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C^*-ALGEBRAS OF REAL RANK ZERO WHOSE K_0'S ARE NOT RIESZ GROUPS

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ABSTRACT. Examples are constructed of stably finite, unital, separable C^*-algebras A of real rank zero such that the partially ordered abelian groups $K_0(A)$ do not satisfy the Riesz decomposition property. This contrasts with the result of Zhang that projections in C^*-algebras of real rank zero satisfy Riesz decomposition. The construction method also produces a stably finite, unital, separable C^*-algebra of real rank zero which has the same K-theory as an approximately finite dimensional C^*-algebra, but is not itself approximately finite dimensional.

1. **Introduction.** The Riesz decomposition property has long been an important tool in the K-theory of C^*-algebras. We recall the definition of this property for a partially ordered abelian group G: whenever $x, y_1, y_2 \in G^+$ with $x \leq y_1 + y_2$, there is a decomposition $x = x_1 + x_2$ with $x_i \in G^+$ such that $x_i \leq y_i$ for $i = 1, 2$. A directed, partially ordered abelian group satisfying Riesz decomposition is called a *Riesz group* for short. For example, Riesz decomposition was the key to the Effros-Handelman-Shen theorem [7] determining precisely which ordered groups appear as K_0-groups in Elliott's classification of AF (approximately finite dimensional) C^*-algebras [8]. It was also the fundamental property used by Handelman, Lawrence, and the author in building a structure theory for K_0 of finite Rickart C^*-algebras [15]. More recently, Riesz decomposition has played an important role in the analysis and classification of C^*-algebras of real rank zero. Zhang proved in [26, Corollary 1.3] that the projections in any C^*-algebra A of real rank zero satisfy Riesz decomposition with respect to Murray-von Neumann equivalence, that is, whenever p, q_1, q_2 are projections in A with $p \lesssim q_1 \oplus q_2$, there is an orthogonal decomposition $p = p_1 \oplus p_2$ for some projections $p_i \in A$ such that $p_i \lesssim q_i$ for $i = 1, 2$. This result was a key tool in his investigations of the structure of such algebras and their corona and multiplier algebras (*e.g.*, [26, 27]). In Elliott's recent classification of certain C^*-algebras of real rank zero [10], the Riesz decomposition property again made possible an axiomatic characterization of the range of the invariant: namely, every countable, weakly unperforated, graded Riesz group appears as $K_*(A) = K_0(A) \oplus K_1(A)$ for a separable, nuclear C^*-algebra A of real rank zero and stable rank one [9, Theorem 8.2].

If A is a C^*-algebra of real rank zero, Zhang's result implies that the monoid $V(A)$ of Murray-von Neumann equivalence classes of projections from $M_\infty(A)$ (*cf.* [1, 5.1.2]) has Riesz decomposition. In case the projections in A have cancellation (in particular, in case A has stable rank 1), it follows that $K_0(A)$ has Riesz decomposition. *A priori*,

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however, the Riesz decomposition property can be lost in the passage from the monoid $V(A)$ to the ordered group $K_0(A)$. This phenomenon was exhibited in the context of von Neumann regular rings by Moncasi [19, Example], who constructed examples of stably finite, unital regular rings whose K_0's are not Riesz groups. We show that a C^*-analog of Moncasi's construction produces stably finite, unital C^*-algebras which have the same ordered K_0's as his examples. However, the algebras thus constructed do not have real rank zero. One final construction step is necessary to produce the examples announced in our title.

As an unexpected byproduct, our construction also produces an example of a stably finite, unital C^*-algebra of real rank zero which has the same K_0- and K_1-groups as an AF C^*-algebra, yet is not AF itself. This algebra is in a sense 'just outside' the class addressed by Elliott's classification – it is an extension of a Bunce-Deddens algebra by a direct sum of two copies of the algebra of compact operators, but it has stable rank 2 and so cannot be an inductive limit of direct sums of Elliott's building block algebras. In fact, it is not an approximately homogeneous algebra. The first example of such a 'bad' extension was constructed by Putnam (see [16, 2]). Our example may be an indication that the class of all stably finite, separable, nuclear C^*-algebras with real rank zero is too large to be classified by K-theory. In particular, stable finiteness may be too weak a finiteness condition (outside the class of simple C^*-algebras), and so for classification purposes a stronger condition should be imposed. One obvious candidate is stable rank one; another, suggested by Loring, is the 'extremal richness' property studied in [4]. However, S. Eilers and N. Larsen have put a dent in the latter suggestion by showing that the example constructed here is extremally rich [personal communication].

It should be emphasized that the question whether Riesz decomposition holds in K_0 is most natural in the setting of stably finite algebras, because for such algebras the natural pre-order relation on K_0 is a partial ordering [1, Proposition 6.3.3]. Failure of Riesz decomposition would be less surprising in a K_0-group which is not partially ordered.

For notation and definitions of unexplained terms, we refer the reader to [1] and [12]. Following a suggestion of the referee, we write $n \cdot e$ rather than $n. e$ for the orthogonal sum of n copies of a projection e (that is, the $n \times n$ diagonal matrix $\mathrm{diag}(e, e, \ldots, e)$).

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2. **Construction.**

2.1. Fix a positive integer n, and let u_n denote the n-fold right shift on the Hilbert space ℓ^2, that is, the n-th power of the unilateral shift $(\alpha_1, \alpha_2, \ldots) \longmapsto (0, \alpha_1, \alpha_2, \ldots)$. Then u_n is an isometry in $\mathcal{B}(\ell^2)$ and $1 - u_n u_n^*$ is a rank n projection. Set T_n equal to the (separable, unital) sub-C^*-algebra of $\mathcal{B}(\ell^2)$ generated by u_n together with the ideal $\mathcal{K}$ of compact operators on ℓ^2. In case $n = 1$, we have the standard Toeplitz algebra; as is well known, $T_1 / \mathcal{K} \cong C(\mathbb{T})$ by an isomorphism that links the coset $u_1 + \mathcal{K}$ with the standard unitary $z \in C(\mathbb{T})$, namely the inclusion map $z: \mathbb{T} \hookrightarrow \mathbb{C}$.

An alternative description of T_n can be obtained by adapting a construction used by Plastiras [21] and Bures [5]. Let $\Delta_n: T_1 \longrightarrow M_n(T_1)$ be the diagonal map, and set $T_n' =$

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$\Delta_n(T_1) + M_n(\mathcal{K})$. It is clear that $T_n \cong T_n'$ by an isomorphism sending $\mathcal{K}$ to $M_n(\mathcal{K})$ and u_n to $\Delta_n(u_1)$. From this description we see that $T_n/\mathcal{K} \cong C(\mathbb{T})$ by an isomorphism that sends $u_n + \mathcal{K}$ to z. Let $\pi\colon T_n \longrightarrow C(\mathbb{T})$ denote the corresponding quotient map.

2.2. Choose a rank 1 projection $e \in \mathcal{K}$, so that $1 - u_n u_n^* \sim n \cdot e$. Since $u_n u_n^* \sim 1$, it follows that $1 \sim 1 \oplus n \cdot e$. Hence, $n[e] = 0$ in $K_0(T_n)$.

The groups $K_1\big(C(\mathbb{T})\big)$, $K_0(\mathcal{K})$, and $K_0\big(C(\mathbb{T})\big)$ are all infinite cyclic, generated by the classes $[z]$, $[e]$, and $[1_{C(\mathbb{T})}]$ respectively. Since $K_1(\mathcal{K}) = 0$, the standard 6-term exact sequence for the extension $0 \longrightarrow \mathcal{K} \xrightarrow{j} T_n \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0$ looks as follows:

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \xrightarrow{j_*} & K_0(T_n) & \xrightarrow{\pi_*} & K_0\big(C(\mathbb{T})\big) \\ \partial\big\uparrow & & & & \big\downarrow \\ K_1\big(C(\mathbb{T})\big) & \longleftarrow & K_1(T_n) & \longleftarrow & 0 \end{array}$$

In particular, π_* is surjective, and hence $K_0(T_n) = \mathbb{Z}[1_{T_n}] \oplus \ker(\pi_*)$. Further, $\ker(\pi_*) = \operatorname{im}(j_*)$ is generated by the class $[e] \in K_0(T_n)$. Since the unitary $z \in C(\mathbb{T})$ lifts to the isometry $u_n \in T_n$, we compute that $\partial([z]) = [1 - u_n^* u_n] - [1 - u_n u_n^*] = -n[e]$ (*cf.* [1, Section 8.3.2]). Hence, ∂ is injective and its image is generated by $n[e] \in K_0(\mathcal{K})$. It follows that $K_1(T_n) = 0$ and that

(1) $\qquad [e] \in K_0(T_n)$ has order n.

In particular, $K_0(T_n) = \mathbb{Z}[1_{T_n}] \oplus \mathbb{Z}[e] \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$.

2.3. The next step is to manufacture a stably finite algebra with a quotient isomorphic to T_n. Moncasi used a trick of G. Bergman's (*cf.* [11, Example 5.10] and [18, Lemma 13]) for this purpose. The analog in our setting is the algebra

$$B_n = \{(x, y) \in T_n \oplus T_n^{\mathrm{op}} \mid \pi(x) = \pi(y)\},$$

where T_n^{op} denotes the opposite algebra of T_n (the C^*-algebra with the same elements, linear structure, adjoint, and norm as T_n, but with the reverse multiplication). Here π is also used to denote the quotient map $T_n^{\mathrm{op}} \longrightarrow C(\mathbb{T})^{\mathrm{op}} = C(\mathbb{T})$, and the commutativity of $C(\mathbb{T})$ is crucial to ensure that B_n is closed under the multiplication in $T_n \oplus T_n^{\mathrm{op}}$. The algebra B_1 was examined by Menal [17], who showed that it is finite and has stable rank 2.

We shall give a different description of B_n that is more natural in the C^*-algebra setting. Note that $\mathcal{B}(\ell^2)^{\mathrm{op}} \cong \mathcal{B}(\ell^2)$ via transposition of matrices (with respect to the standard orthonormal basis of ℓ^2). The transpose sends u_n to u_n^*, and so it restricts to an isomorphism of T_n^{op} onto T_n. Thus, we may replace T_n^{op} by T_n, provided we also replace $C(\mathbb{T})^{\mathrm{op}}$ by a copy of $C(\mathbb{T})$ in which z is replaced by z^*. We account for the latter using the automorphism θ of $C(\mathbb{T})$ given by $f(t) \longmapsto f(t^{-1})$; note that $\theta(z) = z^*$.

Thus, we now present B_n in the form

$$B_n = \{(x, y) \in T_n \oplus T_n \mid \pi(x) = \theta\pi(y)\},$$

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which is a separable, unital sub-C^*-algebra of $T_n \oplus T_n$. (In this form, the case $n = 1$ yields an algebra that has been analyzed by Pedersen in [20, 9.3–9.6].) The subset $I = \mathcal{K} \oplus \mathcal{K}$ is a closed ideal of B_n, and $B_n/I \cong C(\mathbb{T})$. Further, the projections of $T_n \oplus T_n$ onto its factors induce isomorphisms $B_n/(0 \oplus \mathcal{K}) \cong T_n \cong B_n/(\mathcal{K} \oplus 0)$. There are two natural quotient maps $B_n \longrightarrow C(\mathbb{T})$, namely $(x,y) \longmapsto \pi(x)$ and $(x,y) \longmapsto \pi(y) = \theta\pi(x)$. We shall use π' to denote the former; thus $\pi'(u_n, u_n^*) = z$. Note that B_n is nuclear, since nuclearity is preserved in extensions [1, Theorem 15.8.2].

2.4. To compute the K-theory of B_n, we appeal to the standard 6-term exact sequence for the extension $0 \longrightarrow I \xrightarrow{j'} B_n \xrightarrow{\pi'} C(\mathbb{T}) \longrightarrow 0$, namely

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{j'_*} & K_0(B_n) & \xrightarrow{\pi'_*} & K_0\big(C(\mathbb{T})\big) \\ \partial' \uparrow & & & & \downarrow \\ K_1\big(C(\mathbb{T})\big) & \longleftarrow & K_1(B_n) & \longleftarrow & 0 \end{array}$$

As before, $K_0(B_n) = \mathbb{Z}[1_{B_n}] \oplus \ker(\pi'_*)$, but this time $\ker(\pi'_*)$ is generated by the classes $[(e,0)]$ and $[(0,e)]$ in $K_0(B_n)$. In B_n, the element (u_n, u_n^*) is a partial isometry such that $(u_n, u_n^*)(u_n, u_n^*)^* = (u_n u_n^*, 1)$ and $(u_n, u_n^*)^*(u_n, u_n^*) = (1, u_n u_n^*)$; hence, $(u_n u_n^*, 1) \sim (1, u_n u_n^*)$. Since $(1 - u_n u_n^*, 0) \sim n \cdot (e, 0)$ and $(0, 1 - u_n u_n^*) \sim n \cdot (0, e)$, we also have

$$n \cdot (e,0) \oplus (u_n u_n^*, 1) \sim (1,1) \sim n \cdot (0,e) \oplus (1, u_n u_n^*),$$

and thus $n[(e,0)] = n[(0,e)]$ in $K_0(B_n)$. Since K_0 of the left-hand projection $B_n \longrightarrow T_n$ sends $[(e,0)] - [(0,e)]$ to $[e]$, it follows from (1) that $k\big([(e,0)] - [(0,e)]\big) \neq 0$ for $k = 1, \ldots, n-1$. Therefore

(2) $$[(e,0)] - [(0,e)] \text{ has order } n \text{ in } K_0(B_n).$$

Since the unitary $z \in C(\mathbb{T})$ lifts to the partial isometry $(u_n, u_n^*) \in B_n$, we have

(3) $$\begin{aligned} \partial'([z]) &= [1 - (u_n, u_n^*)^*(u_n, u_n^*)] - [1 - (u_n, u_n^*)(u_n, u_n^*)^*] \\ &= [(0, 1 - u_n u_n^*)] - [(1 - u_n u_n^*, 0)] = n[(0,e)] - n[(e,0)] \end{aligned}$$

in $K_0(I)$. Observe that $K_0(I) \cong \mathbb{Z}^2$ with basis $[(e,0)]$, $[(0,e)]$. In view of (3), it follows that ∂' is injective, and that its image is generated by the element $n\big([(e,0)] - [(0,e)]\big)$ in $K_0(I)$. Hence,

(4) $$K_1(B_n) = 0,$$

and $\ker(j'_*) = \operatorname{im}(\partial')$ is contained in the subgroup $\mathbb{Z}\big([(e,0)] - [(0,e)]\big)$ of $K_0(I)$. This subgroup is a direct summand with complement $\mathbb{Z}[(0,e)]$. Consequently,

(5) $$K_0(B_n) = \mathbb{Z}[1_{B_n}] \oplus \mathbb{Z}[(0,e)] \oplus \mathbb{Z}\big([(e,0)] - [(0,e)]\big) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}).$$

(*Cf.* [20, Remark 9.6] for the case $n = 1$.) Under this isomorphism, the class $[1_{B_n}] \in K_0(B_n)$ corresponds to the triple $(1,0,0)$.

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Observe that $K_0(I)^+ = \mathbb{Z}^+[(e,0)] + \mathbb{Z}^+[(0,e)]$, and so $\mathrm{im}(\partial') \cap K_0(I)^+ = \{0\}$. Since I and $C(\mathbb{T})$ are both stably finite, it now follows from [24, Lemma 1.5] that

$$B_n \text{ is stably finite.} \tag{6}$$

2.5. The order structure on $K_0(B_n)$ remains to be computed. We claim that in terms of the isomorphism (5),

$$\begin{aligned} K_0(B_n)^+ &\cong \{(a,b,c+n\mathbb{Z}) \in \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \mid a > 0\} \\ &\cup \{(0,b,c+n\mathbb{Z}) \in \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \mid b \geq c \geq 0\}. \end{aligned} \tag{7}$$

Consider an arbitrary element $p = a[1_{B_n}] + b[(0,e)] + c\big([(e,0)] - [(0,e)]\big)$ in $K_0(B_n)$, where a, b, c are integers. Since π'_* sends p to $a[1_{C(\mathbb{T})}]$, we see that if $p \in K_0(B_n)^+$ then $a \geq 0$. On the other hand, $|c| \cdot (e,0) \oplus |b-c| \cdot (0,e)$ is equivalent to a projection in I, whence

$$-b[(0,e)] - c\big([(e,0)] - [(0,e)]\big) \leq |c| \cdot [(e,0)] + |b-c| \cdot [(0,e)] \leq [1_{B_n}]$$

in $K_0(B_n)$. Thus $p \in K_0(B_n)^+$ in case $a > 0$.

Now assume that $a = 0$. If $p \in K_0(B_n)^+$, then $p = [g]$ for some projection $g \in M_\infty(B_n)$, and $b \cdot (0,e) \oplus c \cdot (e,0) \oplus m \cdot 1_{B_n} \sim c \cdot (0,e) \oplus g \oplus m \cdot 1_{B_n}$ for some positive integer m. Since B_n/I is stably finite, g must lie in $M_\infty(I)$. Thus, the elements of the form $p = b[(0,e)] + c\big([(e,0)] - [(0,e)]\big)$ in $K_0(B_n)^+$ are precisely the $K_0(B_n)$-classes of projections from $M_\infty(I)$. Any such projection is equivalent to $c \cdot (e,0) \oplus d \cdot (0,e)$ for some nonnegative integers c, d, and

$$c[(e,0)] + d[(0,e)] = (c+d)[(0,e)] + c\big([(e,0)] - [(0,e)]\big).$$

Hence, $j'_*(K_0(I)^+) = \big\{b[(0,e)] + c\big([(e,0)] - [(0,e)]\big) \mid b \geq c \geq 0\big\}$, and the claim is verified. Thus $K_0(B_n)$ is isomorphic to the lexicographic direct sum of $\mathbb{Z}$ with the ordered group

$$G_n := \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}), \quad \text{where} \quad G_n^+ := \{(b, c+n\mathbb{Z}) \mid b \geq c \geq 0\}.$$

2.6. The algebra B_n is the C^*-algebra analog of Moncasi's example in [19], and has the same ordered K_0, which (as is easily checked) is not a Riesz group. In Moncasi's construction, the role of $C(\mathbb{T})$ is taken by a field (rational functions in one variable), which ensures that his example is a von Neumann regular ring. In the present setting, however, we do not yet have a large enough supply of projections, since $C(\mathbb{T})$ does not have real rank zero. To manufacture an example with real rank zero, we form an inductive limit similar to those studied in [13], using B_n in place of $C(X)$ as our basic building block.

Choose a dense sequence of points $\{t_1, t_2, \ldots\} \subset \mathbb{T}$. For $k = 1, 2, \ldots$, let

$$\delta_k : B_n \longrightarrow C(\mathbb{T}) \longrightarrow \mathbb{C} \longrightarrow B_n$$

denote the composition of the quotient map $\pi' : B_n \longrightarrow C(\mathbb{T})$ with evaluation at t_k, followed by the unital embedding $\mathbb{C} \hookrightarrow B_n$. We shall also use δ_k to denote the induced maps $M_\bullet(B_n) \longrightarrow M_\bullet(B_n)$. Set A_n equal to the C^*-inductive limit of the sequence

$$B_n \xrightarrow{\phi_1} M_2(B_n) \xrightarrow{\phi_2} M_4(B_n) \xrightarrow{\phi_3} \cdots$$

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where the connecting maps $\phi_k\colon M_{2^{k-1}}(B_n) \longrightarrow M_{2^k}(B_n)$ are block diagonal maps given by

$$\phi_k(x) = \begin{pmatrix} x & 0 \\ 0 & \delta_k(x) \end{pmatrix}. \tag{8}$$

Obviously A_n is separable, unital, and nuclear [1, Theorem 15.8.2]. Further, because of (6) we see that

$$A_n \text{ is stably finite.} \tag{9}$$

Since all the δ_k vanish on I, each ϕ_k maps $M_{2^{k-1}}(I)$ into $M_{2^k}(I)$. The inductive limit of these ideals yields a closed ideal J in A_n, with $J \cong I \otimes \mathcal{K} \cong I$. Thus J is AF; in particular, J has real rank zero and $K_1(J) = 0$. On the other hand, A_n/J is isomorphic to the inductive limit of a sequence

$$C(\mathbb{T}) \longrightarrow M_2\big(C(\mathbb{T})\big) \longrightarrow M_4\big(C(\mathbb{T})\big) \longrightarrow \cdots$$

with connecting maps analogous to (8). By [13, Lemma 1, Theorems 3,9], A_n/J is simple with stable rank 1 and real rank 0. Therefore

$$A_n \text{ has real rank } 0 \tag{10}$$

by [27, Proposition 2.3; 28, Corollary 2.12] (*cf.* [3, Theorem 3.14, Corollary 3.16]). Note also that

$$K_1(A_n) = 0, \tag{11}$$

in view of (4) and the continuity of K_1.

2.7. The inductive limits of the ideals $M_{2^{k-1}}(\mathcal{K}\oplus 0)$ and $M_{2^{k-1}}(0\oplus\mathcal{K})$ yield closed ideals J_1 and J_2 in A_n such that $J_1 + J_2 = J$. Further, T_n, which is isomorphic to $B_n/(\mathcal{K}\oplus 0)$ and to $B_n/(0\oplus\mathcal{K})$, embeds in A_n/J_1 and in A_n/J_2. Hence, A_n/J_1 and A_n/J_2 are infinite. It follows, in particular, that A_n cannot have stable rank 1. On the other hand, since J and A_n/J have stable rank 1, the stable rank of A_n is at most 2 [25, Theorem 4; 22, Corollary 4.12]. Therefore

$$A_n \text{ has stable rank } 2. \tag{12}$$

To compute $K_0(A_n)$, identify each $K_0\big(M_{2^{k-1}}(B_n)\big)$ with $K_0(B_n)$, equipped with the order-unit $2^{k-1}[1_{B_n}]$. Then each connecting map $K_0(\phi_k)$ doubles $[1_{B_n}]$ while fixing $[(e,0)]$ and $[(0,e)]$. In view of (5) and (7), it follows that

$$K_0(A_n) \cong \mathbb{Z}\left[\frac{1}{2}\right] \bigoplus_{\text{lex}} G_n \tag{13}$$

as ordered groups; further, the class $[1_{A_n}] \in K_0(A_n)$ corresponds to the pair $(1,0)$ under this isomorphism. We conclude by observing the same failure of Riesz decomposition noted by Moncasi, in case $n \geq 2$. Namely, the positive elements $x = (0,1,0)$, $y_1 = (0,1,1+n\mathbb{Z})$, and $y_2 = (0,n-1,n-1+n\mathbb{Z})$ satisfy $x \leq y_1 + y_2$ but $x \not\leq y_1$ and $x \not\leq y_2$.

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Since x is an atom in G_n^+, it cannot be the sum of two positive elements dominated by y_1 and y_2. Therefore

(14) $K_0(A_n)$ is not a Riesz group when $n \geq 2$.

2.8. In the case $n = 1$, the statement (14) turns from negative to positive, since

$$K_0(A_1) \cong \mathbb{Z}\Big[\frac{1}{2}\Big] \bigoplus_{\text{lex}} \mathbb{Z},$$

which is a dimension group [12, Proposition 3.3]. Thus A_1 has the same K-theory as an AF-algebra, *i.e.*, there exists an AF-algebra A' with $K_0(A') \cong \mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} \mathbb{Z}$ as ordered groups (see [8, Theorem 5.5] and [7, Theorem 2.2], or [1, Theorem 7.4.1]). However, A_1 is not AF, by virtue of (12).

Another difference between A_1 and the AF-algebra A' is that A' has only one proper nonzero closed ideal (corresponding to the single proper nonzero ideal $0 \oplus \mathbb{Z}$ in $\mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} \mathbb{Z}$), whereas A_1 has three (namely J_1, J_2, J). The ideals J_1 and J_2 are not seen by $K_0(A_1)$ since the ideals of $K_0(A_1)$ correspond just to the stably cofinite ideals of A_1 [14, Theorem 10.9].

2.9. The algebras A_n are also interesting from the point of view of extensions. First note that A_n/J is just the basic Bunce-Deddens algebra of type 2^∞. (For instance, Elliott's invariant $K_*(A_n/J)$ is clearly the graded ordered group $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$ with positive cone $\{(0,0)\} \cup \{(a,b) \mid a > 0\}$ and order-unit $(1,0)$, which is the same as for the Bunce-Deddens algebra, and [10, Theorem 7.1] applies.) Thus

(15) A_n is an extension of the Bunce-Deddens algebra by $\mathcal{K} \oplus \mathcal{K}$.

Since A_n/J_1 is infinite, it cannot be approximately homogeneous (*i.e.*, it is not an inductive limit of finite direct products of matrix algebras over commutative C^*-algebras). It follows immediately that

(16) A_n is not approximately homogeneous.

That extensions of Bunce-Deddens algebras by AF algebras need not be approximately homogeneous was already known via an example of Putnam analyzed in [16, Section 1, end] and [2, Section 1, end]. These examples show that the class of inductive limit algebras analyzed in [10] is not closed under extensions. There also exist extensions of algebras in this class which have stable rank 1 yet are not approximately homogeneous [6, Example 4.5].

3. Summary.

THEOREM 3.1. *(a) The algebras A_n are stably finite, unital, separable, nuclear C^*-algebras with real rank* 0 *and stable rank* 2.

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(b) $K_1(A_n) = 0$ *and* $K_0(A_n) \cong \mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} G_n$*, where* G_n *denotes the partially ordered group* $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$ *with positive cone* $G_n^+ = \{(b, c + n\mathbb{Z}) \mid b \geq c \geq 0\}$*. Under this isomorphism, the class* $[1_{A_n}] \in K_0(A_n)$ *corresponds to the element* $(1,0) \in \mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} G_n$*.*

(c) If $n \geq 2$*, then* $K_0(A_n)$ *fails to satisfy the Riesz decomposition property.*

(d) There exists a unital approximately finite dimensional C^**-algebra* A' *such that* $\big(K_0(A'), [1_{A'}]\big) \cong \big(K_0(A_1), [1_{A_1}]\big)$ *(and* $K_1(A') = 0 = K_1(A_1)$*). However,* $A' \not\cong A_1$*, because* A_1 *is not approximately finite dimensional.*

(e) A_n *is an extension of the Bunce-Deddens algebra by* $\mathcal{K} \oplus \mathcal{K}$*, but* A_n *is not approximately homogeneous.* ∎

3.2. Dădarlăt has pointed out that the existence of C^*-algebras with most of the above properties can be predicted from the Universal Coefficient Theorem of Rosenberg and Schochet [23, Theorem 1.17; 1, Theorem 23.1.1]. Namely, let $\mathcal{B}$ be the Bunce-Deddens algebra, and let f_n be the group homomorphism from $K_1(\mathcal{B}) \cong \mathbb{Z}$ to $K_0(\mathcal{K} \oplus \mathcal{K}) \cong \mathbb{Z}^2$ with matrix $(-n, n)$. By the UCT, there is an element of $KK(\mathcal{B}, \mathcal{K} \oplus \mathcal{K})$ inducing f_n, and this KK-element may be represented by a unital extension $0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow \mathcal{A}_n \longrightarrow \mathcal{B} \longrightarrow 0$ for which the index map $K_1(\mathcal{B}) \longrightarrow K_0(\mathcal{K} \oplus \mathcal{K})$ equals f_n. As above, it follows from [27, 28] that $\mathcal{A}_n$ has real rank zero, from [24] that $\mathcal{A}_n$ is stably finite, and from the 6-term exact sequence that $K_1(\mathcal{A}_n) = 0$ and $K_0(\mathcal{A}_n) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$.

It does not appear that one can completely determine the order structure on $K_0(\mathcal{A}_n)$. However, enough can be determined to see the failure of Riesz decomposition when $n \geq 2$, as follows. Let H_n denote the kernel of K_0 of the quotient map $\mathcal{A}_n \longrightarrow \mathcal{B}$. Since $\mathcal{B}$ is stably finite, H_n is an ideal of $K_0(\mathcal{A}_n)$ [14, Lemma 10.5], and so it suffices to show that Riesz decomposition fails in H_n. It is easily checked that $H_n \cap K_0(\mathcal{A}_n)^+$ equals the image of $K_0(\mathcal{K} \oplus \mathcal{K})^+$ (using again the fact that $\mathcal{B}$ is stably finite). Thus H_n is isomorphic to the quotient of $\mathbb{Z}^2$ (with the product ordering) modulo the image of f_n. In other words, $H_n \cong G_n$ as ordered groups, and we conclude as in (2.7) that H_n does not satisfy Riesz decomposition unless $n = 1$.

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