim_n H'_n$, $H'' = \varinjlim_n H''_n$ be two objects in $(p\text{-div}/S)^{\log}_c$, and we assume that H'' is étale. We denote by

$$\mathfrak{Ext}_{S_{\text{kfl}}}(H'', H') \quad (\text{resp. } \mathfrak{Ext}_{S_{\text{fl}}}(H'', H'))$$

the category of extensions of H'' by H' in $(p\text{-div}/S)^{\log}_r$ (resp. $(p\text{-div}/S)^{\log}_c$). Let

$$H''(1) := \varinjlim_n H''_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}(1),$$

and we denote by

$$\mathfrak{Hom}(H''(1), H') \otimes_{\mathbb{Z}} P^{\text{gp}}$$

the discrete category associated with the set $\mathrm{Hom}_S(H''(1), H') \otimes_{\mathbb{Z}} P^{\text{gp}}$. Let

$$H^{\text{cl}} = \varinjlim_n H^{\text{cl}}_n \in \mathfrak{Ext}_{S_{\text{fl}}}(H'', H'),$$

and $\beta \in \mathfrak{Hom}(H''(1), H') \otimes_{\mathbb{Z}} P^{\text{gp}}$. The element β induces a compatible system

$$\{\beta_n \in \mathfrak{Hom}(H''_n(1), H'_n) \otimes_{\mathbb{Z}} P^{\text{gp}}\}_n.$$

We apply the functor (3.1) to the pair $(H^{\text{cl}}_n, \beta_n)$ for each $n \geq 1$ and change the notation for the functor (3.1) (resp. (3.2)) from Φ (resp. Φ_2) to Φ^n (resp. Φ^n_2) in order to indicate its relation with $(H^{\text{cl}}_n, \beta_n)$. Then we get a compatible system $\{\Phi^n(H^{\text{cl}}_n, \beta_n)\}_n$ with

$$\Phi^n(H^{\text{cl}}_n, \beta_n) = H^{\text{cl}}_n +_{\text{Baer}} \Phi^n_2(\beta_n) \in \mathfrak{Ext}_{S_{\text{kfl}}}(H''_n, H'_n).$$

Note that $\Phi_2(\beta) := \varinjlim_n \Phi^n_2(\beta_n)$ is an object of $(p\text{-div}/S)^{\log}_r$. Therefore,

$$\varinjlim_n \Phi^n(H^{\text{cl}}_n, \beta_n) = \varinjlim_n (H^{\text{cl}}_n +_{\text{Baer}} \Phi^n_2(\beta_n))$$

lies in $(p\text{-div}/S)^{\log}_r$. We denote $\varinjlim_n \Phi^n(H^{\text{cl}}_n, \beta_n)$ by $\Phi(H^{\text{cl}}, \beta)$. The association of $\Phi(H^{\text{cl}}, \beta)$ with the pair (H^{cl}, β) gives rise to a functor

$$(3.12) \qquad \Phi : \mathfrak{Ext}_{S_{\text{fl}}}(H'', H') \times \mathfrak{Hom}(H''(1), H') \otimes_{\mathbb{Z}} P^{\text{gp}} \to \mathfrak{Ext}_{S_{\text{kfl}}}(H'', H').$$

Note that we use the same notation for both the functor (3.12) and the functor (3.1), but the potential confusion between the two functors can be cleared in the context.

Definition 3.12 Let $H', H'' \in (p\text{-div}/S)^{\log}_c$ with H'' étale. An extension H of H'' by H' in the category $(p\text{-div}/S)^{\log}_r$ is called *standard with respect to the given*

chart $P \to M_S$, if it lies in the essential image of the functor (3.12). We denote by $\mathfrak{Ext}_{S_{\mathrm{kfl}}}(H'', H')_{\mathrm{std}}$ the full subcategory of $\mathfrak{Ext}_{S_{\mathrm{kfl}}}(H'', H')$ consisting of the standard extensions, and the functor (3.12) induces a functor

$$(3.13) \qquad \mathfrak{Ext}_{S_{\mathrm{fl}}}(H'', H') \times \mathfrak{Hom}(H''(1), H') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(H'', H')_{\mathrm{std}}.$$

If $H \cong \Phi(H^{\mathrm{cl}}, \beta)$, we call β a *pseudo-monodromy of H with respect to the given chart* $P \to M_S$. If the functor (3.13) is an equivalence, the pseudo-monodromy β is uniquely determined. In this case, we call β the *monodromy of H with respect to the given chart* $P \to M_S$.

Theorem 3.13 (Kato)   *Let S be an fs log scheme whose underlying scheme is* Spec *A for a Noetherian Henselian local ring A, and let* $P \to M_S$ *be a global chart such that the induced map* $P \to M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}^{\times}$ *is an isomorphism for the closed point s of S. Let* $H', H'' \in (p\text{-}div/S)_c^{\log}$ *with H'' étale. Then the functor*

$$\Phi : \mathfrak{Ext}_{S_{\mathrm{fl}}}(H'', H') \times \mathfrak{Hom}(H''(1), H') \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to \mathfrak{Ext}_{S_{\mathrm{kfl}}}(H'', H')$$

from (3.12) *is an equivalence of categories.*

Proof   This follows from the equivalences on the finite levels (see Theorem 3.8).   ∎

Let S be as in Theorem 3.13, and we further assume that the residue characteristic p is positive. Assume that $H \in (p\text{-}div/S)_d^{\log}$. Similarly to Section 3.2, there is an exact connected-étale sequence

$$0 \to H^{\circ} \to H \to H^{\mathrm{\acute{e}t}} \to 0,$$

where H° (resp. $H^{\mathrm{\acute{e}t}}$) is a connected (resp. étale) classical p-divisible group. This one gets by observing that the connected-étale sequences for the finite levels are compatible with the transition maps. Using this and Corollary 3.10, one gets a complete classification of log-p-divisible groups in $(p\text{-}div/S)_d^{\log}$.

Corollary 3.14 (Kato)   *Let S be an fs log scheme whose underlying scheme is* Spec *A for a Noetherian Henselian local ring A with residue characteristic* $p > 0$, *and let* $P \to M_S$ *be a global chart such that the induced map* $P \to M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}$ *is an isomorphism for the closed point s of S. Then any* $H \in (p\text{-}div/S)_d^{\log}$ *is constructed out of a unique object in*

$$\mathfrak{Ext}_{S_{\mathrm{fl}}}(H^{\mathrm{\acute{e}t}}, H^{\circ}) \times \mathfrak{Hom}(H^{\mathrm{\acute{e}t}}(1), H^{\circ}) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$$

from the equivalence of categories (3.12).

Remark 3.15.   Let S be a general fs log scheme whose underlying scheme is locally Noetherian, and let $H = (H_n)_n$ be extension of an étale classical p-divisible group H'' by another classical p-divisible group H'. For each n, we can make H_n standard étale locally on S by Theorem 3.11. With some more effort, we can make all H_n standard at the same time over some étale neighborhood of any point $s \in S$. However, it is not clear to the authors how to construct a compatible system of monodromies at all levels.

We can, however, use the theorem to prove that log p-divisible groups are formally log smooth.

Theorem 3.16 *Let S be a locally Noetherian fs log scheme on which p is locally nilpotent, and let H be a log p-divisible group over S, which lies in $(p\text{-}div/S)_{\mathrm{d}}^{\log}$. Then H is formally log smooth, i.e., for any strict closed square-zero thickening $T_0 \hookrightarrow T$ in (fs/S), any element of $H(T_0)$ can be lifted to an element of $H(T)$ étale locally on T.*

Proof We may assume that T is affine. Now we work locally around a geometric point $\bar{t}$ of T. Let $\overline{T}$ be the spectrum of the étale stalk of the structure sheaf $\mathcal{O}_{T,\bar{t}}$, which is a strict Henselian local ring. Then $\overline{T} = \varprojlim_{(V,\bar{t})} V$ is the limit along the étale neighborhoods of $\bar{t}$. Now $H = \varinjlim H_n$, where H_n are finite kfl log group schemes. So the H_n are in particular of finite presentation and therefore limit preserving functors. Here, on the étale neighborhoods, we put the log structure that is induced from the log structure on T. Hence, we get

$$
\begin{aligned}
H(\overline{T}) &= \varinjlim_n H_n(\overline{T}) \\
&= \varinjlim_n H_n(\varprojlim_V V) \\
&= \varinjlim_n \varinjlim_V H_n(V) \\
&= \varinjlim_V H(V).
\end{aligned}
$$

Here, the first equality is due to quasi-compactness of $\overline{T}$ and the last equality follows formally since colimits commute with colimits. In particular, any section of $H(\overline{T})$ extends to an étale neighborhood.

From this, we see that we can assume that T is strict Henselian local. Further, by replacing H by $H \times_S T$, we may work with the base T. It suffices to prove the surjectivity of the map $H(T) \to H(T_0)$.

Fix a neat chart P of $\mathcal{M}_T$, such that $P \cong M_{T,\bar{t}}/\mathcal{O}_{T,\bar{t}}^\times = M_{T_0,\bar{t}}/\mathcal{O}_{T_0,\bar{t}}^\times$, where now $\bar{t}$ denotes the closed point of T. Now let $0 \to H^\circ \to H \to H^{\text{ét}} \to 0$ be the connected-étale sequence of H, and we have that H° and $H^{\text{ét}}$ are classical. By Corollary 3.14, H admits a decomposition $H_T = H^{\text{cl}} +_{\text{Baer}} H^N$ into the Baer sum of a classical p-divisible group $H^{\text{cl}} \in \mathfrak{Ext}_{S_{\text{fl}}}(H^{\text{ét}}, H^\circ)$ and a log p-divisible group H^N constructed out of the monodromy $N \in \mathfrak{Hom}(H^{\text{ét}}(1), H^\circ) \otimes_{\mathbb{Z}} P^{\text{gp}}$ of H. We have the following canonical commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^\circ(T) & \longrightarrow & H(T) & \longrightarrow & H^{\text{ét}}(T) & \overset{\delta}{\longrightarrow} & H^1_{\text{kfl}}(T, H^\circ) \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & {\scriptstyle \cong}\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \lambda} \\
0 & \longrightarrow & H^\circ(T_0) & \longrightarrow & H(T_0) & \longrightarrow & H^{\text{ét}}(T_0) & \overset{\delta_0}{\longrightarrow} & H^1_{\text{kfl}}(T_0, H^\circ)
\end{array}
$$

with exact rows. Since both H° and $H^{\text{ét}}$ are classically formally smooth by [16, Chapter II, 3.3.13], both α and γ are surjective. Moreover, since $H^{\text{ét}}$ is étale, γ is even an isomorphism. To show the surjectivity of β, it suffices to show that the restriction $\lambda|_{\text{im}(\delta)}$ of λ to $\text{im}(\delta)$ is injective by the five lemma.

Let $\varepsilon : (\mathrm{fs}/T)_{\mathrm{kfl}} \to (\mathrm{fs}/T)_{\mathrm{fl}}$ be the canonical forgetful map between the two sites. By Proposition D.1, the choice of P also provides a splitting

$$(3.14) \qquad H^1_{\mathrm{kfl}}(T, H^\circ) \cong H^1_{\mathrm{fl}}(T, H^\circ) \oplus \mathrm{Hom}_T(\mathbb{Z}_p(1), H^\circ) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

For all $n \geq 1$, the sheaf $\mathcal{H}om_T(\mathbb{Z}/p^n\mathbb{Z}(1), H^\circ_n)$ is representable by an étale group scheme over T by Lemma A.1. Thus, we have

$$\Gamma(T, \mathcal{H}om_T(\mathbb{Z}/p^n\mathbb{Z}(1), H^\circ_n)) = \Gamma(T_0, \mathcal{H}om_T(\mathbb{Z}/p^n\mathbb{Z}(1), H^\circ_n)).$$

Therefore, the map λ is injective on $\mathrm{Hom}_T(\mathbb{Z}_p(1), H^\circ) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$.

Recall that we have $H = H^{\mathrm{cl}} +_{\mathrm{Baer}} H^N$. Let $\mathrm{pr}_{\mathrm{cl}}$ (resp. pr_N) be the projection from H^{cl} (resp. H^N) to $H^{\mathrm{\acute{e}t}}$. For any $x \in H^{\mathrm{\acute{e}t}}(T)$, $\delta(x)$ can be represented by the sum of the two torsors $\mathrm{pr}_{\mathrm{cl}}^{-1}(x) \in H^1_{\mathrm{fl}}(T, H^\circ)$ and $\mathrm{pr}_N^{-1}(x) \in H^1_{\mathrm{kfl}}(T, H^\circ)$, and this sum is the direct decomposition with respect to (3.14). Let x be such that $\delta(x) \in \ker(\lambda)$. Since λ is injective on $\mathrm{Hom}_T(\mathbb{Z}_p(1), H^\circ) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$, it follows that $\delta(x)$ is represented by its classical part $\mathrm{pr}_{\mathrm{cl}}^{-1}(x)$, and $0 = \lambda(\delta(x)) = \lambda(\mathrm{pr}_{\mathrm{cl}}^{-1}(x))$. Since H^{cl} is formally smooth by [16, Chapter II, 3.3.13], the homomorphism β^{cl} in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^\circ(T) & \longrightarrow & H^{\mathrm{cl}}(T) & \longrightarrow & H^{\mathrm{\acute{e}t}}(T) & \xrightarrow{\delta^{\mathrm{cl}}} & H^1_{\mathrm{fl}}(T, H^\circ) \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta^{\mathrm{cl}}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \lambda_1} \\
0 & \longrightarrow & H^\circ(T_0) & \longrightarrow & H^{\mathrm{cl}}(T_0) & \longrightarrow & H^{\mathrm{\acute{e}t}}(T_0) & \xrightarrow{\delta^{\mathrm{cl}}_0} & H^1_{\mathrm{fl}}(T_0, H^\circ)
\end{array}
$$

is surjective. An easy diagram chasing tells us that $\delta(x) = \delta^{\mathrm{cl}}(x)$ is trivial (here the bijectivity of γ is used). Hence, $\lambda|_{\mathrm{im}(\delta)}$ is injective. This finishes the proof. ∎

4 Log p-divisible groups associated with log 1-motives

In this section, we want to study the finite Kummer flat group log schemes associated with log 1-motives. In this case, we will see that the splittings (in the sense of the previous section) are induced from the monodromy pairings of log 1-motives (see Definition 4.1).

Let S be an fs log scheme with its underlying scheme locally Noetherian. Let G be an extension of an abelian scheme by a torus T, and let G_{log} be the pushout of G in the category of sheaves of abelian groups on $(\mathrm{fs}/S)_{\mathrm{kfl}}$ along the log-enlargement $T \to T_{\mathrm{log}}$. Here, $T_{\mathrm{log}} = \mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}})$ with X the character sheaf of the torus T. Then a log 1-motive $\mathbf{M} = [Y \xrightarrow{u} G_{\mathrm{log}}]$ over S is a morphism of commutative group sheaves, where Y is a lattice, i.e., an étale locally finite free abelian group sheaf. See [8, Definition 2.2] for the details about log 1-motives. Note that G_{log} in [8] is defined on the site $(\mathrm{fs}/S)_{\mathrm{\acute{e}t}}$, and agrees with ours by [26, Proposition 2.1].

The composition

$$Y \xrightarrow{u} G_{\mathrm{log}} \to G_{\mathrm{log}}/G = T_{\mathrm{log}}/T = \mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})$$

corresponds to a pairing

$$\langle -, - \rangle_{\mathbf{M}} : X \times Y \to \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}.$$

Definition 4.1 We call the pairing $\langle -, - \rangle_{\mathbf{M}}$ the *monodromy pairing* of the log 1-motive **M**.

Proposition 4.2 *Suppose G is an extension of an abelian scheme B by a torus T over the underlying scheme of S. Then étale locally on S we have a decomposition $u = u_1 + u_2$, where $u_i : Y \to G_{\log}$ for $i = 1, 2$, such that u_1 factorizes as $Y \to G \hookrightarrow G_{\log}$ and u_2 factorizes as $Y \to T_{\log} \hookrightarrow G_{\log}$.*

Proof Let $\bar{u}$ be the composition

$$Y \xrightarrow{u} G_{\log} \to (G_{\log}/G)_{S_{\text{ét}}} = (T_{\log}/T)_{S_{\text{ét}}}.$$

The short exact sequence $0 \to T \to T_{\log} \to (T_{\log}/T)_{S_{\text{ét}}} \to 0$ gives rise to an exact sequence

$$\to \operatorname{Hom}_S(Y, T_{\log}) \to \operatorname{Hom}_S(Y, T_{\log}/T) \xrightarrow{\delta} \operatorname{Ext}^1_{S_{\text{ét}}}(Y, T) \to .$$

Étale locally, $\operatorname{Ext}^1_{S_{\text{ét}}}(Y, T) = \operatorname{Ext}^1_{S_{\text{ét}}}(\mathbb{Z}^r, \mathbb{G}_m^t) = \operatorname{Ext}^1_{S_{\text{ét}}}(\mathbb{Z}, \mathbb{G}_m)^{rt} \cong \operatorname{Pic}(S)^{rt}$. Since a line bundle is Zariski locally trivial, $\delta(\bar{u})$ is zero étale locally on S. It follows that there exists $u_2 \in \operatorname{Hom}_S(Y, T_{\log})$ lifting $\bar{u}$ étale locally on S. Let $u_1 := u - u_2$ as a homomorphism from Y to $G_{\log}$, and it is easy to see that u_1 factors through G. This finishes the proof. ∎

Proposition 4.3 *Assume that S admits a global chart $\alpha : P \to M_S$, where P is an fs monoid. Any bilinear map $\langle -, - \rangle : X \times Y \to P_S^{\text{gp}}$ which lifts the monodromy pairing $\langle -, - \rangle_{\mathbf{M}}$ of **M**, gives rise to a splitting $u = u_1 + u_2$, where $u_2 : Y \to T_{\log} \hookrightarrow G_{\log}$ is the map induced by $\langle -, - \rangle$ and $u_1 = u - u_2 : Y \to G_{\log}$ factors through $G \hookrightarrow G_{\log}$.*
And, such a bilinear map always exists étale locally.

Proof The chart $\alpha : P \to M_S$ gives rise to a pushout diagram

$$
\begin{array}{ccc}
\alpha^{-1}(\mathcal{O}_S^\times) & \longrightarrow & \mathcal{O}_S^\times \\
\downarrow & & \downarrow \\
P & \longrightarrow & M_S
\end{array}
$$

Since taking group envelope commutes with colimits, we get another pushout diagram

$$
\begin{array}{ccc}
\alpha^{\text{gp},-1}(\mathcal{O}_S^\times) & \longrightarrow & \mathcal{O}_S^\times \\
\downarrow & & \downarrow \\
P_S^{\text{gp}} & \longrightarrow & M_S^{\text{gp}}
\end{array}
$$

where $\alpha^{gp} : P_S^{\text{gp}} \to M_S^{\text{gp}}$ denotes the homomorphism induced from α. This pushout diagram gives a short exact sequence

$$0 \to \alpha^{\text{gp},-1}(\mathcal{O}_S^\times) \to P_S^{\text{gp}} \to \overline{M}_S^{\text{gp}} \to 0$$

on the small étale site of S with $\overline{M}_S^{\mathrm{gp}} := M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$. Applying $\mathrm{Hom}_{S_{\text{ét}}}(Y \otimes_{\mathbb{Z}} X, -)$ to this short exact sequence, we get an exact sequence

$$\mathrm{Hom}_S(Y \otimes_{\mathbb{Z}} X, P_S^{\mathrm{gp}}) \to \mathrm{Hom}_S(Y \otimes_{\mathbb{Z}} X, \overline{M}_S^{\mathrm{gp}}) \xrightarrow{\delta} \mathrm{Ext}^1_{S_{\text{ét}}}(Y \otimes_{\mathbb{Z}} X, \alpha^{\mathrm{gp},-1}(\mathcal{O}_S^\times)).$$

Let $\bar{\gamma} : Y \otimes_{\mathbb{Z}} X \to (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\text{ét}}}$ be the homomorphism induced by the canonical pairing. Since both X and Y are étale locally constant, the map $\bar{\gamma}$ is determined by its induced map $Y \otimes_{\mathbb{Z}} X \to \overline{M}_S^{\mathrm{gp}}$, which we still denote by $\bar{\gamma}$ by abuse of notation. Since

$$\mathrm{Ext}^1_{S_{\text{ét}}}(Y \otimes_{\mathbb{Z}} X, \alpha^{\mathrm{gp},-1}(\mathcal{O}_S^\times)) = H^1(S_{\text{ét}}, \mathcal{H}om_S(Y \otimes_{\mathbb{Z}} X, \alpha^{\mathrm{gp},-1}(\mathcal{O}_S^\times))),$$

the element $\delta(\bar{\gamma})$ is étale locally trivial on S. Therefore, étale locally on S the map $\bar{\gamma}$ lifts to a homomorphism $\gamma : Y \otimes_{\mathbb{Z}} X \to P_S^{\mathrm{gp}}$. The map γ gives rise to a homomorphism $Y \otimes_{\mathbb{Z}} X \to \mathbb{G}_{\mathrm{m,log}}$ which corresponds to a homomorphism $u_2 : Y \to T_{\mathrm{log}}$. Obviously, u_2 lifts $\bar{u}$. Let $u_1 := u - u_2$, and we have that u_1 factors through $G \hookrightarrow G_{\mathrm{log}}$. This finishes the proof. ∎

Definition 4.4 For any positive integer n, let

$$T_n(\mathbf{M}) := H^{-1}(\mathbf{M} \otimes^L \mathbb{Z}/n\mathbb{Z}).$$

Proposition 4.5 *Let S be a locally Noetherian fs log scheme, let $M = [Y \xrightarrow{u} G_{\mathrm{log}}]$ be a log 1-motive over S, and let n be a positive integer. Then we have the following.*

(1) *$T_n(M)$ fits into the following exact sequence*

$$0 \to G[n] \to T_n(\mathbf{M}) \to Y/nY \to 0$$

of sheaves of abelian groups on $(\mathrm{fs}/S)_{\mathrm{kfl}}$.
(2) *$T_n(\mathbf{M}) \in (\mathrm{fin}/S)_{\mathrm{d}}$.*
(3) *Let m be another positive integer. Then the map $T_{mn}(\mathbf{M}) \to T_n(\mathbf{M})$ induced by $\mathbb{Z}/mn\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}$ is surjective.*

Proof The map $Y \xrightarrow{n} Y$ is clearly injective. Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & G_{\mathrm{log}} & \longrightarrow & \mathcal{H}om_{S_{\mathrm{kfl}}}(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\
0 & \longrightarrow & G & \longrightarrow & G_{\mathrm{log}} & \longrightarrow & \mathcal{H}om_{S_{\mathrm{kfl}}}(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}) & \longrightarrow & 0
\end{array}
$$

with exact rows. By [13, Proposition 4.2], we also have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z}(1) & \longrightarrow & \mathbb{G}_{\mathrm{m}} & \xrightarrow{n} & \mathbb{G}_{\mathrm{m}} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z}(1) & \longrightarrow & \mathbb{G}_{\mathrm{m,log}} & \xrightarrow{n} & \mathbb{G}_{\mathrm{m,log}} & \longrightarrow & 0 \, .
\end{array}
$$

From the last diagram, one sees that

$$\mathcal{H}om_{S_{\mathrm{kfl}}}(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}) \xrightarrow{n} \mathcal{H}om_{S_{\mathrm{kfl}}}(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})$$

is an isomorphism. Moreover, $G \xrightarrow{n} G$ is surjective, so we get that $G[n] \cong G_{\mathrm{log}}[n]$ and that $G_{\mathrm{log}} \xrightarrow{n} G_{\mathrm{log}}$ is surjective. By [23, Section 3.1], we get a short exact sequence

$$0 \to G[n] \to T_n(\mathbf{M}) \to Y/nY \to 0.$$

This finishes the proof of part (1).

For part (2), note that by [12, Proposition 2.3] and the short exact sequence from part (1), we have that $T_n(\mathbf{M}) \in (\mathrm{fin}/S)_{\mathrm{r}}$. Let $\mathbf{M}^{\vee}$ be the dual log 1-motive of $\mathbf{M}$ (see [8, Section 2.7]). Then we can identify the Cartier dual of $T_n(\mathbf{M})$ with $T_n(\mathbf{M}^{\vee})$. Since we also have $T_n(\mathbf{M}^{\vee}) \in (\mathrm{fin}/S)_{\mathrm{r}}$, we get $T_n(\mathbf{M}) \in (\mathrm{fin}/S)_{\mathrm{d}}$.

Part (3) is clearly true for the two log 1-motives $[Y \to 0]$ and $[0 \to G_{\mathrm{log}}]$. It follows that it also holds for $\mathbf{M}$. ∎

Definition 4.6 By Proposition 4.5,

$$\mathbf{M}[p^{\infty}] := \varinjlim_{n} T_{p^n}(\mathbf{M})$$

is an object of $(p\text{-div}/S)_{\mathrm{d}}^{\log}$, and we call it the *log p-divisible group associated with* $\mathbf{M}$.

Proposition 4.7 *Assume that S admits a global chart $\alpha : P \to M_S$, where P is an fs monoid. Let $\mathbf{M} = [Y \xrightarrow{u} G_{\mathrm{log}}]$ be a log 1-motive over S, such that X and Y are constant. Let n be a positive integer.*

Assume further that there exists a bilinear map as in Proposition 4.3

$$\langle -, - \rangle : X \times Y \to P_S^{\mathrm{gp}},$$

such that the homomorphism $u_2 : Y \xrightarrow{\beta} T_{\mathrm{log}} \hookrightarrow G_{\mathrm{log}}$, which is induced from the pairing $\langle -, - \rangle$, gives rise to a decomposition $u = u_1 + u_2$ with $u_1 : Y \xrightarrow{\alpha} G \hookrightarrow G_{\mathrm{log}}$. Then we have the following.

(1) *$T_n(\mathbf{M})$ is a standard extension of Y/nY by $G[n]$ with respect to the given chart.*
(2) *Let $f_1, \ldots, f_m$ (resp. $e_1, \ldots, e_r$) be a basis of X (resp. Y), and let $f_1^*, \ldots, f_m^*$ (resp. $e_1^*, \ldots, e_r^*$) be the corresponding dual basis of $X^* = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (resp. $Y^* = \mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Z})$). The pairing $\langle , \rangle : X \times Y \to P_S^{\mathrm{gp}}$ gives rise to a canonical pseudo-monodromy of $T_n(M)$, namely the element $\sum_{i,j} \delta_{ij} \otimes \langle f_i, e_j \rangle$ in the group*

$$\mathrm{Hom}_S(Y/nY(1), T[n]) \otimes P^{\mathrm{gp}} = \mathrm{Hom}_S(X/nX, Y^*/nY^*) \otimes P^{\mathrm{gp}}.$$

Here, δ_{ij} denotes the map $\bar{f}_k \mapsto \begin{cases} 0, & \text{if } k \neq i, \\ \bar{e}_j^*, & \text{if } k = i. \end{cases}$

(3) *For varying n, the pseudo-monodromies from (2) are compatible with each other.*

Proof Let $\mathbf{M}_1 := [Y \xrightarrow{u_1} G_{\mathrm{log}}]$, $\mathbf{M}_{\alpha} := [Y \xrightarrow{\alpha} G]$, $M_2 = [Y \xrightarrow{u_2} G_{\mathrm{log}}]$, and $\mathbf{M}_{\beta} := [Y \xrightarrow{\beta} T_{\mathrm{log}}]$. Then $T_n(\mathbf{M}_1)$ is canonically isomorphic to $T_n(\mathbf{M}_{\alpha})$, $T_n(M_2)$ is given by the pushout of $T_n(M_{\beta})$ along the canonical map $T[n] \hookrightarrow G[n]$, and

$T_n(\mathbf{M}) = T_n(\mathbf{M}_1) +_{\text{Baer}} T_n(\mathbf{M}_2)$. Since $T_n(\mathbf{M}_1) = T_n(\mathbf{M}_\alpha) \in (\text{fin}/S)_c$, we are reduced to show that $T_n(\mathbf{M}_2)$ is a standard extension of Y/nY by $G[n]$.

We first deal with the case that $G = T$. Applying the functor ε_* to the short exact sequence $0 \to T[n] \to T_n(\mathbf{M}_2) \to Y/nY \to 0$ on $(\text{fs}/S)_{\text{kfl}}$, we get a long exact sequence

$$0 \to T[n] \to T_n(\mathbf{M}_2) \to Y/nY \xrightarrow{\delta} R^1\varepsilon_* T[n]$$

on $(\text{fs}/S)_{\text{fl}}$. For any $1 \le j \le r$, $\delta(\bar{e}_j)$ is represented by the $T[n]$-torsor

$$\{a \in T_{\log} \mid a^n = u_2(e_j)\}.$$

Let $\bar{u}$ be the composition $Y \xrightarrow{u_2} T_{\log} \to T_{\log}/T \cong \mathcal{H}om_{S_{\text{kfl}}}(X, \mathbb{G}_{\text{m,log}}/\mathbb{G}_{\text{m}})$. Then $\bar{u}(e_j)(f_i) = p_{ij}$, where $p_{ij} \in P^{\text{gp}}$ is such that $\langle f_i, e_j \rangle = p_{ij}$ for each $1 \le i \le m$. Therefore, under the identification

$$R^1\varepsilon_* T[n] \cong \mathcal{H}om_{S_{\text{fl}}}(\mathbb{Z}/n\mathbb{Z}(1), T[n]) \otimes (\mathbb{G}_{\text{m,log}}/\mathbb{G}_{\text{m}})_{S_{\text{fl}}}$$
$$\cong X^*/nX^* \otimes (\mathbb{G}_{\text{m,log}}/\mathbb{G}_{\text{m}})_{S_{\text{fl}}},$$

we have $\delta(\bar{e}_j) = \sum_{i=1}^m \bar{f}_i^* \otimes p_{ij}$. It follows that δ can be lifted canonically to a homomorphism

$$\bar{\delta} : Y/nY \to \mathcal{H}om_{S_{\text{fl}}}(\mathbb{Z}/n\mathbb{Z}(1), T[n]) \otimes P^{\text{gp}} = X^*/nX^* \otimes P^{\text{gp}}, \bar{e}_j \mapsto \sum_{i=1}^m \bar{f}_i \otimes p_{ij}.$$

Therefore, $T_n(\mathbf{M}_2)$ is standard with respect to the given chart by Proposition 3.5.

In general, applying the functor ε_* to the pushout diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T[n] & \longrightarrow & T_n(\mathbf{M}_\beta) & \longrightarrow & Y/nY & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G[n] & \longrightarrow & T_n(\mathbf{M}_2) & \longrightarrow & Y/nY & \longrightarrow & 0
\end{array}
$$

we get the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T[n] & \longrightarrow & T_n(\mathbf{M}_\beta) & \longrightarrow & Y/nY & \longrightarrow & R^1\varepsilon_* T[n] \\
 & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & G[n] & \longrightarrow & T_n(\mathbf{M}_2) & \longrightarrow & Y/nY & \longrightarrow & R^1\varepsilon_* G[n]
\end{array}
$$

with exact rows. Therefore, we are reduced to the case $G = T$ by Proposition 3.5. This finishes the proof of part (1). The searched canonical pseudo-monodromy in part (2) is given by the canonical lift $\bar{\delta}$ of the map δ. The compatibility in part (3) is clear from the construction of the canonical pseudo-monodromy. ∎

Remark 4.8. If we have a decomposition of $\mathbf{M}$ into u_1 and u_2 as above, it is easy to see that for any $n \ge 1$, we also get a decomposition $T_n(\mathbf{M}) = T_n(u_1) +_{\text{Baer}} T_n(u_2)$, where

$T_n(u_1)$ is classical. The proposition shows that this decomposition is the same as the one obtained from the proof of Proposition 3.5, if one uses the lift

$$\delta_{T_n(\mathbf{M})} : Y/nY \xrightarrow{\tilde{\delta}} \mathcal{H}om_{S_{\mathrm{fl}}}(\mathbb{Z}/n\mathbb{Z}(1), T[n]) \otimes P^{\mathrm{gp}} \to \mathcal{H}om_{S_{\mathrm{fl}}}(\mathbb{Z}/n\mathbb{Z}(1), G[n]) \otimes P^{\mathrm{gp}}$$

of $Y/nY \to R^1\varepsilon_* G[n]$.

Assume that the underlying scheme of S is the spectrum of a Henselian local ring R and $P \to M_S$ is a global chart which is neat at the closed point s of S. By the same argument as in Lemma 3.7, we have

$$\mathrm{Hom}_S(X \otimes_{\mathbb{Z}} Y, P^{\mathrm{gp}}) \xrightarrow{\cong} \mathrm{Hom}_S(X \otimes_{\mathbb{Z}} Y, (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}});$$

therefore, the lifting of the canonical pairing is uniquely determined by the choice of the chart, and, further, the pure log part $u_2 : Y \to T_{\log} \hookrightarrow G_{\log}$ is uniquely determined by the choice of the chart. When R is further a complete discrete valuation ring, choosing a uniformizer determines a neat chart, and therefore determines a decomposition $u = u_1 + u_2$. Such a determination is compatible with Raynaud's decomposition of strict 1-motives over $\mathrm{Frac}(R)$ with integral geometric monodromy (see [23, Corollary 4.5.1] and [28, Section 4.2]). In the case of a DVR endowed with its canonical log structure, the compatibility with the work of Kato has also already been treated in [3].

5 The Serre–Tate theorem for log abelian varieties with constant degeneration

In this section, we want to show how one can adapt Drinfeld's proof of the classical Serre–Tate theorem to the situation of log abelian varieties with constant degeneration using the results from the previous sections.

5.1 Log abelian varieties with constant degeneration

We first recall the notion of a log abelian variety with constant degeneration from [8, Section 3]. Let S be an fs log scheme. Let $\mathbf{M} = [Y \xrightarrow{u} G_{\log}]$ be a log 1-motive over S with its dual log 1-motive $\mathbf{M}^* = [X \xrightarrow{u^*} G^*_{\log}]$. Consider the subsheaf

$$\mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})^{(Y)}$$

of $\mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})$ on the site $(\mathrm{fs}/S)_{\mathrm{\acute{e}t}}$, whose sections over $V \in (\mathrm{fs}/S)$ are defined to be

$$\{\phi \in \mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})(V) : \text{for any } v \in V \text{ and } x \in X_{\bar{v}}, \text{ there exist } y_{v,x}, y'_{v,x} \in Y_{\bar{v}},$$
$$\text{such that } \langle x, y_{v,x}\rangle_{\bar{v}} | \phi(x)_{\bar{v}} | \langle x, y'_{v,x}\rangle_{\bar{v}} \}.$$

Here, $\langle -, - \rangle : X \times Y \to \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}$ denotes the monodromy pairing of M. We then define $G^{(Y)}_{\log}$ as the preimage of $\mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})^{(Y)}$ under the natural map

$$G_{\log} \to G_{\log}/G \cong \mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}).$$

A log abelian variety with constant degeneration over S is defined to be a sheaf of abelian groups on $(\mathrm{fs}/S)_{\text{ét}}$, which is isomorphic to $G_{\log}^{(Y)}/Y$ for a pointwise polarizable log 1-motive $[Y \xrightarrow{u} G_{\log}]$. For the notion of being pointwise polarizable for a log 1-motive, we refer to [8, Definition 2.8]. One of the main results of [8] is then the following.

Theorem 5.1 [8, Theorem 3.4] *The association*

$$[Y \xrightarrow{u} G_{\log}] \mapsto G^{(Y)}/Y$$

defines an equivalence of categories from pointwise polarizable log 1-motives over S to log abelian varieties with constant degeneration over S.

Remark 5.2.   Pointwise polarizable log 1-motives satisfy effective étale descent. For this, assume that $\tilde{S} \to S$ is étale and $[\tilde{u} : \tilde{Y} \to \tilde{G}_{\log}]$ is a pointwise polarizable log 1-motive on $\tilde{S}$, with a gluing datum on $\tilde{S} \times_S \tilde{S}$. Then, by [8, Proposition 2.5], one also gets a descent datum on the group scheme $\tilde{G}$, which is an extension of an abelian scheme by a torus. Hence, $\tilde{G}$ descends to such an extension G over S. Moreover, $\tilde{Y}$ also descends to some Y over S and then $\tilde{u}$ glues to a morphism $[u : Y \to G_{\log}]$. By the definition of being pointwise polarizable, u is still pointwise polarizable.

By this, one then also gets effective étale descent for log abelian varieties with constant degeneration.

First, we remark that the log p-divisible group of a log abelian variety with constant degeneration coincides with the one coming from the log 1-motive.

Lemma 5.3   *Let A be a weak log abelian variety with constant degeneration over S (see [9, Section 1.7]). By [9, Section 1.7], there exists an admissible and nondegenerate log 1-motive $\mathbf{M} = [Y \to G_{\log}]$ such that $A = G_{\log}^{(Y)}/Y$. Then we have the following.*

(1) *For any positive integer n, $A[n]$ is canonically identified with $T_n(\mathbf{M})$.*
(2) *$A[p^\infty] := \bigcup_{n>0} A[p^n]$ is canonically identified with $\mathbf{M}[p^\infty]$.*

Proof   Consider the following two commutative diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & G_{\log}^{(Y)} & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\
0 & \longrightarrow & Y & \longrightarrow & G_{\log}^{(Y)} & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & G_{\log} & \longrightarrow & G_{\log}/Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\
0 & \longrightarrow & Y & \longrightarrow & G_{\log} & \longrightarrow & G_{\log}/Y & \longrightarrow & 0
\end{array}
$$

with exact rows. By Lemma [26, Lemma 3.2], we have that $G_{\log}^{(Y)} \xrightarrow{n} G_{\log}^{(Y)}$ is surjective with kernel $G[n]$. We also have that $G_{\log} \xrightarrow{n} G_{\log}$ is surjective with kernel $G[n]$.

It follows that $A[n]$ and $(G_{\log}/Y)[n]$ fit into the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G[n] & \longrightarrow & A[n] & \longrightarrow & Y/nY & \longrightarrow & 0 \\
& & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & G[n] & \longrightarrow & (G_{\log}/Y)[n] & \longrightarrow & Y/nY & \longrightarrow & 0
\end{array}
$$

with exact rows. Hence, $A[n]$ is canonically identified with $(G_{\log}/Y)[n]$. The canonical quasi-isomorphism

$$
[Y \to G_{\log}] \to G_{\log}/Y
$$

gives rise to canonical isomorphism $T_n(\mathbf{M}) \cong (G_{\log}/Y)[n]$. It follows that $A[n]$ is canonically identified with $T_n(\mathbf{M})$. ∎

Lemma 5.4 *Let $F_1 \in (\mathrm{fin}/S)_c$, and let F_2 be a subobject of F_1 in $(\mathrm{fin}/S)_r$. Then we have $F_2 \in (\mathrm{fin}/S)_c$. In other words, the category $(\mathrm{fin}/S)_c$ is closed under subobjects in $(\mathrm{fin}/S)_r$.*

Proof We need to show that the log scheme representing F_2 has log structure induced from S. This can be verified on each fiber over S. Hence, we are reduced to the case that S is a log point, which is [26, Lemma 3.1] ∎

The following lemma generalizes [26, Lemma 3.2].

Lemma 5.5 *Let $\mathbf{M} = [Y \to G_{\log}], \mathbf{M}' = [Y' \to G'_{\log}]$ be two nondegenerate log 1-motives over S, $(f_{-1}, f_0) : \mathbf{M} \to \mathbf{M}'$ a homomorphism of log 1-motives, and $f_c : G \to G'$ the map induced by f_0. Let X (resp. X') be the character group of the torus part T (resp. T') of G (resp. G'), $\mathcal{Q}$ (resp. $\mathcal{Q}'$) the sheaf $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ (resp. $\mathcal{H}om(X', \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y')}$), $f_1 : X' \to X$ the map induced by f_c, and $\tilde{f}_d : \mathcal{Q} \to \mathcal{Q}'$ the map induced by f_1. If f_c is an isogeny, then the map $\tilde{f} : G_{\log}^{(Y)} \to G_{\log}'^{(Y')}$ induced by (f_{-1}, f_0) is surjective with kernel $\mathrm{Ker}(f_c)$, and the map $\tilde{f}_d$ is bijective.*

Proof Without loss of generality, we may assume that the underlying scheme of S is Noetherian. Then the proof of [26, Lemma 3.2] works also here. ∎

The following proposition generalizes [26, Proposition 3.2].

Proposition 5.6 *Let A be a log abelian variety with constant degeneration over S, and let $F \in (\mathrm{fin}/S)_r$ be a subgroup sheaf of A. Let n be a positive integer which kills F, and let $\mathbf{M} = [Y \xrightarrow{u} G_{\log}]$ be the pointwise polarizable log 1-motive corresponding to A. Let $F' := \mathrm{ker}(F \to A[n] \to Y/nY)$ and $F'' := F/F'$. Assume that $F' \in (\mathrm{fin}/S)_r$. Then:*

(1) *Both F' and F'' lie in $(\mathrm{fin}/S)_c$.*

(2) *There exist a pointwise polarizable log 1-motive $\mathbf{M}' = [Y' \xrightarrow{u'} G'_{\log}]$ and a morphism $(f_{-1}, f_0) : \mathbf{M} \to \mathbf{M}'$ such that $F' = \mathrm{ker}(G \xrightarrow{f_c} G')$ and $F'' \cong Y'/Y$.*

(3) *The morphism (f_{-1}, f_0) induces an isogeny $A \to A'$ of log abelian varieties with constant degeneration with kernel F, where A' is the one associated with $\mathbf{M}'$.*

Proof The proof is very similar to the proof of [26, Proposition 3.2]. By construction, F' and F'' fit into the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \,. \\
& & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & G[n] & \longrightarrow & A[n] & \longrightarrow & Y/nY & \longrightarrow & 0
\end{array}
$$

Then part (1) follows from Lemma 5.4, using that $G[n]$ and Y/nY are classical.

For part (2), let E be the pullback of $G_{\log}^{(Y)}$ along $F \subset A$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\
& & \big\| & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & G_{\log}^{(Y)} & \longrightarrow & A & \longrightarrow & 0,
\end{array}
$$

and let E_{tor} be the torsion subsheaf of E and $Y' = E/E_{\mathrm{tor}}$. Now,

$$
G_{\log}^{(Y)}/G = \mathcal{H}om(X, \mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})^{(Y)}
$$

is torsion-free and E_{tor} maps into G. Therefore, we have $E_{\mathrm{tor}} = F'$ and $Y'/Y = F'' \subset Y/nY$, and Y' is étale locally constant. Let $G' := G/E_{\mathrm{tor}} = G/F'$. Then the inclusion $E \hookrightarrow G_{\log}^{(Y)} \hookrightarrow G_{\log}$ yields a homomorphism $Y' \to G_{\log}/F' = G'_{\log}$ by taking the quotients by F'. In this way, we get a log 1-motive $\mathbf{M}' := [Y' \xrightarrow{u'} G'_{\log}]$ together with a homomorphism

$$
(f_{-1}, f_0) : \mathbf{M} := [Y \xrightarrow{u} G_{\log}] \to \mathbf{M}'.
$$

To show that $\mathbf{M}'$ is pointwise polarizable, we may assume that S is a log point with its underlying field separably closed. The existence of a polarization in this case is shown in the proof of [26, Proposition 3.2]. Let now A' be the log abelian variety with constant degeneration associated with $\mathbf{M}'$. Then, by [8, Theorem 3.4], the homomorphism (f_{-1}, f_0) gives rise to a homomorphism $f : A \to A'$ of log abelian varieties with constant degeneration, which fits into a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & G_{\log}^{(Y)} & \longrightarrow & A & \longrightarrow & 0 \\
& & \big\downarrow{\scriptstyle f_{-1}} & & \big\downarrow{\scriptstyle \tilde{f}} & & \big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Y' & \longrightarrow & G_{\log}'^{(Y')} & \longrightarrow & A' & \longrightarrow & 0
\end{array}
$$

with exact rows. It then follows from Lemma 5.5 that f is an isogeny with kernel F. ∎

5.2 Serre–Tate theorem

In this subsection, we closely follow [15].

Let R be a ring which is killed by an integer $N \geq 1$, and let I be an ideal of R satisfying $I^{\nu+1} = 0$. Let S be an fs log scheme with its underlying scheme Spec R, (fs/S) the category of fs log schemes over S, and (fsAff/S) the full subcategory of (fs/S) consisting of fs log schemes whose underlying scheme is affine. We denote R/I (resp. $\text{Spec}(R/I)$) by R_0 (resp. S_0), and we endow S_0 with the induced log structure from S.

Definition 5.7 Let G be a functor from (fsAff/S) to the category of abelian groups. The subgroup functor G_I of G is defined by

$$G_I(U) = \ker(G(U) \to G(U \otimes_R R_0))$$

for $U \in (\text{fsAff}/S)$, where $U \otimes_R R_0$ is endowed with the induced log structure from U.
The subgroup functor $\hat{G}$ of G is defined by

$$\hat{G}(U) = \ker(G(U) \to G(U^{\text{red}})),$$

for $U \in (\text{fsAff}/S)$, where U^{red} is the reduced closed subscheme of U endowed with the induced log structure from U.

Lemma 5.8 *Let G be a commutative formal Lie group over R, and we endow G with the log structure induced from S. Then the subgroup functor G_I is killed by N^{ν}.*

Proof See [15, Lemma 1.1.1]. ∎

Lemma 5.9 *Let G be a sheaf of abelian groups on $(\text{fs}/S)_{\text{kfl}}$ such that $\hat{G}$ is Kummer log flat locally representable by a classical formal Lie group over R. Then N^{ν} kills G_I.*

Proof See [15, Lemma 1.1.2]. ∎

Lemma 5.10 *Let G and H be two sheaves of abelian groups on $(\text{fs}/S)_{\text{kfl}}$ such that:*
(a) *G is N-divisible.*
(b) *$\hat{H}$ is Kummer log flat locally representable by a classical formal Lie group.*
(c) *H is formally log smooth.*
Let G_0 (resp. H_0) denote the pullback of G (resp. H) to $(\text{fs}/S_0)_{\text{kfl}}$. Then we have:
(1) *The groups $\text{Hom}_S(G, H)$ and $\text{Hom}_{S_0}(G_0, H_0)$ have no N-torsion.*
(2) *The natural map "reduction mod I" $\text{Hom}_S(G, H) \to \text{Hom}_{S_0}(G_0, H_0)$ is injective.*
(3) *For any homomorphism $f_0 : G_0 \to H_0$, there exists a unique homomorphism "$N^{\nu}f$" $: G \to H$ which lifts $N^{\nu}f_0$.*
(4) *In order that a homomorphism $f_0 : G_0 \to H_0$ lifts to a (necessarily unique) homomorphism $f : G \to H$, it is necessary and sufficient that the homomorphism "$N^{\nu}f$" : $G \to H$ annihilates the subgroup $G[N^{\nu}] := \ker(G \xrightarrow{N^{\nu}} G)$ of G.*

Proof Properties (1), (2), and (4) follow from Lemma 5.9 just as in [15, Lemma 1.1.3]. Only (3) requires a bit more care. Let T be an fs log scheme over S, and let T_0 be the closed subscheme defined by I together with the induced log structure from T. We will construct a map $G(T) \to H(T)$ lifting $N^{\nu}f_0$ and its uniqueness will follow from (2).

For any $t \in G(T)$, let $t_0 \in G(T_0)$ be its reduction mod I. Then, by formal log smoothness of H, there is an étale cover $\tilde{T} \to T$ such that the image of $f_0(t_0)$ in $H(\tilde{T}_0)$ lifts to some element $\tilde{h} \in H(\tilde{T})$. Now $N^{\nu}\tilde{h}$ is unique, so, in particular, on the overlap $\tilde{T} \times_T \tilde{T}$, we have $\text{pr}_1^*(N^{\nu}\tilde{h}) = \text{pr}_2^*(N^{\nu}\tilde{h})$. Hence, $N^{\nu}\tilde{h}$ descends to a unique $h \in H(T)$. One readily checks that this construction is natural in T. ∎

Now let N be p^r for a prime number p and a positive integer r. Let A be a log abelian variety with constant degeneration over S, and $A[p^\infty]$ the log p-divisible group associated with A. We are going to show that both A and $A[p^\infty]$ satisfy the conditions (a)–(c) from Lemma 5.10.

Proposition 5.11 *The conditions (a)–(c) from Lemma 5.10 hold for A.*

Proof The multiplication by n map $n_A : A \to A$ is an isogeny by [27, Corollary 2.1] for any positive integer n. In particular, A is p-divisible.

By [9, Theorem 4.1], A is log smooth over S. In particular, it is formally log smooth over S.

We are left with checking the condition (b). Assume that $A \cong G_{\log}^{(Y)}/Y$ for a pointwise polarizable log 1-motive $[Y \to G_{\log}]$ over S. By [26, Theorem 2.1], A fits into the following short exact sequence

$$(5.1) \qquad 0 \to G \to A \to \mathcal{H}om_S(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y \to 0$$

of sheaves of abelian groups over $(\mathrm{fs}/S)_{\mathrm{kfl}}$. We abbreviate $\mathcal{H}om_S(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ as $\mathcal{Q}$. Let $U \in (\mathrm{fsAff}/S)$, and we have the following two commutative diagrams

$$(5.2) \qquad \begin{array}{ccccccc}
0 & \longrightarrow & G(U) & \longrightarrow & A(U) & \longrightarrow & (\mathcal{Q}/Y)(U) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G(U^{\mathrm{red}}) & \longrightarrow & A(U^{\mathrm{red}}) & \longrightarrow & (\mathcal{Q}/Y)(U^{\mathrm{red}})
\end{array}$$

and

$$(5.3)$$
$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y(U) & \longrightarrow & \mathcal{Q}(U) & \longrightarrow & (\mathcal{Q}/Y)(U) & \longrightarrow & H^1_{\mathrm{kfl}}(U, Y) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y(U^{\mathrm{red}}) & \longrightarrow & \mathcal{Q}(U^{\mathrm{red}}) & \longrightarrow & (\mathcal{Q}/Y)(U^{\mathrm{red}}) & \longrightarrow & H^1_{\mathrm{kfl}}(U^{\mathrm{red}}, Y)
\end{array}$$

with exact rows. By [26, Lemma 2.4][3], we have $H^1_{\mathrm{kfl}}(U, Y) \cong H^1_{\mathrm{fl}}(U, Y)$ and $H^1_{\mathrm{kfl}}(U^{\mathrm{red}}, Y) \cong H^1_{\mathrm{fl}}(U^{\mathrm{red}}, Y)$. Since Y is a smooth group scheme over S, we further get $H^1_{\mathrm{fl}}(U, Y) \cong H^1_{\mathrm{ét}}(U, Y)$ and $H^1_{\mathrm{fl}}(U^{\mathrm{red}}, Y) \cong H^1_{\mathrm{ét}}(U^{\mathrm{red}}, Y)$. Since the small étale site of U and the small étale site of U^{red} are equivalent, we get $H^1_{\mathrm{ét}}(U, Y) \cong H^1_{\mathrm{ét}}(U^{\mathrm{red}}, Y)$ and $Y(U) \cong Y(U^{\mathrm{red}})$. Hence, the left vertical map and the right vertical map of (5.3) are both isomorphisms. Since the formation of $\mathbb{G}_{m,\log}/\mathbb{G}_m$ is compatible with strict base change of log schemes, we get that the canonical map $\mathcal{Q}(U) \to \mathcal{Q}(U^{\mathrm{red}})$ is an isomorphism. Applying the five lemma to (5.3), we get that the canonical map $\mathcal{Q}/Y(U) \to \mathcal{Q}/Y(U^{\mathrm{red}})$ is injective. By diagram (5.2), we have that the canonical map $\hat{G}(U) \to \hat{A}(U)$ is an isomorphism. Since $\hat{G}$ is representable by a formal Lie group, so is $\hat{A}$. ∎

---

[3] The proof of [26, Lemma 2.4] makes use of fpqc descent of schemes which probably does not always hold and deserves a precise reference (see [24, Tag 0APK] for the situation there). Alternatively, see [29, Theorem 1.3].

Proposition 5.12 *Let S_0 be an fs log scheme where p is locally nilpotent. Let $\mathbf{M}_0 = [Y_0 \to G_{0,\log}]$ be a log 1-motive over S_0. Let $S_0 \to S$ be strict closed immersion of fs log schemes defined by a nilpotent sheaf of ideals, and let H be a log p-divisible group over S such that $H_0 := H \times_S S_0 = \mathbf{M}_0[p^\infty]$. Then $\hat{H}_0$ (resp. $\hat{H}$) is locally a formal Lie group over S_0 (resp. over S).*

Proof The log p-divisible group H_0 fits into a short exact sequence

$$0 \to G_0[p^\infty] \to H_0 \to Y_0 \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Since $Y_0 \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$ is classical étale, the subgroup functor $\hat{H}_0$ of H_0 coincides with $\hat{G}_0 = \widehat{G_0[p^\infty]}$ which is locally a formal Lie group.

Since the small étale site of S_0 is equivalent to the small étale site of S, Y_0 extends to a locally constant sheaf Y over S. Let $H_n := H[p^n]$ and $H_{0,n} := H_0[p^n]$. Consider the commutative diagram

$$
\begin{array}{ccccc}
H_{0,n} \times_{S_0} H_{0,n} & \xrightarrow{\bar{m}_n} & H_{0,n} & \xrightarrow{i_n \circ \bar{p}_n} & Y/p^n Y \\
\big\downarrow & & {\scriptstyle P_n}\big\downarrow \quad {\scriptstyle p_n} & & \big\downarrow \\
H_n \times_S H_n & \xrightarrow{m_n} & H_n & \xrightarrow{} & S
\end{array}
$$

of solids arrows, where $\bar{m}_n$ (resp. m_n) is the group law of $H_{0,n}$ (resp. H_n), $\bar{p}_n$ is the composition $H_{0,n} = T_{p^n}(\mathbf{M}_0) \to Y_0/p^n Y_0$, and i_n is the closed immersion $Y_0/p^n Y_0 \hookrightarrow Y/p^n Y$. Since $Y/p^n Y$ is log étale (in fact even classical étale) over S and $H_{0,n} \to H_n$ is a strict closed immersion defined by a nilpotent ideal, $i_n \circ \bar{p}_n$ and $i_n \circ \bar{p}_n \circ \bar{m}_n$ lift to unique maps p_n and P_n, respectively, such that the diagram remains commutative. Let μ_n be the group law of $Y/p^n Y$. Since $(i_n \circ \bar{\mu}_n) \circ \bar{p}_n \times \bar{p}_n = (i_n \circ \bar{p}_n) \circ \bar{m}_n$, the outer square of the commutative diagram

$$
\begin{array}{ccccc}
H_{0,n} \times_{S_0} H_{0,n} & \xrightarrow{\bar{p}_n \times \bar{p}_n} & Y_0/p^n Y_0 \times_{S_0} Y_0/p^n Y_0 & \xrightarrow{i_n \circ \bar{\mu}_n} & Y/p^n Y \\
\big\downarrow & & {\scriptstyle P_n}\big\downarrow \quad {\scriptstyle \mu_n} & & \big\downarrow \\
H_n \times_S H_n & \xrightarrow{p_n \times p_n} & Y/p^n Y \times_S Y/p^n Y & \xrightarrow{} & S
\end{array}
$$

of solid arrows coincides with the outer square of the last diagram. Therefore, the map P_n from last diagram is also the unique lifting of $(i_n \circ \bar{\mu}_n) \circ \bar{p}_n \times \bar{p}_n$ such that the diagram remains commutative. It follows that $\mu_n \circ (p_n \times p_n) = P_n = p_n \circ m_n$. Therefore, p_n is a homomorphism of group log schemes. By the uniqueness of p_n, it is also clear that $(p_n)_{n \geq 1}$ gives rise to a homomorphism

$$p : H \to Y \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p.$$

Let $H'_n := \ker(H_n \xrightarrow{p_n} Y/p^n Y)$. Clearly, H'_n is representable. Since $S_0 \to S$ is a strict nil immersion and

$$(H'_n) \times_S S_0 = G_0[p^n] \in (\text{fin}/S_0)_c,$$

H'_n is strict over S. Since $G_0[p^n]$ is classically flat, in particular log flat, $(p_n)_0$ is log flat over S_0 by descent. Since H_n is log flat over S, p_n is log flat by Corollary B.2. Hence, $H'_n = \ker(p_n)$ is log flat. Since it is also strict over S, it is even classically flat. Moreover, $H'_n \to S$ agrees with

$$H'_n = H_n \times_{Y/p^n Y} S \to Z \to H_n \to S,$$

where Z denotes the fiber product of $H_n \to Y/p^n Y \xleftarrow{e} S$ in the category of schemes. The first map is finite by [21, Chapter III, Corollary 2.1.6]. Since the unit section e of $Y/p^n Y$ is a closed immersion, the second map is also finite. The last map is finite by Proposition 2.4. It follows that $H'_n \in (\mathrm{fin}/S)_c$.

Thus, we get a classical p-divisible group $H' := \varinjlim_n H'_n$ such that $H' \times_S S_0 = G_0[p^\infty]$. Using the exact sequence

$$0 \to H' \to H \xrightarrow{p} Y \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p,$$

we get that $\hat{H} = \hat{H}'$ is locally a formal Lie group. ∎

Proposition 5.13 *Let S_0, S, H_0, and H be as in Proposition 5.12. Then H_0 and H satisfy the conditions (a)–(c) from Lemma 5.10.*

Proof Condition (a) is trivial, condition (b) follows from Proposition 5.12, and condition (c) follows from Theorem 3.16. ∎

We denote by LAVwCD_S the category of log abelian varieties with constant degeneration over S, and by $\mathrm{Def}(S, S_0)$ the category of triples

$$(A_0, H, \varepsilon)$$

consisting of a log abelian variety with constant degeneration A_0 over S_0, an object H of $(p\text{-div}/S)_d^{\log}$, and an isomorphism $\varepsilon : H_0 \xrightarrow{\cong} A_0[p^\infty]$ of log p-divisible groups over S_0.

Theorem 5.14 (Serre–Tate for LAVwCD) *Let $R, R_0, S, S_0, I, N,$ and ν be as in the beginning of this subsection. We further assume that N is a power of a prime number p. Then the canonical functor*

$$\mathrm{LAVwCD}_S \to \mathrm{Def}(S, S_0), A \mapsto (A_0, A[p^\infty], \mathrm{id}_{A_0[p^\infty]}),$$

where A_0 denotes the base change of A to S_0, is an equivalence of categories.

Proof Step 1: Full faithfulness. Let A, A' be log abelian varieties over S, $f_{p^\infty} : A[p^\infty] \to A'[p^\infty]$ a homomorphism of log p-divisible groups over S, and $f_0 : A_0 \to A'_0$ a homomorphism of log abelian varieties over S_0 such that $f_0[p^\infty]$ coincides with $f_{p^\infty} \times_S S_0$. To finish the proof of full faithfulness, it suffices to show that there exists a unique homomorphism $f : A \to A'$ which induces both f_{p^∞} and f_0. The uniqueness follows easily from Proposition 5.11 and Lemma 5.10(2).

To construct the homomorphism f, consider the canonical lifting "$N^\nu f$" $: A \to A'$ of $N^\nu f_0$ guaranteed by Lemma 5.10(3). Since "$N^\nu f$" lifts $N^\nu f_0$, the induced map "$N^\nu f$"$[p^\infty]$ on log p-divisible groups lifts $N^\nu(f_0[p^\infty])$. By part (2) of Lemma 5.10, we have "$N^\nu f$"$[p^\infty] = N^\nu f_{p^\infty}$. Hence, "$N^\nu f$" kills $A[N^\nu]$. By part (4) of Lemma 5.10,

there exists a homomorphism $f : A \to A'$ lifting f_0 such that "$N^\nu f$" $= N^\nu f$. Since f lifting f_0 implies $f[p^\infty]$ lifting $f_0[p^\infty]$, we get $f[p^\infty] = f_{p^\infty}$ again by part (2) of Lemma 5.10. This finishes the proof of the full faithfulness.

Step 2: Essential surjectivity. Let (A_0, H, α_0) be an object of $\mathrm{Def}(S, S_0)$, and we look for a log abelian variety A which gives rise to (A_0, H, α_0) up to isomorphism. By [10, Theorem 1.6] and [11, Corollary 3.11], A_0 lifts to a log abelian variety with constant degeneration A' over S. The isomorphism

$$\alpha_0 : A'_0 := A' \times_S S_0 \xrightarrow{\cong} A_0$$

of log abelian varieties over S_0 induces an isomorphism

$$\alpha_0[p^\infty] : A'_0[p^\infty] \to A_0[p^\infty]$$

of log p-divisible groups over S_0. By part (3) of Lemma 5.8, $N^\nu \alpha_0[p^\infty]$ has a unique lifting to a homomorphism

$$\text{``}N^\nu \alpha[p^\infty]\text{''} : A'[p^\infty] \to H$$

of log p-divisible groups over S. The same procedure applied to $\beta_0 := \alpha_0^{-1}$, and we get a homomorphism

$$\text{``}N^\nu \beta[p^\infty]\text{''} : H \to A'[p^\infty]$$

of log p-divisible groups over S. Since the composition $(N^\nu \beta_0[p^\infty]) \circ (N^\nu \alpha_0[p^\infty])$ (resp. $(N^\nu \alpha_0[p^\infty]) \circ (N^\nu \beta_0[p^\infty])$) is the multiplication by $N^{2\nu}$ map on $A'_0[p^\infty]$ (resp. H_0), the composition "$N^\nu \beta[p^\infty]$" $\circ$ "$N^\nu \alpha[p^\infty]$" (resp. "$N^\nu \alpha[p^\infty]$" $\circ$ "$N^\nu \beta[p^\infty]$") is the multiplication by $N^{2\nu}$ map on $A'[p^\infty]$ (resp. H). Let $K := \ker(\text{``}N^\nu \alpha[p^\infty]\text{''})$ and $Q := \ker(\text{``}N^\nu \beta[p^\infty]\text{''})$. Thus, we get a short exact sequence

$$0 \to K \to A[N^{2\nu}] \xrightarrow{u} Q \to 0$$

using the kernel–cokernel exact sequences for the compositions

$$\text{``}N^\nu \beta[p^\infty]\text{''} \circ \text{``}N^\nu \alpha[p^\infty]\text{''} = N^{2\nu}$$

and

$$\text{``}N^\nu \alpha[p^\infty]\text{''} \circ \text{``}N^\nu \beta[p^\infty]\text{''} = N^{2\nu}.$$

Claim $K = \ker(\text{``}N^\nu \alpha[p^\infty]\text{''})$ *lies in* $(\mathrm{fin}/S)_\mathrm{r}$. $\blacksquare$

Since K is also the kernel of $A'[N^{2\nu}] \to H[N^{2\nu}]$, it is representable by an fs log scheme. Clearly, both $K_0 := K \times_S S_0$ and $Q_0 := Q \times_S S_0$ lie in $(\mathrm{fin}/S_0)_\mathrm{r}$. Then $u_0 : A'_0[N^{2\nu}] \to Q_0$ as a K_0-torsor is log flat by Kummer log flat descent (see [7, Theorem 0.1]). By the log flatness of $A'[N^{2\nu}] \to S$ and Corollary B.2, u is log flat. From this, it then follows that K is log flat over S. Moreover, the Kummer property extends to infinitesimal lifts by Lemma 5.15. Therefore, K is also of Kummer type over S. Being the kernel of $A'[N^{2\nu}] \to H[N^{2\nu}]$, the group K is also finite over S, by the same argument as in the proof of the finiteness of the group H'_n in the proof of Proposition 5.12. But then, K being finite Kummer log flat over S, one has $K \in (\mathrm{fin}/S)_\mathrm{r}$ (see Proposition 2.4). This finishes the proof of the claim.

Let $[Y \to G_{\log}]$ be the log 1-motive corresponding to A'. Let

$$K' := \ker(K \to A'[N^{2v}] \to Y/N^{2v}Y).$$

Since $K_0 = \ker(N^v \alpha_0[p^\infty]) = A'_0[N^v]$, we have

$$K'_0 = \ker(A'_0[N^v] \to Y/N^{2v}Y) = G_0[N^v].$$

Replacing u by $K \to Y/N^{2v}Y$ and K by K', the same proof as that of the claim implies that $K' \in (\mathrm{fin}/S)_r$. Now, applying Proposition 5.6, we get that $A' \to A'/K$ is an isogeny of log abelian varieties with constant degeneration with kernel K. Let $A := A'/K$. Apparently,

$$A \times_S S_0 = A'_0/K_0 = A'_0/A'_0[N^v] = A'_0 \cong A_0,$$

and "$N^v \alpha[p^\infty]$" induces an isomorphism $A[p^\infty] \xrightarrow{\cong} H$. This finishes the proof of the essential surjectivity.

In the proof above, we have used the following lemma.

Lemma 5.15 *Assume that $S_0 \subset S$ is a strict nilpotent thickening of fs log schemes, and let $f : X \to Y$ be a morphism of fs log schemes over S, such that $f_0 = f \times_S S_0$ is Kummer. Then f is also Kummer.*

Proof For any log scheme Z, we denote $M_Z/\mathcal{O}_Z^\times$ by $\overline{M}_Z$.

Let x be any point of X, and let $y = f(x)$. It suffices to show that the canonical map $\overline{M}_{Y,\bar{y}} \to \overline{M}_{X,\bar{x}}$ of fs monoids is Kummer. Let $X_0 := X \times_S S_0$ and $Y_0 := Y \times_S S_0$. Consider the following commutative diagram

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
Y_0 & \longrightarrow & Y
\end{array}
$$

with horizontal maps strict. This square induces another commutative square

$$
\begin{array}{ccc}
\overline{M}_{X_0,\bar{x}} & \xleftarrow{\cong} & \overline{M}_{X,\bar{x}} \\
\uparrow & & \uparrow \\
\overline{M}_{Y_0,\bar{y}} & \xleftarrow{\cong} & \overline{M}_{Y,\bar{y}}
\end{array}
$$

Since f_0 is Kummer, the left vertical map in the above square is Kummer. If follows that the right vertical map is also Kummer. ∎

A A lemma on finite flat group schemes

Lemma A.1 *Let S be a scheme, F a finite flat group scheme over S which is of multiplicative type, and F' a finite flat group scheme over S. Then the fppf sheaf $H := \mathcal{H}om_S(F, F')$ is representable by an étale quasi-finite separated group scheme over S.*

Proof   Let $0 \to F' \to G_1 \to G_2 \to 0$ be the canonical smooth resolution of F' with G_1, G_2 affine smooth group schemes over S (see [18, Theorem A.5]). Then we have an exact sequence

$$0 \to \mathcal{H}om_S(F, F') \to \mathcal{H}om_S(F, G_1) \xrightarrow{\alpha} \mathcal{H}om_S(F, G_2)$$

of fppf sheaves of abelian groups over S. By [1, Exposé XI, Corollary 4.2], the sheaves $\mathcal{H}om_S(F, G_i)$, for $i = 1, 2$, are representable by smooth separated group schemes over S. Hence, $H = \mathcal{H}om_S(F, F')$ is the kernel of a morphism of representable sheaves and thus representable itself. Let n be a positive integer which kills F. We then have

$$\mathcal{H}om_S(F, G_i) = \mathcal{H}om_S(F, G_i)[n],$$

where $\mathcal{H}om_S(F, G_i)[n]$ denotes the n-torsion subgroup scheme of $\mathcal{H}om_S(F, G_i)$. The fibers of the group scheme $\mathcal{H}om_S(F, G_i)$ over S are finite by the structure theorem [1, Exposé XVII, Theorem 7.2.1] of commutative group schemes and [1, Exposé XVII, Proposition 2.4]. It follows that the group schemes $\mathcal{H}om_S(F, G_i)$, for $i = 1, 2$, are étale and quasi-finite over S. By [5, Corollary 17.3.5], α is étale. It is also separated. It follows that H is étale separated and quasi-finite. This finishes the proof. ∎

B Fiberwise flatness for algebraic stacks

Lemma B.1 *Let $\mathcal{S}$ be a locally Noetherian algebraic stack. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of locally Noetherian algebraic stacks over $\mathcal{S}$, where $\mathcal{X}$ is flat over $\mathcal{S}$. Assume that $f_s : \mathcal{X}_s \to \mathcal{Y}_s$ is flat for all points (valued in fields) $s : \mathrm{Spec}(k(s)) \to \mathcal{S}$. Then f is flat.*

Proof   As being locally Noetherian and flatness of a morphism are local for the smooth topology, everything reduces to the case of schemes, where the relevant statement is [24, Tag 039D].

More precisely, we have a commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & S \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} & \longrightarrow & \mathcal{S}
\end{array} \quad ,
$$

where the vertical morphisms are smooth covers by locally Noetherian schemes. We then let $s' : \mathrm{Spec}(k(s)) \to S$ be a point of S. It maps to a point s of $\mathcal{S}$. We then get a commutative diagram

$$
\begin{array}{ccc}
X_{s'} & \xrightarrow{\tilde{f}_{s'}} & Y_{s'} \\
\downarrow & & \downarrow \\
\mathcal{X}_s & \xrightarrow{f_s} & \mathcal{Y}_s
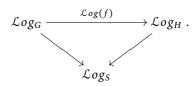\end{array} \quad ,
$$

where again the vertical maps are smooth coverings by schemes. Now f_s is flat by assumption. But this is the same thing as saying that $\tilde{f}_{s'}$ is flat (see, for example, [24, Tag 06FN] and note that flatness is local on the source and the target). Hence, $\tilde{f}_{s'}$ is flat

for all $s' \in S$. So, by the fiberwise flatness criterion for schemes, $\tilde{f}$ is flat as well. This, in turn, implies that f is flat. ∎
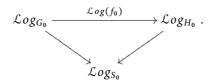
Corollary B.2 *Let S_0 be a Noetherian fs log scheme, and let $S_0 \to S$ be a strict closed immersion defined by a nilpotent sheaf of ideals I. Let $f : G \to H$ be a morphism of Noetherian fs log schemes over S with $G \to S$ log flat. Assume that $f_0 := f \times_S S_0 : G_0 \to H_0$ is log flat. Then f is log flat.*

Proof Consider the following diagram of algebraic stacks:

$$\mathcal{L}og_G \xrightarrow{\mathcal{L}og(f)} \mathcal{L}og_H \; .$$
$$\searrow \qquad\qquad \swarrow$$
$$\mathcal{L}og_S$$

Here, if Z is a log scheme, $\mathcal{L}og_Z$ denotes the stack of log structures introduced in [22, Section 1]. It is an algebraic stack. If Z is locally Noetherian, then so is $\mathcal{L}og_Z$ (by [22, Corollary 5.25]). Furthermore, a morphism $f : Z \to Z'$ of fine log schemes is log flat if and only if the induced morphism $\mathcal{L}og(f) : \mathcal{L}og_Z \to \mathcal{L}og_{Z'}$ is flat (see [22, Theorem 4.6 and Remark 4.7]).

Moreover, by [22, Proposition 3.20], the diagram above reduces modulo I to

$$\mathcal{L}og_{G_0} \xrightarrow{\mathcal{L}og(f_0)} \mathcal{L}og_{H_0} \; .$$
$$\searrow \qquad\qquad \swarrow$$
$$\mathcal{L}og_{S_0}$$

Now, as f_0 is log flat, $\mathcal{L}og(f_0)$ is flat. Thus, by the fiberwise flatness criterion for algebraic stacks (see Lemma B.1), $\mathcal{L}og(f)$ is flat as well, which means that f is log flat. ∎

C Kummer log flat descent of finiteness

Following [19, Definition 1.9], we say a morphism $f : X \to S$ of fs log scheme is finite (resp. quasi-finite, resp. separated, resp. proper, resp. universally closed, resp. locally of finite presentation, surjective), if the underlying morphism of schemes is so.

Remark C.1. According to [22, Lemma 4.8], $f : X \to S$ is locally of finite presentation if and only if the induced morphism $\mathcal{L}og(f) : \mathcal{L}og_X \to \mathcal{L}og_S$ of algebraic stacks is locally of finite presentation, where $\mathcal{L}og_T$, for a log scheme T, is the stack of log structures as in [22, Section 1].

By [24, Tag 02LA], the finiteness of morphisms of schemes descends in the fpqc topology. According to [13, Theorem 7.1], the descent of the finiteness of morphisms of fs log schemes in the Kummer flat topology has been shown in Tani's thesis [25]. However, the thesis is in Japanese, so we present a proof here.
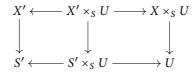
Let $X \to S$ and $T \to S$ be two morphisms of fs log schemes. We denote by $\mathring{X} \times_{\mathring{S}} \mathring{T}$ the fiber product of the underlying schemes. Note that in this article $X \times_S T$ denotes the fiber product in the category of fs log schemes.

Proposition C.2 Let $f : X \to S$ be a morphism of fs log schemes, and let $g : S' \to S$ be a Kummer log flat cover (see Definition 2.1). Let $f' : X' := X \times_S S' \to S'$ be the base change of f. Then f is finite if and only if f' is finite.

Proof If f is finite, so is f' by [19, Section 1.10].
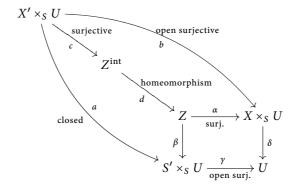
Conversely, assume that f' is finite. To show that f is finite, we need to show that f is (1) quasi-finite, (2) universally closed, (3) separated, and (4) of finite type.

(1) The quasi-finiteness of f follows from Nakayama's "fourth point lemma" (see [19, Section 2.2.2]) and the finiteness of f'.

(2) To show that f is universally closed, it suffices to show that for any strict morphism $U \to S$, the base change $X \times_S U \to U$ is closed. Consider the following diagram

$$
\begin{array}{ccccc}
X' & \longleftarrow & X' \times_S U & \longrightarrow & X \times_S U \\
\downarrow & & \downarrow & & \downarrow \\
S' & \longleftarrow & S' \times_S U & \longrightarrow & U
\end{array}
$$

with the two squares Cartesian. Since $X' \to S'$ is universally closed and $S' \times_S U \to S'$ is strict, the map $X' \times_S U \to S' \times_S U$ is closed. Since $S' \to S$ is a Kummer log flat cover, so are $S' \times_S U \to U$ and $X' \times_S U \to X \times_S U$. By [13, Proposition 2.5], the underlying topological maps of $S' \times_S U \to U$ and $X' \times_S U \to X \times_S U$ are open and surjective.

Let (logSch/S) be the category of log schemes over S, and let (int/S) be the full subcategory consisting of integral log schemes over S. Recall that (fs/S) is the full subcategory consisting of fs log schemes over S. We denote by Z the fiber product of $S' \times_S U$ and $X \times_S U$ over U in (logSch/S). Then the integral log scheme Z^{int} associated with Z is the fiber product of $S' \times_S U$ and $X \times_S U$ over U in (int/S), and the fs log scheme $Z^{\mathrm{fs}} := (Z^{\mathrm{int}})^{\mathrm{sat}}$ associated with Z^{int} is the fiber product of $S' \times_S U$ and $X \times_S U$ over U in (fs/S) (see [21, Chapter III, Proposition 2.1.5 and Corollary 2.1.6]). Note that the fiber product Z^{fs} of $S' \times_S U$ and $X \times_S U$ over U in (fs/S) is nothing but the fiber product $X' \times_S U$ (in (fs/S)). The map $X' \times_S U = Z^{\mathrm{fs}} \to Z^{\mathrm{int}}$ is finite and surjective by [21, Chapter III, Proposition 2.1.5(2)], and the map $Z^{\mathrm{int}} \to Z$ is a nil immersion by [21, Chapter III, Corollary 2.2.4]. Now we put the information useful for us in the following commutative diagram

with the two squares Cartesian.

of topological spaces. To prove that $X \times_S U \to U$ is closed, it suffices to show that $\delta(V)$ is closed for any closed subset V of $X \times_S U$. Since c is surjective, one can check that $\beta(\alpha^{-1}(V)) = a(b^{-1}(V))$. Since the map a is closed, the set $\beta(\alpha^{-1}(V))$ is closed. Since α is surjective, one sees that $\delta(V) = \gamma(\beta(\alpha^{-1}(V)))$. By the Cartesian property of the inner square in the category of schemes, one sees that $\gamma^{-1}(\delta(V)) = \beta(\alpha^{-1}(V))$ by [6, Lemma 4.28]. Since γ is open and $\beta(\alpha^{-1}(V))$ is closed, we get that $\delta(V)$ is closed.

(3) Let $\Delta_{X'/S'} : X' \to X' \times_{S'} X'$ (resp. $\Delta_{\mathring{X}'/\mathring{S}'} : \mathring{X}' \to \mathring{X}' \times_{\mathring{S}'} \mathring{X}'$) be the diagonal map in the category of fs log schemes (resp. category of schemes). Similarly for $\Delta_{X/S}$ and $\Delta_{\mathring{X}/\mathring{S}}$. We have $X' \times_{S'} X' = (X \times_S X) \times_S S'$ (resp. $\mathring{X}' \times_{\mathring{S}'} \mathring{X}' = (\mathring{X} \times_{\mathring{S}} \mathring{X}) \times_{\mathring{S}} \mathring{S}'$) and $\Delta_{X'/S'} = \Delta_{X/S} \times_S S'$ (resp. $\Delta_{\mathring{X}'/\mathring{S}'} = \Delta_{\mathring{X}/\mathring{S}} \times_{\mathring{S}} \mathring{S}'$). In the commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\Delta_{X'/S'}} & X' \times_{S'} X' \ , \\
& \Delta_{\mathring{X}'/\mathring{S}'} \searrow & \downarrow \\
& & \mathring{X}' \times_{\mathring{S}'} \mathring{X}'
\end{array}
$$

the map $\Delta_{\mathring{X}'/\mathring{S}'}$ is a closed immersion by the separatedness of $X' \to S'$, the map $X' \times_{S'} X' \to \mathring{X}' \times_{\mathring{S}'} \mathring{X}'$ is finite by [19, Section 1.10]; hence, $\Delta_{X'/S'}$ is a closed immersion. In particular, $\Delta_{X'/S'}$ is universally closed. By the descent of universal closeness along Kummer log flat covers from the last part, we have that $\Delta_{X/S}$ is universally closed. The finite morphism $X \times_S X \to \mathring{X} \times_{\mathring{S}} \mathring{X}$ is clearly universally closed; hence, $\Delta_{\mathring{X}/\mathring{S}}$ as the composition $X \xrightarrow{\Delta_{X/S}} X \times_S X \to \mathring{X} \times_{\mathring{S}} \mathring{X}$ is also universally closed. By [24, Tag 01KJ], $\Delta_{\mathring{X}/\mathring{S}}$ is an immersion. It follows that $\Delta_{\mathring{X}/\mathring{S}}$ is a closed immersion, i.e., X is separated over S.

(4) By [24, Tag 04XU], the universally closed map $f : X \to S$ is quasi-compact. Thus, to show that f is of finite type, it suffices to show that it is locally of finite type. By [24, Tag 02KX], being locally of finite type is local on the target for the fppf topology. Therefore, we may assume that the Kummer log flat cover g is as in the local situation of [7, Proposition 1.3], i.e., g admits a chart $(Q \to M_{S'}, P \to M_S, P \xrightarrow{h} Q)$ with P, Q fs monoids, and h Kummer, and thus g admits a factorization $S' \xrightarrow{g_1} S'' \xrightarrow{g_2} S$ of g with $S'' := S \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$ and g_1 a classical fppf cover. Consider the following diagram

$$
\begin{array}{ccccc}
X' & \longrightarrow & X \times_S S'' & \longrightarrow & X \\
\downarrow{f'} & & \downarrow{f''} & & \downarrow{f} \\
S' & \xrightarrow{g_1} & S'' & \xrightarrow{g_2} & S
\end{array}
$$

with the two small squares Cartesian. Since f' is of finite type, so is f'' by [24, Tag 02KZ]. By [7, Lemma 2.1], g_2 satisfies the condition on the base change map in [7, Lemma 2.3]. From this, we further get that f is locally of finite type. This finishes the proof. ∎

D Decomposition of Kummer log flat torsors under classical finite flat group schemes over Henselian local base

Let S be a locally Noetherian fs log scheme, and let $\varepsilon : (\mathrm{fs}/S)_{\mathrm{kfl}} \to (\mathrm{fs}/S)_{\mathrm{fl}}$ be the forgetful map between these two sites. Let G be a finite flat group scheme over the underlying scheme of S, and we endow it with the induced log structure from S. The Leray spectral sequence $H^i_{\mathrm{fl}}(S, R^j \varepsilon_* G) \Rightarrow H^{i+j}_{\mathrm{kfl}}(S, G)$ gives rise to an exact sequence

$$(\mathrm{D.1}) \qquad 0 \to H^1_{\mathrm{fl}}(S, G) \to H^1_{\mathrm{kfl}}(S, G) \xrightarrow{\alpha} H^0_{\mathrm{fl}}(S, R^1 \varepsilon_* G).$$

Proposition D.1 *Assume that the underlying scheme of S is the spectrum of a Henselian local ring and that the log structure of S admits a chart $P \to M_S$ for an fs monoid P such that the induced map $P \to M_{S,\bar{s}}/\mathcal{O}^\times_{S,\bar{s}}$ is an isomorphism, where s denotes the closed point of S. Then we have the following.*

(1) *Let n be a positive integer which kills G. Then*

$$\begin{aligned} H^0_{\mathrm{fl}}(S, R^1 \varepsilon_* G) &= H^0_{\mathrm{fl}}(S, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}) \\ &= \mathrm{Hom}_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}. \end{aligned}$$

(2) *The short exact sequence $0 \to \mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{G}_{\mathrm{m,log}} \xrightarrow{n} \mathbb{G}_{\mathrm{m,log}} \to 0$ on $(\mathrm{fs}/S)_{\mathrm{kfl}}$ gives rise to a canonical map $\delta : H^0_{\mathrm{kfl}}(S, \mathbb{G}_{\mathrm{m,log}}) \to H^1_{\mathrm{kfl}}(S, \mathbb{Z}/n\mathbb{Z}(1))$. For any $a \in P^{\mathrm{gp}}$, let T_a denote the Kummer log flat $\mathbb{Z}/n\mathbb{Z}(1)$-torsor over S given by $\delta(a)$. Then the map α from the exact sequence (D.1) admits a canonical section given by*

$$\beta : \mathrm{Hom}_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to H^1_{\mathrm{kfl}}(S, G), \quad h \otimes_{\mathbb{Z}} a \mapsto h_* T_a.$$

(3) *We have a canonical (after having fixed P) decomposition*

$$(\mathrm{D.2}) \qquad H^1_{\mathrm{kfl}}(S, G) \cong H^1_{\mathrm{fl}}(S, G) \oplus \mathrm{Hom}_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

Proof (1) Let $\mathcal{G} := (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\mathrm{fl}}}$. By Kato's theorem (see [13, Theorem 4.1] or [20, Theorem 3.12]), we have an isomorphism

$$(\mathrm{D.3}) \qquad \gamma : \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \mathcal{G} \xrightarrow{\cong} R^1 \varepsilon_* G,$$

which is locally given by the map β (see [13, Section 4.3]). Consider the following canonical commutative diagram:

$$(\mathrm{D.4}) \quad \begin{array}{ccc} \mathrm{Hom}_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} & \longrightarrow & H^0_{\mathrm{fl}}(S, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \mathcal{G}) \ . \\ \downarrow & & \downarrow \\ \mathrm{Hom}_s(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} & \longrightarrow & H^0_{\mathrm{fl}}(s, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \mathcal{G}) \end{array}$$

By [24, Tag 09ZI], we have

$$H^0_{\mathrm{fl}}(S, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \mathcal{G})$$
$$= H^0_{\text{ét}}(S, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\text{ét}}})$$
$$\cong H^0_{\text{ét}}(s, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{S_{\text{ét}}})$$
$$= H^0_{\mathrm{fl}}(s, \mathcal{H}om_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} \mathcal{G})$$

and

$$\mathrm{Hom}_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \cong \mathrm{Hom}_s(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

Let k be the residue field of S. The property of the given chart forces the induced log structure on s to be the one associated with $P \mapsto k, p \mapsto \begin{cases} 0, & \text{if } p \neq 0, \\ 1, & \text{otherwise.} \end{cases}$ Hence, the lower horizontal map in (D.4) is clearly an isomorphism. Then part (1) follows from the commutative diagram (D.4).

(2) Since the map γ from (D.3) is locally constructed in the same way as β, the diagram

$$
\begin{array}{ccc}
 & \mathrm{Hom}_S(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} & \\
 \overset{\beta}{\swarrow} & & \cong \bigg\downarrow H^0_{\mathrm{fl}}(S,\gamma) \\
H^1_{\mathrm{kfl}}(S, G) \xrightarrow{\ \alpha\ } & H^0_{\mathrm{fl}}(S, R^1\varepsilon_* G) &
\end{array}
$$

is commutative. Therefore, β is a section to α after the identification $H^0_{\mathrm{fl}}(S, \gamma)$.

Part (3) follows from part (2) and the exact sequence (D.1).   ■

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