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A Property of Lie Group Orbits

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Abstract. Let G be a real Lie group and X a real analytic manifold. Suppose that G acts analytically on X with finitely many orbits. Then the orbits are subanalytic in X. As a consequence we show that the micro-support of a G-equivariant sheaf on X is contained in the conormal variety of the G-action.

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Subanalytic sets were introduced by Hironaka. They play a prominent role in various areas of mathematics. In this note we give examples of subanalytic sets that arise naturally in the representation theory of Lie groups. We show that if a Lie group acts on a real analytic manifold with finitely many orbits then the orbits are subanalytic in the ambient manifold. The main motivation for this note was to prove that the micro-support of an equivariant sheaf is contained in the union of conormal bundles of the orbits. This result is known in the case of a complex algebraic group acting on a complex algebraic variety [1]. However in the real analytic case, to our knowledge, its proof is absent from the literature (although it may be known to some specialists). The proof of Theorem 1 is elementary and uses only the basic properties of subanalytic sets.

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All manifold and maps will be real analytic. We refer to [2] for the theory of subanalytic sets. We fix at the beginning a manifold X. Let G be a connected real Lie group acting on X with finitely many orbits. Let

$$X = \bigsqcup_{1 \le i \le r} Q_i$$

be the orbit stratification on X. We remark that the orbits Q_i are locally closed in X. This fact was pointed out to me by D. Miličić (the proof can be found for example in [4, Lemma 5.2.4.1]). The aim of this note is to prove the following result:

Theorem 1 *The orbits Q_i, $i = 1, \ldots, r$ of G-action on X are subanalytic in X.*

The proof of the theorem will be presented in a number of steps. First we need some notation. We denote by $\mathfrak{g}$ the Lie algebra of G. For $x \in X$ let $\mathfrak{g}_x$ be the Lie algebra of the stabilizer of x in G. Denote by $p\colon X \times \mathfrak{g} \to X$ the projection. Set

$$\tilde{\mathfrak{g}} = \{(x, \xi) \in X \times \mathfrak{g} : \xi \in \mathfrak{g}_x\} \quad \text{and} \quad \tilde{Q}_i = \tilde{\mathfrak{g}} \cap p^{-1}(Q_i),\ 1 \le i \le r.$$

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Clearly $\tilde{Q}_i$ is a vector bundle over the orbit Q_i. Since the orbits Q_i are locally closed they are submanifolds of X. This implies that $\tilde{Q}_i$ is a submanifold of $X \times \mathfrak{g}$.

Lemma 2 *$\tilde{\mathfrak{g}}$ is subanalytic in $X \times \mathfrak{g}$.*

Proof Given a manifold Y we denote its tangent bundle by TY. We write $\mathrm{Aff}(TY, TY)$ for the bundle over Y whose fiber at $y \in Y$ is the set of affine endomorphisms of the tangent space T_yY. Denote by $\mathrm{Hom}(TY, TY)$ the subbundle of $\mathrm{Aff}(TY, TY)$ defined by the fiberwise linear endomorphisms of TY.

We view the Lie algebra $\mathfrak{g}$ as the tangent space to G at the identity $e \in G$. The action of G on X determines the map $\alpha_x \colon \mathfrak{g} \times T_xX \to T_xX$, $x \in X$. Denote by 0 the zero vector in $\mathfrak{g}$ and by 0_x the zero vector in T_xX. Then the formula

$$\Phi(x, \xi)(v_x) = \alpha_x(\xi, 0_x) + \alpha_x(0, v_x), \quad x \in X,\ \xi \in \mathfrak{g},\ v_x \in T_xX,$$

defines an affine endomorphism of T_xX. Hence we obtain a real analytic map $\Phi \colon X \times \mathfrak{g} \to \mathrm{Aff}(TX, TX)$ of bundles over X. The above formula implies that the linearity of $\Phi(x, \xi)$ is equivalent to $\xi \in \mathfrak{g}_x$. Thus the set $\tilde{\mathfrak{g}}$ is equal to the preimage under Φ of $\mathrm{Hom}(TX, TX)$. Since $\mathrm{Hom}(TX, TX)$ is a closed submanifold of $\mathrm{Aff}(TX, TX)$, the claim of the lemma follows. ■

Next we summarize the properties of subanalytic sets that will be used below. Let Y be a manifold and Z a subanalytic subset of Y. Then the connected components of Z are locally finite and subanalytic. Recall that the set of smooth points Z_{reg} is defined as the subset of points $y \in Z$ such that there exists a neighborhood U of y such that $U \cap Z$ is a closed submanifold of U. Then the sets Z_{reg} and $Z \setminus Z_{\mathrm{reg}}$ are subanalytic in Y and $\dim(Z \setminus Z_{\mathrm{reg}}) < \dim Z$.

Lemma 3 *Set $n = \dim G$. Then:*

(1) $\dim \tilde{\mathfrak{g}} = n$;
(2) $\tilde{\mathfrak{g}}_{\mathrm{reg}} \cap \tilde{Q}_i$ *is open in* $\tilde{\mathfrak{g}}_{\mathrm{reg}}$ *for* $1 \le i \le r$;
(3) $\tilde{\mathfrak{g}}_{\mathrm{reg}} \cap \tilde{Q}_i \neq \varnothing$ *for* $1 \le i \le r$.

Proof Let $y \in \tilde{\mathfrak{g}}_{\mathrm{reg}}$. We choose a neighborhood U of y in $X \times \mathfrak{g}$ such that $U \cap \tilde{\mathfrak{g}}$ is a closed submanifold of U. Then if $U \cap \tilde{Q}_i \neq \varnothing$, it is a submanifold of $U \cap \tilde{\mathfrak{g}}$ of dimension n. Take i such that $y \in \tilde{Q}_i$. By shrinking U if necessary we may assume that $U \cap \tilde{Q}_i$ is closed in U. If $U \cap \tilde{\mathfrak{g}} \setminus U \cap \tilde{Q}_i \neq \varnothing$ we repeat the previous argument. Since $U \cap \tilde{g} \setminus U \cap \tilde{Q}_i = \bigcup_{i \neq j} U \cap \tilde{Q}_j$ we may use induction on r to conclude $\dim U \cap \tilde{\mathfrak{g}} = n$. This proves (1). (2) is an immediate consequence of (1). In order to prove (3) assume that $\tilde{\mathfrak{g}}_{\mathrm{reg}} \cap \tilde{Q}_i = \varnothing$. Then we would have $\tilde{Q}_i \subset \tilde{\mathfrak{g}} \setminus \tilde{\mathfrak{g}}_{\mathrm{reg}}$. Since $\dim(\tilde{\mathfrak{g}} \setminus \tilde{\mathfrak{g}}_{\mathrm{reg}}) \le n - 1$ and $\dim \tilde{Q}_i = n$ we have a contradiction. This proves (3). ■

We endow $X \times \mathfrak{g}$ with G resp. $\mathbb{R}^+$ action (here $\mathbb{R}^+$ is the set of strictly positive real numbers) as follows:

$$g.(x, \xi) = \big(gx, Ad(g)\xi\big) \quad \text{resp.} \quad c.(x, \xi) = (x, c\xi), \quad g \in G,\ c \in \mathbb{R}^+,\ x \in X,\ \xi \in \mathfrak{g}.$$

Lemma 4 *Any connected component of $\tilde{\mathfrak{g}}_{\mathrm{reg}}$ is stable for the G resp. $\mathbb{R}^+$ action on $X \times \mathfrak{g}$.*

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Proof Clearly $\tilde{\mathfrak{g}}$ is stable for the G-action and for $g \in G$ the map $(x, \xi) \mapsto \big(gx, Ad(g)\xi\big)$ is an analytic isomorphism of $X \times \mathfrak{g}$. This implies that $\tilde{\mathfrak{g}}_{\text{reg}}$ is invariant for the G-action. Let Z be a connected component of $\tilde{\mathfrak{g}}_{\text{reg}}$. Then we have $Z \subset G.Z \subset \tilde{\mathfrak{g}}_{\text{reg}}$ and $G.Z$ is connected since G is connected. It follows that $Z = G.Z$, as claimed. The proof for the $\mathbb{R}^+$-action is analogous. ■

Lemma 5 *Let $\pi\colon E \to Y$ be a real vector bundle over the manifold Y. Suppose that Z is a subanalytic $\mathbb{R}^+$-invariant subset of E. Then $\pi(Z)$ is subanalytic.*

Proof A slightly more general statement can be found in [3, 8.3.8]. For the convenience of the reader we reproduce the proof. Let U be a subanalytic open neighborhood of the zero-section of E such that $\bar{U} \to X$ is proper. Clearly we have $Z = \mathbb{R}^+.(Z \cap U)$ and therefore $\pi(Z) = \pi(Z \cap U)$. On the other hand $Z \cap U$ is subanalytic in U and the restriction of π to $\bar{U}$ is proper and hence $\pi(Z \cap U)$ is subanalytic in X. ■

Now we can prove the theorem:

Proof of 1 We fix i and choose a connected component Z of $\tilde{\mathfrak{g}}_{\text{reg}}$ such that $Z \cap \tilde{Q}_i \neq \varnothing$. This is possible by Lemma 3(3). We have $Z = \bigcup_{1 \leq j \leq r} (\tilde{Q}_j \cap Z)$ and by 3(2) any $\tilde{Q}_j \cap Z$ is open in Z. This implies further that $Z = Z \cap \tilde{Q}_i$, *i.e.*, $Z \subset \tilde{Q}_i$. It suffices now to apply Lemmas 4 and 5 to conclude that $p(Z) = Q_i$ is subanalytic in X. ■

2

In this section we show that the micro-support of a sheaf on X constructible with respect to the orbit stratification is contained in the union of conormal bundles of the orbits. First we need some preparation. Our approach will be based on [3].

Let X be a manifold and $A, B \subset X$. We choose a local coordinate system at $x \in X$. The tangent cone $C_x(A, B)$ is defined as the set of limits of the sequences $c_n(x_n - y_n)$, where $c_n \in \mathbb{R}$, $c_n > 0$, $x_n \in A$, $y_n \in B$ and $\lim x_n = x = \lim y_n$. We regard $C_x(A, B)$ as a subset of $T_x X$.

We write $T_x^* X$ for the cotangent space at x and $T^* X$ for the cotangent bundle of X. We denote by $\pi\colon T^* X \to X$ the natural projection. Let $M \subset X$ be a submanifold. We denote by $T_M^* X$ the conormal bundle to M in X. Then the fiber of $T_M^* X$ over $x \in M$ is equal to $(T_x M)^\perp$, *i.e.*, the set of linear forms on $T_x X$ vanishing on $T_x M$. For a morphism of manifolds $f\colon X \to Y$ we denote by $T_x(f)\colon T_x X \to T_{f(x)} Y$ the tangent map at $x \in X$ and by ${}^t T_x(f)\colon T_{f(x)}^* Y \to T_x^* X$ the cotangent map.

Let $H\colon T(T^* X) \to T^*(T^* X)$ be the Hamiltonian isomorphism defined using the canonical symplectic stucture on $T^* X$ [3, A.2]. Let $M, N \subset X$ be submanifolds of X. We say that the pair (M, N) satisfies the μ-condition [3, 6.2.4, 8.3.19] if for any $p \in \pi^{-1} N$ we have

$$\big({}^t T_p(\pi)\big)^{-1} \big(H_p C_p(T_M^* X, T_N^* X)\big) \subset (T_{\pi(p)} N)^\perp.$$

We remark that the μ condition is closely related to the Whitney conditions [3, exercise VIII.12].

Lemma 6 *With the same assumptions as in Theorem 1, let M and N be the G-orbits. Then (M, N) satisfies the μ-condition.*

Proof We begin the proof with a general remark. Let $f\colon X \to Y$ be a morphism of manifolds. A simple computation in local coordinates yields $T_x(f)\big(C_x(A,B)\big) \subset C_{f(x)}\big(f(A), f(B)\big)$, $A, B \subset X$. In particular, if A and B are invariant for an automorphism f of X we have $T_x(f)\big(C_x(A,B)\big) = C_{f(x)}(A,B)$. We apply this formula to the automorphism of T^*X determined by the action of an element $g \in G$. Furthermore, a short computation shows that the Hamiltonian isomorphism intertwines isomorphisms on $T(T^*X)$ and $T^*(T^*X)$ induced by the action of $g \in G$. It follows that the μ-condition is G-invariant in $p \in \pi^{-1}(N)$.

Since M and N are subanalytic in X the μ-condition is satisfied at some point in N [3, 8.3.20]. By the equivariance, the μ-condition is then satisfied for any point in N. ■

We say that a partition $X = \bigsqcup_{i\in I} X_i$ is a μ-stratification if the following conditions are satisfied:

(1) for any $i \in I$, X_i is a locally closed subanalytic submanifold of X;
(2) the family $(X_i, i \in I)$ is locally finite and if $\bar{X}_i \cap X_j \neq \emptyset$, $i, j \in I$, then $X_j \subset \bar{X}_i$;
(3) any pair (X_i, X_j) such that $X_j \subset \bar{X}_i \setminus X_i$ satisfies the μ-condition.

Proposition 7 *The orbit stratification $X = \bigsqcup_{1\leq i\leq r} Q_i$ is a μ-stratification. In particular, if $\mathcal{F}$ is a sheaf of complex vector spaces on X whose restrictions to the orbits are locally constant then the micro-support $SS(\mathcal{F})$ of $\mathcal{F}$ [3, 5.1.2] satisfies:*

$$SS(\mathcal{F}) \subset \bigsqcup_{1\leq i\leq r} T^*_{Q_i} X.$$

Proof The first statement follows from 1 and 6. The second statement is [3, 8.4.1]. ■

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