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Weighted Norm Inequalities for a Maximal Operator in Some Subspace of Amalgams

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Abstract. We give weighted norm inequalities for the maximal fractional operator $\mathcal{M}_{q,\beta}$ of Hardy–Littlewood and the fractional integral I_γ. These inequalities are established between $(L^q, L^p)^\alpha(X, d, \mu)$ spaces (which are superspaces of Lebesgue spaces $L^\alpha(X, d, \mu)$ and subspaces of amalgams $(L^q, L^p)(X, d, \mu)$) and in the setting of space of homogeneous type (X, d, μ). The conditions on the weights are stated in terms of Orlicz norm.

1 Introduction

Consider the fractional maximal operator $\mathfrak{m}_{q,\beta}$ $(1 \le q \le \beta \le \infty)$ defined on $\mathbb{R}^n$ by

$$\mathfrak{m}_{q,\beta} f(x) = \sup_{Q \in \mathcal{Q}: x \in Q} |Q|^{\frac{1}{\beta} - \frac{1}{q}} \| f \chi_Q \|_q,$$

where $\mathcal{Q}$ is the set of all cubes Q of $\mathbb{R}^n$ with edges parallel to the coordinate axes, $|E|$ stands for the Lebesgue measure of the subset E of $\mathbb{R}^n$ and $\| \cdot \|_q$ denotes the usual norm on the Lebesgue space $L^q(\mathbb{R}^n, dx)$. Weighted norm inequalities for $\mathfrak{m}_{1,\beta}$ have been extensively studied in the setting of Lebesgue, weak-Lebesgue and Morrey spaces (see [3, 14, 15] and the references therein). The following result is contained in [14].

Theorem 1.1 *Assume that $1 \le q < \beta \le \infty$, $\frac{1}{t} = \frac{1}{q} - \frac{1}{\beta}$ and v is a weight function satisfying*

$$\sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{\beta} - 1} \| v \chi_Q \|_t \| v^{-1} \chi_Q \|_{q'} < \infty. \quad \left(\tfrac{1}{q'} + \tfrac{1}{q} = 1 \right).$$

Then there exists a constant C such that for any Lebesgue measurable function f

$$\Big(\int_{\{x \in \mathbb{R}^n : \mathfrak{m}_{1,\beta} f(x) > \lambda\}} v(y)^t dy \Big)^{1/t} \le C \lambda^{-1} \| f v \|_q \quad \lambda > 0.$$

The spaces $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$ $(1 \le q \le \alpha \le p \le \infty)$ have been defined in [7] as follows:

- $I_k^r = \prod_{i=1}^n [k_i r, (k_i + 1) r), \quad k = (k_i)_{1 \le i \le n} \in \mathbb{Z}^n, r > 0,$
- $J_x^r = \prod_{i=1}^n (x_i - \frac{r}{2}, x_i + \frac{r}{2}), \quad x = (x_i)_{1 \le i \le n} \in \mathbb{R}^n, r > 0,$

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- a Lebesgue measurable function f belongs to $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$ if $\|f\|_{q,p,\alpha} < \infty$, where

$$\|f\|_{q,p,\alpha} = \sup_{r>0} r^{n(\frac{1}{\alpha}-\frac{1}{q})} {}_r\|f\|_{q,p},$$

$${}_r\|f\|_{q,p} = \begin{cases} \left[\sum_{k\in\mathbb{Z}^n} \left(\|f\chi_{I_k^r}\|_q\right)^p\right]^{\frac{1}{p}} & \text{if } p < \infty, \\ \sup_{x\in\mathbb{R}^n} \|f\chi_{J_x^r}\|_q & \text{if } p = \infty. \end{cases}$$

The $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$ have been introduced in connection with Fourier multiplier problems, but they are also linked to $L^q - L^p$ multiplier problems. We refer the readers to [11], where spaces of Radon measures containing $(L^1, \ell^p)^\alpha(\mathbb{R}^n)$ are considered. Notice that these spaces are subspaces of amalgam spaces introduced by Wiener and studied by many authors (see [9] and the references therein).

It has been proved in [6] that given $1 \le q \le \alpha < \infty$, $\{(L^q, \ell^p)^\alpha(\mathbb{R}^n)\}_{p\ge\alpha}$ is a monotone increasing family of Banach spaces, $(L^q, \ell^\alpha)^\alpha(\mathbb{R}^n) = L^\alpha(\mathbb{R}^n)$ and $(L^q, \ell^\infty)^\alpha(\mathbb{R}^n)$ is clearly the classical Morrey space denoted by $L^{q,n(1-\frac{q}{\alpha})}(\mathbb{R}^n)$ in [3]. Moreover, if $q < \alpha < p$, then the weak-$L^\alpha(\mathbb{R}^n)$ space is embedded in $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$. Due to this remarkable link between the spaces $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$ and the Lebesgue ones, it is tempting to look for an extension of Theorem 1.1 to the setting of $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$ space. The following result is contained in [8].

Theorem 1.2 *Assume*

- $1 \le q \le \alpha \le p$ *and* $0 < \frac{1}{s} = \frac{1}{\alpha} - \frac{1}{\beta}$,
- $q \le q_1 \le \alpha_1 \le p_1$ *and* $0 < \frac{1}{t} = \frac{1}{q_1} - \frac{1}{\beta} \le \frac{1}{p_1}$,
- v *is a weight function satisfying*

$$\sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{\beta}-\frac{1}{q}} \|v\chi_Q\|_t \|v^{-1}\chi_Q\|_{1/(\frac{1}{q}-\frac{1}{q_1})} < \infty.$$

Then there exists a real constant $C > 0$ such that

$$\Big(\int_{\{x\in\mathbb{R}^n : \mathfrak{M}_{1,\beta}f(x)>\lambda\}} v(y)^t\,dy\Big)^{1/t} \le C\lambda^{-1}\|fv\|_{q_1,p_1,\alpha_1}(\lambda^{-1}\|f\|_{q,\infty,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}$$

for any real $\lambda > 0$ and Lebesgue measurable function f on $\mathbb{R}^n$.

It turns out that the $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$ setting is particularly well adapted for the search of controls on Lebesgue norm of fractional maximal functions $\mathfrak{M}_{q,\beta}f$. Actually we have the following result whose first part is a consequence of Theorem 1.2 (see [8]).

Theorem 1.3 *Assume that $1 \le q \le \alpha \le \beta$ and $\frac{1}{s} = \frac{1}{\alpha} - \frac{1}{\beta}$.*

(i) *If $\alpha \le p$ and $\frac{1}{q} - \frac{1}{\beta} \le \frac{1}{p}$, then there is a real constant C such that for all Lebesgue measurable functions f on $\mathbb{R}^n$,*

$$\|\mathfrak{M}_{q,\beta}f\|_{s,\infty}^* \equiv \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^n : \mathfrak{M}_{q,\beta}f(x) > \lambda\}|^{1/s} \le C\|f\|_{q,p,\alpha}.$$

(ii) *If* $1 \leq u \leq s \leq v$, *then there is a real constant* C *such that for any Lebesgue measurable function* f *on* $\mathbb{R}^n$,

$$\|f\|_{q,p,\alpha} \leq C \|\mathfrak{m}_{q,\beta} f\|_{u,v,s}. \tag{1.1}$$

It follows from inequality (1.1) and the embedding of the weak-$L^s(\mathbb{R}^n)$ space into $(L^u, \ell^v)^s(\mathbb{R}^n)$ for $u < s < v$ that f has its fractional maximal function $\mathfrak{m}_{q,\beta} f$ in a weak Lebesgue space only if it belongs to some $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$.

Let $X = (X, d, \mu)$ be a space of homogeneous type which is separable and satisfies a reverse doubling condition (see (2.2) in Section 2 for a definition).

For $1 \leq q \leq \beta \leq \infty$ we set, for any μ-measurable function f on X,

$$\mathcal{M}_{q,\beta} f(x) = \sup_B \mu(B)^{\frac{1}{\beta} - \frac{1}{q}} \|f \chi_B\|_q \quad x \in X,$$

where the supremum is taken over all balls B in X containing x and $\|\cdot\|_q$ denotes the norm on the Lebesgue space $L^q = L^q(X, d, \mu)$. As we can see, $\mathcal{M}_{q,\beta}$ is clearly a generalization of $\mathfrak{m}_{q,\beta}$. In the last decades, much work has been dedicated to obtaining Morrey and Lebesgue norm inequalities for $\mathcal{M}_{q,\beta}$ and other operators of fractional maximal type on spaces of homogeneous type. We refer the reader to [1,2,4,16,17,19,21] and the references therein.

As in the Euclidean case, Lebesgue and Morrey spaces on homogeneous type spaces may be viewed as the end points of a chain of Banach function spaces $(L^q, L^p)^\alpha(X)$ defined as follows: a μ-measurable function f represents an element of $(L^q, L^p)^\alpha(X)$ if

$$\|f\|_{q,p,\alpha} = \sup_{r>0}, \quad {}_r\|f\|_{q,p,\alpha} < \infty,$$

where

$${}_r\|f\|_{q,p,\alpha} = \begin{cases} \Big[\int_X \big(\mu(B_{(y,r)})^{\frac{1}{\alpha} - \frac{1}{p} - \frac{1}{q}} \|f \chi_{B_{(y,r)}}\|_q \big)^p d\mu(y) \Big]^{\frac{1}{p}} & \text{if } p < \infty, \\ \sup\operatorname{ess}_{y \in X} \mu(B_{(y,r)})^{\frac{1}{\alpha} - \frac{1}{q}} \|f \chi_{B_{(y,r)}}\|_q & \text{if } p = \infty. \end{cases}$$

The $(L^q, L^p)^\alpha(X)$ are generalizations of the $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$, and the main properties extend to them (see [5]).

In this paper we are interested in continuity properties of $\mathcal{M}_{q,\beta}$ and the fractional integral operator I_γ (as defined by relation (1.3)) involving the spaces $(L^q, L^p)^\alpha(X)$ and weights fulfilling condition of $\mathcal{A}_\infty$ type stated in terms of Orlicz norm as in [16].

The main result is Theorem 2.3, which is an extension of Theorem 1.2 and contains, as a special case, the following result.

Theorem 1.4 *Assume*

- *there is a positive non decreasing function* φ *defined on* $[0, \infty)$ *and positive constants* $\mathfrak{a}$ *and* $\mathfrak{b}$ *such that*

$$\mathfrak{a}\varphi(r) \leq \mu(B_{(x,r)}) \leq \mathfrak{b}\varphi(r) \quad x \in X \, 0 < r, \tag{1.2}$$

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- *q, α, p, and β are elements of $[1, \infty]$ such that $q \leq \alpha \leq p$ and $0 < \frac{1}{s} = \frac{1}{\alpha} - \frac{1}{\beta} \leq \frac{1}{q} - \frac{1}{\beta} \leq \frac{1}{p}$.*

Then there is a real constant C such that, for any μ-measurable function f on X we have

$$\|\mathcal{M}_{q,\beta} f\|^*_{s,\infty} \equiv \sup_{\theta>0} \theta\mu(\{x \in X : \mathcal{M}_{q,\beta} f(x) > \theta\})^{1/s} \leq C\|f\|_{q,p,\alpha}.$$

Note that condition (1.2) is satisfied in the following cases:

- X is an Ahlfors n regular metric space, *i.e.,* there is a positive integer n and a positive constant C which is independent of the main parameters such that $C^{-1}r^n \leq \mu(B_{(x,r)}) \leq Cr^n$,
- X is a Lie group with polynomial growth equipped with a left Haar measure μ and the Carnot–Carathéodory metric d associated with a Hörmander system of left invariant vector fields (see [10, 13, 20]).

Let us assume the hypotheses of Theorem 1.4 and that $q < \alpha < p$. Theorems 2.11 and 2.12 of [5] assert that weak-$L^\alpha(X)$ is strictly included in $(L^q, L^p)^\alpha(X)$. So we may find an element f_0 in $(L^q, L^p)^\alpha(X)$ which is not in weak-$L^\alpha(X)$ space. Theorem 1.4 asserts that $\mathcal{M}_{q,\beta} f_0$ belongs to the weak-L^s space, while Theorem 2-7 of [16] gives no control on it. This remark shows that, even if $\mathcal{M}_{q,\beta}$ is a particular case of the maximal operator $\mathcal{M}_\psi$ under consideration in Theorem 2-7 of [16], the range of application of this last theorem is different from that of our Theorem 2.3.

It is worth noting that $\mathcal{M}_{q,\beta}$ satisfies a norm inequality similar to (1.1) (see Theorem 2.4). This implies that if the maximal function $\mathcal{M}_{q,\beta} f$ belongs to some weak-Lebesgue space, then f is in some $(L^q, L^p)^\alpha(X)$.

Let us consider the following fractional operator I_γ $(0 < \gamma < 1)$ defined by

$$I_\gamma f(x) = \int_X \frac{f(y)d\mu(y)}{\mu(B(x, d(x, y)))^{1-\gamma}}. \tag{1.3}$$

This operator is clearly an extension of the classical Riesz potential operator in $\mathbb{R}^n$. As in the Euclidean case, I_γ is controlled in norm by $\mathcal{M}_{1,\beta}$ where $\beta = \frac{1}{\gamma}$ (see Theorem 3.1). Thus from the weight norm inequality on $\mathcal{M}_{1,\beta}$ stated in Theorem 2.3, we may deduce a similar one on I_γ.

The remainer of the paper is organized as follows: Section 2 is devoted to continuity properties of $\mathcal{M}_{q,\beta}$ and also contains background elements on homogeneous spaces, Young functions, and $(L^q, L^p)^\alpha(X)$ spaces. In Section 3 we extend the results on $\mathcal{M}_{q,\beta}$ to I_γ. Throughout the paper, we will denote by C a positive constant which is independent of the main parameters, but may vary from line to line. Constants with subscripts such as C_μ, do not change in different occurrences.

2 Continuity of the Fractional Maximal Operators $\mathcal{M}_{q,\beta}$

Let $X = (X, d, \mu)$ be a space of homogeneous type: (X, d) is a quasi-metric space endowed with a non negative Borel measure μ satisfying the following doubling condition

$$\mu(B_{(x,2r)}) \leq C\mu(B_{(x,r)}) < \infty, \quad x \in X, r > 0, \tag{2.1}$$

where $B_{(x,r)} = \{y \in X : d(x, y) < r\}$ is the ball of center x and radius r in X. If B is an arbitrary ball, then we denote by x_B its center and $r(B)$ its radius, and for any real number $\delta > 0$, δB denotes the ball centered at x_B with radius $\delta r(B)$.

Since d is a quasimetric, there exists a constant $\kappa \geq 1$ such that

$$d(x, z) \leq \kappa(d(x, y) + d(y, z)), \quad x, y, z \in X.$$

If C'_μ is the smallest constant for which (2.1) holds, then $D_\mu = \log_2 C'_\mu$ is called the doubling order of μ. It is known [2, 21] that for all balls $B_2 \subset B_1$ of (X, d)

$$\frac{\mu(B_1)}{\mu(B_2)} \leq C_\mu \Big(\frac{r(B_1)}{r(B_2)}\Big)^{D_\mu},$$

where $C_\mu = C'_\mu(2\kappa)^{D_\mu}$. A quasimetric δ on X is said to be equivalent to d if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 d(x, y) \leq \delta(x, y) \leq C_2 d(x, y), \quad x, y \in X.$$

We observe that topologies defined by equivalent quasimetrics on X are equivalent. It is shown [12] that there is a quasimetric δ equivalent to d for which balls are open sets.

In the sequel we assume that $X = (X, d, \mu)$ is a fixed space of homogeneous type and

- all balls $B_{(x,r)} = \{y \in X : d(x, y) < r\}$ are open subsets of X endowed with the d-topology and (X, d) is separable,
- $\mu(X) = \infty$,
- $B_{(x,R)} \setminus B_{(x,r)} \neq \varnothing$, $0 < r < R < \infty$, and $x \in X$, so that as proved in [22], there exist two constants $\tilde{C}_\mu > 0$ and $\delta_\mu > 0$ such that

$$\frac{\mu(B_1)}{\mu(B_2)} \geq \tilde{C}_\mu \Big(\frac{r(B_1)}{r(B_2)}\Big)^{\delta_\mu} \text{ for all balls } B_2 \subset B_1 \text{ of } X. \tag{2.2}$$

Now we recall some concepts necessary to express the conditions we impose on our weights.

Definition 2.1 Let Φ be a non negative function on $[0, \infty)$.

(i) Φ is a Young function if it is continuous, non decreasing, convex and satisfies the conditions $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$.

(ii) Assume that Φ is a Young function:

   (a) It is doubling if there is $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t \geq 0$.

   (b) It satisfies the B_p condition ($1 \leq p < \infty$) if there is a number $\tilde{a} > 0$ such that
   $$\int_{\tilde{a}}^{\infty} \frac{\Phi(t)}{t^p}\frac{dt}{t} < \infty.$$

   (c) Its conjugate Φ^*, is defined by $\Phi^*(u) = \sup\{tu - \Phi(t) : t \in \mathbb{R}_+\}$.

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(d) For any μ-measurable function f on X,

$$\|f\|_{\Phi,B} = \inf\Big\{\lambda > 0 : \frac{1}{\mu(B)} \int_B \Phi(\lambda^{-1}|f|)d\mu \leq 1\Big\}$$

for any ball B in X, and $M_\Phi f(x) = \sup_{\text{ball } B \ni x} \|f\|_{\Phi,B}$.

It is proved in [17, Theorem 5.1] that a doubling Young function Φ belongs to the class B_p with $1 < p < \infty$ if and only if there exists a constant $C > 0$ such that

$$\int_X (M_\Phi f(x))^p d\mu(x) \leq C \int_X f(x)^p d\mu(x) \tag{2.3}$$

for all non negative f. We also have the local version of the generalized Hölder inequality

$$\frac{1}{\mu(B)} \int_B |fg| d\mu \leq \|f\|_{\Phi,B}\|g\|_{\Phi^*,B},$$

which is valid for all measurable functions f and g, and for all balls B. For more information about Young function, see [18].

We will need the following covering lemma stated and proved in [2].

Lemma 2.2 *Let $\mathfrak{F}$ be a family of balls with bounded radii. Then there exists a countable subfamily of disjoint balls $\{B_{(x_i,r_i)}, i \in J\}$ such that each ball in $\mathfrak{F}$ is contained in one of the balls $B_{(x_i,3\kappa^2 r_i)}$, $i \in J$.*

We are now ready to state and prove our main result.

Theorem 2.3 *Let $q, \alpha, p, q_1, \alpha_1, p_1, \beta$ be elements of $[1, \infty]$ such that*

$$1 \leq q \leq \alpha \leq p \text{ with } 0 < \frac{1}{\alpha} - \frac{1}{\beta} = \frac{1}{s},$$

and

$$q < q_1 \leq \alpha_1 \leq p_1 < \infty \text{ with } 0 < \frac{1}{q_1} - \frac{1}{\beta} = \frac{1}{t} \leq \frac{1}{p_1}.$$

Let (w, v) be a pair of weights for which there exists a constant A such that

$$\mu(B)^{-1/t}\|w\chi_B\|_t\|v^{-q}\|^{1/q}_{\Phi,B} \leq A$$

for all balls B in (X, d), where Φ is a doubling Young function whose conjugate function Φ^* satisfies the $B_{q_1/q}$ condition. Then there is a constant C such that for any μ-measurable function f, and $\theta > 0$, we have

$$\Big(\int_{\Pi_\theta} w^t(x)d\mu(x)\Big)^{1/t} \leq C\theta^{-1}\|fv\|_{q_1}, \tag{2.4}$$

and if we assume that μ satisfies condition (1.2), *then*

$$\Big(\int_{\Pi_\theta} w^t(x)d\mu(x)\Big)^{1/t} \leq C\big(\theta^{-1}\|fv\|_{q_1,p_1,\alpha_1}\big)\,(\theta^{-1}\|f\|_{q,p,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}, \tag{2.5}$$

where $\Pi_\theta = \{x \in X : \mathcal{M}_{q,\beta} f(x) > \theta\}$.

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Proof Inequality (2.4) is immediate from [16, Theorem 2.7]. We just need to prove inequality (2.5).

Let f be an element of $(L^q, L^p)^\alpha(X)$. Fix $\theta > 0$. For x in Π_θ, there exists r_x such that

$$\mu(B_{(x,r_x)})^{\frac{1}{\beta}-\frac{1}{q}}\|f\chi_{B_{(x,r_x)}}\|_q > \theta, \tag{2.6}$$

and therefore

$$\mu(B_{(x,r_x)}) \leq (\theta^{-1}\|f\|_{q,\infty,\alpha})^s. \tag{2.7}$$

Fix a ball $B_{(x_0,R)}$ in X and set $\Pi_\theta^R = \Pi_\theta \cap B_{(x_0,R)}$. For any x in Π_θ^R we have $B_{(x_0,R)} \subset B_{(x,r_x)}$ provided $r_x > 2\kappa R$. It follows from the reverse doubling property (2.2) and (2.7) that

$$r_x^{\delta_\mu} \leq C_\mu^{-1}\frac{R^{\delta_\mu}}{\mu(B_{(x_0,R)})}(\theta^{-1}\|f\|_{q,\infty,\alpha})^s.$$

So we obtain that for any x in Π_θ^R,

$$r_x^{\delta_\mu} \leq \max\Big\{2\kappa R, C_\mu^{-1}\frac{R^{\delta_\mu}}{\mu(B_{(x_0,R)})}(\theta^{-1}\|f\|_{q,\infty,\alpha})^s\Big\} < \infty.$$

Thus by Lemma 2.2, the family $\mathcal{F} = \{B_{(x,r_x)} : x \in \Pi_\theta^R\}$ has a countable subfamily $\{B_i : i \in J\}$ of disjoint balls such that each element B of $\mathcal{F}$ is contained in some $3\kappa^2 B_i$.

Let i be an element of J. By (2.6) and the generalized Hölder inequality we have

$$\begin{aligned}\theta^q &\leq \mu(B_i)^{q/\beta}\Big(\frac{1}{\mu(B_i)}\int_{B_i}|f\nu\nu^{-1}|d\mu\Big)\\ &\leq C\mu(B_i)^{q/\beta}\|(f\nu\chi_{B_i})^q\|_{\Phi^*,3\kappa^2B_i}\|\nu^{-1}\chi_{B_i}\|_{\Phi,3\kappa^2B_i}\\ &\leq C\mu(B_i)^{q/\beta}M_{\Phi^*}(f\nu\chi_{B_i})^q(y)\|\nu^{-q}\chi_{B_i}\|_{\Phi,3\kappa^2B_i}\end{aligned}$$

for any y in B_i. So we obtain

$$\theta^q\mu(B_i) \leq C\mu(B_i)^{q/\beta}\int_{B_i}M_{\Phi^*}(f\nu\chi_{B_i})^q(y)d\mu(y)\|\nu^{-q}\chi_{B_i}\|_{\Phi,3\kappa^2B_i}.$$

Applying the Hölder inequality and (2.3) we get

$$\begin{aligned}\theta^q &\leq C\mu(B_i)^{-q/t}[\int_{B_i}\{M_{\Phi^*}(f\nu\chi_{B_i})^q(y)\}^{q_1/q}d\mu(y)]^{q/q_1}\|\nu^{-q}\chi_{B_i}\|_{\Phi,3\kappa^2B_i}\\ &\leq C\mu(B_i)^{-q/t}\|f\nu\chi_{B_i}\|_{q_1}^q\|\nu^{-q}\chi_{B_i}\|_{\Phi,3\kappa^2B_i},\end{aligned}$$

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that is,

$$1 \le C\theta^{-1}\mu(B_i)^{-1/t}\|f\nu\chi_{B_i}\|_{q_1}\|\nu^{-q}\chi_{B_i}\|^{1/q}_{\Phi,3\kappa^2 B_i}.$$

As $\Pi^R_\theta \subset \bigcup_{i\in J} 3\kappa^2 B_i$ and $\frac{p_1}{t} \le 1$, we have

$$\begin{aligned}\|w\chi_{\Pi^R_\theta}\|_t &\le \Big(\sum_{i\in J}\|w\chi_{3\kappa^2 B_i}\|^{p_1}_t\Big)^{1/p_1}\\ &\le C\theta^{-1}\Big[\sum_{i\in J}(\mu(B_i)^{-1/t}\|f\nu\chi_{B_i}\|_{q_1}\|\nu^{-q}\chi_{B_i}\|^{1/q}_{\Phi,3\kappa^2 B_i}\|w\chi_{3\kappa^2 B_i}\|_t)^{p_1}\Big]^{1/p_1}.\end{aligned}$$

Thus, according to assumption (2.7),

$$\|w\chi_{\Pi^R_\theta}\|_t \le C\theta^{-1}\Big(\sum_{i\in J}\|f\nu\chi_{B_i}\|^{p_1}_{q_1}\Big)^{1/p_1}. \tag{2.8}$$

Let n be a positive integer and set

- $J_n = \{i \in J : \frac{1}{n} \le r(B_i)\}$,
- m_n and $\bar{k}$ the integers satisfying respectively

$$\rho^{m_n+1} \le \frac{1}{2\kappa n} < \rho^{m_n+2} \quad \text{and} \quad \rho^{\bar{k}+1} \le \frac{r}{2\kappa} < \rho^{\bar{k}+2},$$

where $r = \sup\{r(B_i), i \in J\}$ and $\rho = 8\kappa^5$.

It was proved in [19] that there are points x^k_j and Borel sets E^k_j, $1 \le j < N_k$, $k \ge m_n$ (where $N_k \in \mathbb{N} \cup \{\infty\}$), such that

(i) $B_{(x^k_j,\rho^k)} \subset E^k_j \subset B_{(x^k_j,\rho^{k+1})}$, $1 \le j < N_k$, $k \ge m_n$,

(ii) $X = \bigcup_j E^k_j$, $k \ge m_n$, and $E^k_j \cap E^k_i = \varnothing$ if $i \ne j$,

(iii) given i, j, k, ℓ with $m_n \le k < \ell$, then either $E^k_j \subset E^\ell_i$ or $E^k_j \cap E^\ell_i = \varnothing$.

Let i be an element of J_n. Denote by k_i the integer satisfying

$$\rho^{k_i+1} \le \frac{r(B_i)}{2\kappa} < \rho^{k_i+2}$$

and set $L_i = \{j : 1 \le j < N_{k_i},\ E^{k_i}_j \cap B_i \ne \varnothing\}$.

We know that the number of elements of L_i is less than a constant $\mathfrak{N}$ depending only on the structure constants $(\kappa, C_\mu, D_\mu, \tilde{C}_\mu, \delta_\mu)$ (see [5, (3.3) and (4.3)]). Denoting by j_i an element of L_i satisfying

$$\big\|f\nu\chi_{B_i\cap E^{k_i}_{j_i}}\big\|_{q_1} = \max_{j\in L_i}\big\|f\nu\chi_{B_i\cap E^{k_i}_j}\big\|_{q_1},$$

we have

$$\|f\nu\chi_{B_i}\|_{q_1} \le \mathfrak{N}\big\|f\nu\chi_{B_i\cap E^{k_i}_{j_i}}\big\|_{q_1}.$$

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Hence

$$\Big(\sum_{i\in J_n}\|fv\chi_{B_i}\|_{q_1}^{p_1}\Big)^{1/p_1} \leq \mathfrak{N}\Big(\sum_{i\in J_n}\|fv\chi_{E_{j_i}^{k_i}\cap B_i}\|_{q_1}^{p_1}\Big)^{1/p_1}$$
$$= \mathfrak{N}\Big(\sum_{\ell=1}^{N_{\bar{k}}}\sum_{i\in J_n:E_{j_i}^{k_i}\subset E_\ell^{\bar{k}}}\|fv\chi_{E_{j_i}^{k_i}\cap B_i}\|_{q_1}^{p_1}\Big)^{1/p_1}$$
$$\leq \mathfrak{N}\Big(\sum_{\ell=1}^{N_{\bar{k}}}\|fv\chi_{E_\ell^{\bar{k}}\cap(\cup_{i\in J_n}B_i)}\|_{q_1}^{p_1}\Big)^{1/p_1}$$
$$= \mathfrak{N}\Big[\sum_{\ell=1}^{N_{\bar{k}}}(\mu(E_\ell^{\bar{k}})^{\frac{1}{\alpha_1}-\frac{1}{q_1}}\|fv\chi_{E_\ell^{\bar{k}}\cap(\cup_{i\in J_n}B_i)}\|_{q_1})^{p_1}\mu(E_\ell^{\bar{k}})^{\frac{p}{q_1}-\frac{p}{\alpha_1}}\Big]^{1/p_1}.$$

Notice that for any $1\leq\ell<N_{\bar{k}}$, we have

$$\mu(E_\ell^{\bar{k}})\leq\mu(B_{(x_\ell^{\bar{k}},\rho^{\bar{k}+1})})\leq\mathfrak{b}\varphi(\rho^{\bar{k}+1})\leq\mathfrak{b}\varphi\Big(\frac{r}{2\kappa}\Big)\leq\mathfrak{b}\sup_{i\in J}\varphi\Big(\frac{r(B_i)}{2\kappa}\Big)$$
$$\leq\mathfrak{b}\sup_{i\in J}\mathfrak{a}^{-1}\mu\Big(\frac{1}{2\kappa}B_i\Big)\leq\mathfrak{b}\mathfrak{a}^{-1}(\theta^{-1}\|f\|_{q,\infty,\alpha})^s.$$

Therefore,

$$\text{(2.9)}\quad\Big(\sum_{i\in J_n}\|fv\chi_{B_i}\|_{q_1}^{p_1}\Big)^{1/p_1}$$
$$\leq C\Big[\sum_{\ell=1}^{N_{\bar{k}}}\big(\mu(E_\ell^{\bar{k}})^{\frac{1}{\alpha_1}-\frac{1}{q_1}}\|fv\chi_{E_\ell^{\bar{k}}}\|_{q_1}\big)^{p_1}\Big]^{1/p_1}(\theta^{-1}\|f\|_{q,\infty,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}.$$

Since the last formula does not depend on n, we get from (2.8) and (2.9)

$$\|w\chi_{\Pi_\theta^R}\|_t\leq C\theta^{-1}\Big[\sum_{\ell=1}^{N_{\bar{k}}}\big(\mu(E_\ell^{\bar{k}})^{\frac{1}{\alpha_1}-\frac{1}{q_1}}\|fv\chi_{E_\ell^{\bar{k}}}\|_{q_1}\big)^{p_1}\Big]^{1/p_1}(\theta^{-1}\|f\|_{q,\infty,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}.$$

We recall that Proposition 4.1 of [5] asserts that there are positive constants C_1 and C_2 not depending on r and fv such that

$$C_1\,{}_r\|fv\|_{q_1,p_1,\alpha_1}\leq\Big[\sum_{\ell=1}^{N_{\bar{k}}}\big(\mu(E_\ell^{\bar{k}})^{\frac{1}{\alpha_1}-\frac{1}{q_1}}\|fv\chi_{E_\ell^{\bar{k}}}\|_{q_1}\big)^{p_1}\Big]^{1/p_1}\leq C_2\,{}_r\|fv\|_{q_1,p_1,\alpha_1}.$$

So we have

$$\|w\chi_{\Pi_\theta^R}\|_t\leq C\theta^{-1}\,{}_r\|fv\|_{q_1,p_1,\alpha_1}(\theta^{-1}\|f\|_{q,\infty,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}$$
$$\leq C\theta^{-1}\|fv\|_{q_1,p_1,\alpha_1}(\theta^{-1}\|f\|_{q,\infty,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}.$$

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As (x_0, R) is arbitrary in $X \times (0, \infty)$, we obtain

$$\|w\chi_{\Pi_\theta}\|_t \leq C\theta^{-1}\|fv\|_{q_1,p_1,\alpha_1}(\theta^{-1}\|f\|_{q,\infty,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})}. \qquad \blacksquare$$

In the proof of the above theorem, the condition $q < q_1$ is needed only when we have to use the $B_{q_1/q}$ characterization. When $w = v = 1$, this characterization is not necessary. So Theorem 1.4 follows immediately from Theorem 2.3.

The next theorem is some kind of reverse for Theorem 1.4

Theorem 2.4 *Let q, α, u, and v be elements of $[1, \infty]$ such that*

$$q \leq \alpha, \quad 0 \leq \frac{1}{\alpha} - \frac{1}{\beta} = \frac{1}{s}, \quad \text{and} \quad u \leq s \leq v.$$

Then there is a constant D such that for any μ-measurable function f

$$\|f\|_{q,v,\alpha} \leq D\|\mathcal{M}_{q,\beta}f\|_{u,v,s}.$$

Proof Let f be such that $\|\mathcal{M}_{q,\beta}f\|_{u,v,s} < \infty$. We notice that under the hypothesis, we have $q \leq \alpha \leq s \leq v$ and $\alpha \leq \beta$.

Case 1: $q = \infty$. Then $\alpha = \beta = s = v = \infty$ and therefore, it follows from the definitions that

$$\|f\|_{\infty,\infty,\infty} = \|f\|_\infty = \|\mathcal{M}_{\infty,\infty}f\|_\infty \leq C\|\mathcal{M}_{\infty,\infty}f\|_{u,\infty,\infty}.$$

Case 2: $q < \infty$.

(i) If $u = \infty$, then $s = v = \infty$, $\alpha = \beta$ and therefore,

$$\begin{aligned}\|f\|_{q,\infty,\alpha} &= \sup_{r>0}\operatorname*{sup\,ess}_{x\in X}\mu(B_{(x,r)})^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{B_{(x,r)}}\|_q \\ &= \sup_{r>0}\operatorname*{sup\,ess}_{x\in X}\mu(B_{(x,r)})^{\frac{1}{\beta}-\frac{1}{q}}\|f\chi_{B_{(x,r)}}\|_q = \|\mathcal{M}_{q,\beta}f\|_\infty = \|\mathcal{M}_{q,\beta}f\|_{\infty,\infty,\infty}.\end{aligned}$$

(ii) Suppose that $u < \infty$, and consider two positive real numbers r and r_1 satisfying $r_1 = \frac{r}{2\kappa}$. For any $y \in X$ and $x \in B_{(y,r_1)}$, we have $B_{(y,r_1)} \subset B_{(x,r)}$ and therefore, by the doubling condition

$$\mathcal{M}_{q,\beta}f(x) \geq C_\mu^{\frac{1}{\beta}-\frac{1}{q}}\mu(B_{(y,r_1)})^{\frac{1}{\beta}-\frac{1}{q}}\|f\chi_{B_{(y,r_1)}}\|_q.$$

From this, it follows that for any $y \in X$, we have

$$\|\mathcal{M}_{q,\beta}f\chi_{B_{(y,r_1)}}\|_u \geq C_\mu^{\frac{1}{\beta}-\frac{1}{q}}\mu(B_{(y,r_1)})^{\frac{1}{u}+\frac{1}{\beta}-\frac{1}{q}}\left\|f\chi_{B_{(y,r_1)}}\right\|_q$$

and therefore,

$$\mu(B_{(y,r_1)})^{\frac{1}{s}-\frac{1}{v}-\frac{1}{u}}\left\|\mathcal{M}_{q,\beta}f\chi_{B_{(y,r_1)}}\right\|_u \geq C_\mu^{\frac{1}{\beta}-\frac{1}{q}}\mu(B_{(y,r_1)})^{\frac{1}{\alpha}-\frac{1}{v}-\frac{1}{q}}\left\|f\chi_{B_{(y,r_1)}}\right\|_q.$$

This yields immediately the desired inequality. $\blacksquare$

3 Continuity of the Fractional Integral $I_\alpha f$

It is known in the Euclidean case that the fractional integral $I_\gamma f$ is controlled in norm by the fractional maximal function $\mathfrak{m}_{1,\frac{1}{\gamma}} f$ (see [14, Theorem 1]). We give the analogue of this control in the setting of spaces of homogeneous type.

Theorem 3.1 *Let $0 < q < \infty$, $0 < \gamma < 1$ and a weight w in $\mathcal{A}_\infty$. There is a constant C such that for any μ-measurable function f*

$$\sup_{\lambda>0} \lambda^q \int_{E_a} w(x)\, d\mu(x) \le C \sup_{\lambda>0} \lambda^q \int_{F_\lambda} w(x)\, d\mu(x),$$

where $E_\lambda = \{x \in X : |I_\gamma f(x)| > \lambda\}$ and $F_\lambda = \{x \in X : \mathcal{M}_{1,\frac{1}{\gamma}} f(x) > \lambda\}$.

Proof In our argumentation, we shall adapt the proof of [14, Theorem 1], keeping in mind that we do not have a Whitney decomposition available.

(1) Let f be a μ-measurable, non negative, bounded function, with a support included in a ball $B_0 = B_{(x_0,k_0)}$. According to [21, Lemma 6], there exists a constant $C_0 > 0$ not depending on f, such that $I_\gamma f \le \mathcal{M}(I_\gamma f) \le C_0 I_\gamma f$, where $\mathcal{M} = \mathcal{M}_{1,\infty}$. Let θ be a positive number and set

$$\widetilde{E}_\theta = \{x \in X : \mathcal{M}(I_\gamma f)(x) > \theta\} \quad \text{and} \quad E_\theta = \{x \in X : I_\gamma f(x) > \theta\}.$$

The set E_θ is included in $\widetilde{E}_\theta$ which is opened and satisfies $\mu(\widetilde{E}_\theta) < \infty$. According to [21, Lemma 8], there exists a countable family $\{B_{(x_i,r_i)}\ ;\ i \in J\}$ of pairwise disjoint balls and two positive constants M and c depending only on the structure constants of X, such that

$$\text{(3.1)} \qquad \begin{gathered} \widetilde{E}_\theta = \bigcup_{i\in J} B_{(x_i,cr_i)}, \qquad \sum_{i\in J} \chi_{B_{(x_i,2\kappa cr_i)}} \le M\chi_{\widetilde{E}_\theta}, \\ B_{(x_i,4\kappa^2 cr_i)} \cap (X \setminus \widetilde{E}_\theta) \ne \varnothing \quad \text{for all } i \in J. \end{gathered}$$

Let us consider an element (a,ε) of $(1,\infty) \times (0,1]$, and set

$$F_{\theta\varepsilon} = \{x \in X : \mathcal{M}_{1,1/\gamma} f(x) > \theta\varepsilon\}, \quad J_1 = \{i \in J : B_{(x_i,cr_i)} \subset F_{\theta\varepsilon}\},$$
$$J_2 = I \setminus J_1 = \{i \in J : B_{(x_i,cr_i)} \setminus F_{\theta\varepsilon} \ne \varnothing\}.$$

Arguing as in the proof of [14, Lemma 1], we obtain two constants $K > 0$ and $B > 1$ depending only on the structure constants of X, such that if $a \ge B$ and $i \in J_2$, then

$$\text{(3.2)} \qquad \mu\Big(\{x \in B_{(x_i,cr_i)} : I_\gamma f(x) > a\theta\}\Big) \le K\mu(B_{(x_i,cr_i)})\Big(\frac{\epsilon}{a}\Big)^{\frac{1}{1-\gamma}}.$$

Since $E_{\theta a} \subset E_\theta \subset \widetilde{E}_\theta = \bigcup_{i\in J} B_{(x_i,cr_i)}$, we have

$$E_{\theta a} = \Big[\bigcup_{i\in J_1} (E_{\theta a} \cap B_{(x_i,cr_i)})\Big] \cup \Big[\bigcup_{i\in J_2} (E_{\theta a} \cap B_{(x_i,cr_i)})\Big] \subset F_{\theta\varepsilon} \cup \Big[\bigcup_{i\in J_2} (E_{\theta a} \setminus F_{\theta\epsilon}) \cap B_{(x_i,cr_i)}\Big],$$

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and therefore

$$\int_{E_{\theta a}} w(x)\,d\mu(x) \le \int_{F_{\theta\varepsilon}} w(x)\,d\mu(x) + \sum_{i\in J_2} \int_{(E_{\theta a}\setminus F_{\theta\varepsilon})\cap B_{(x_i,cr_i)}} w(x)\,d\mu(x). \tag{3.3}$$

Now fix $a \ge B$ and $\rho > 0$. Since w is in $\mathcal{A}_\infty$, there exists $\delta > 0$ such that for any ball B in X and any subset E of B satisfying $\mu(E) \le \delta\mu(B)$, we have

$$\int_E w(x)\,d\mu(x) \le \rho \int_B w(x)\,d\mu(x).$$

Choose $\overline{\varepsilon} \in (0,1]$ such that $K(\frac{\overline{\varepsilon}}{a})^{\frac{1}{1-\gamma}} < \delta$ and take $0 < \varepsilon < \min(\overline{\varepsilon}, \frac{1}{C_0 L})$, where $L = C_\mu(2\kappa + 4\kappa^2)^{(1-\gamma)D_\mu}$. According to (3.2) we have for any $i \in J_2$,

$$\mu(B_{(x_i,cr_i)} \cap E_{\theta a}) < \delta\mu(B_{(x_i,cr_i)}),$$

and therefore

$$\int_{B_{(x_i,cr_i)}\cap E_{\theta a}} w(x)\,d\mu(x) \le \rho \int_{B_{(x_i,cr_i)}} w(x)\mu(x).$$

From this inequality, (3.3), and (3.1) we obtain

$$\int_{E_{\theta a}} w(x)\,d\mu(x) \le \int_{F_{\theta\varepsilon}} w(x)\,d\mu(x) + \rho M \int_{\widetilde{E}_\theta} w(x)\,d\mu(x). \tag{3.4}$$

Let $x \in X \setminus 3\kappa B_0$. Assume that $0 < t < \frac{1}{2}\inf_{y\in B_0} d(x,y)$ and $u_t \in B_0$ satisfies

$$d(x,u_t) - t \le d(x,y),\ \ y \in B_0.$$

We have $2r(B_0) \le d(x,y) \le \kappa[d(x,u_t) + 2\kappa r(B_0)],\ y \in B_0$ and therefore

$$\begin{aligned}
I_\gamma f(x) &\le \int_{B_0} \frac{f(y)}{\mu(B_{(x,d(x,y))})^{1-\gamma}}\,d\mu(y)\\
&\le \frac{1}{\mu(B_{(x,d(x,u_t)-t)})^{1-\gamma}} \int_{B_{(x,\kappa(d(x,u_t)+2\kappa r(B_0)))}} f(y)\,d\mu(y)\\
&\le C_\mu\Big[\frac{\kappa(d(x,u_t) + 2\kappa r(B_0))}{d(x,u_t) - t}\Big]^{(1-\gamma)D_\mu} \mathcal{M}_{1,\frac{1}{\gamma}} f(x) \le L\mathcal{M}_{1,\frac{1}{\gamma}} f(x).
\end{aligned}$$

Hence, $\widetilde{E}_\theta \subset E_{\theta/C_0} \subset (E_{\theta/C_0} \cap 3\kappa B_0) \cup F_{\theta/C_0 L}$. We obtain from (3.4)

$$\begin{aligned}
\int_{E_{\theta a}} w(x)\,d\mu(x) &\le \int_{F_{\theta\varepsilon}} w(x)\,d\mu(x) + \rho M \int_{E_{\frac{\theta}{C_0}}\cap 3\kappa B_0} w(x)d\mu(x)\\
&\qquad + \rho M \int_{F_{\frac{\theta}{C_0 L}}} w(x)\,d\mu(x)\\
&\le (1+\rho M)\int_{F_{\theta\varepsilon}} w(x)\,d\mu(x) + \rho M \int_{E_{\frac{\theta}{C_0}}\cap 3\kappa B_0} w(x)\,d\mu(x).
\end{aligned}$$

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That is,

$$(\theta a)^q \int_{E_{\theta a}} w(x)\, d\mu(x) \le (1+\rho M)\Big(\frac{a}{\varepsilon}\Big)^q (\theta\varepsilon)^q \int_{F_{\theta\varepsilon}} w(x)\, d\mu(x) + \rho M\Big(\frac{\theta}{C_0}\Big)^q (C_0 a)^q \int_{E_{\frac{\theta}{C_0}} \cap 3\kappa B_0} w(x)\, d\mu(x).$$

Let N be a positive integer. From the preceding inequality we obtain

$$\sup_{0<s<N} s^q \int_{E_s} w(x)\, d\mu(x) \le (1+\rho M)\Big(\frac{a}{\varepsilon}\Big)^q \sup_{0<s<N\frac{\varepsilon}{a}} s^q \int_{F_s} w(x)\, d\mu(x) + \rho M (C_0 a)^q \sup_{0<s<\frac{N}{aC_0}} s^q \int_{E_s \cap 3\kappa B_0} w(x)\, d\mu(x).$$

As

$$\sup_{0<s<\frac{N}{aC_0}} s^q \int_{E_s \cap 3\kappa B_0} w(x)\, d\mu(x) \le \sup_{0<s<N} s^q \int_{E_s \cap 3\kappa B_0} w(x)\, d\mu(x) < \infty,$$

by taking $\rho = \frac{1}{2M(C_0 a)^q}$ in the last inequality, we get

$$\frac{1}{2} \sup_{0<s<N} s^q \int_{E_s} w(x)\, d\mu(x) \le \Big(1 + \frac{1}{2(C_0 a)^q}\Big)\Big(\frac{a}{\varepsilon}\Big)^q \sup_{0<s<N\frac{\varepsilon}{a}} s^q \int_{F_s} w(x)\, d\mu(x).$$

The desired inequality follows by letting N go to infinity.

(2) Let f be an arbitrary μ-measurable function f. For any positive integer k, set $f_k = f\chi_{E_k}$ with $E_k = \{x \in B_{(x_0,k)} : |f(x)| \le k\}$. By part (1) of the proof, for any $k > 0$, we have

$$\frac{1}{2} \sup_{0<s<N} s^q \int_{\{x\in X: I_\gamma f_k(x) > s\}} w(x)\, d\mu(x) \le \Big(1 + \frac{1}{2(C_0 a)^q}\Big)\Big(\frac{a}{\varepsilon}\Big)^q \sup_{0<s<N\frac{\varepsilon}{a}} s^q \int_{\{x\in X: \mathcal{M}_{1,\frac{1}{\gamma}} f_k(x) > s\}} w(x)\, d\mu(x).$$

So letting k go to infinity, we obtain the result. ■

Remark 3.2 Assume that

- μ satisfies condition (1.2),
- $q, \alpha, p, p_1, q_1, \alpha_1, \gamma$ are elements of $[0,\infty]$ such that

$$1 \le q \le \alpha \le p \text{ with } 0 < \frac{1}{\alpha} - \gamma = \frac{1}{s}$$

and

$$q < q_1 \le \alpha_1 \le p_1 < \infty \text{ with } 0 < \frac{1}{q_1} - \gamma = \frac{1}{t} \le \frac{1}{p_1},$$

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- Φ is a doubling Young function whose conjugate function Φ^* satisfies the $B_{q_1/q}$ condition,
- v and w are two weights for which there exists a constant A such that

$$\mu(B)^{-1/t}\|w\chi_B\|_t\|v^{-q}\|_{\Phi,B}^{1/q} \leq A, \quad B \text{ ball}$$

and w^t satisfies $\mathcal{A}_\infty$ condition.

Then there is a constant C such that for any μ-measurable function f and $\theta > 0$, we have

$$\Big(\int_{E_\theta} w^t(x)\,d\mu(t)\Big)^{1/t} \leq C(\theta^{-1}\|fv\|_{q_1,p_1,\alpha_1})(\theta^{-1}\|f\|_{q,p,\alpha})^{s(\frac{1}{q_1}-\frac{1}{\alpha_1})},$$

where $E_\theta = \{x \in X \,/\, |I_\gamma f(x)| > \theta\}$.

Proof Let f be a μ-measurable function. From Theorem 2.3, it follows that there exists a constant C such that

$$\sup_{\theta>0}\theta^{1+s(\frac{1}{q_1}-\frac{1}{\alpha_1})}\Big(\int_{E_\theta} w^t(x)\,d\mu(x)\Big)^{1/t} \leq C\sup_{\theta>0}\theta^{1+s(\frac{1}{q_1}-\frac{1}{\alpha_1})}\Big(\int_{F_\theta} w^t(x)\,d\mu(x)\Big)^{1/t},$$

with $F_\theta = \{x \in X : \mathcal{M}_{1,\frac{1}{\gamma}} f(x) > \theta\}$. Since $\mathcal{M}_{1,\beta} \leq \mathcal{M}_{q,\beta}$ for $q > 1$, the result follows from Theorem 2.3. ∎

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