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SKEIN HOMOLOGY

DOUG BULLOCK, CHARLES FROHMAN AND JOANNA
KANIA-BARTOSZYŃSKA

ABSTRACT. A new class of homology groups associated to a 3-manifold is defined. The theories measure the syzygies between skein relations in a skein module. We investigate some of the properties of the homology theory associated to the Kauffman bracket.

1**. Introduction.** It is possible to introduce, for each skein module, a chain complex whose homology is a 3-manifold invariant and whose 0-th level is the original module. In some sense the homology theories we construct measure skein relations among skein relations, which mimics Hilbert's theory of syzygies.

Skein modules were first introduced by Przytycki and Turaev [5, 6], with different motivations leading to various constructions. One in particular generalizes the Kauffman bracket polynomial [3], itself a reformulation of the Jones polynomial [2]. We will explicitly define the homology modules based on the Kauffman bracket—a procedure easily generalized to other skein modules—and demonstrate their nontriviality. More detailed computations are difficult because the usual tools, excision and exact sequences, are absent. The reason is simple: there is no workable definition of relative skein homology (although one seems apparent from the work of W. LoFaro).

The natural examples with which to begin investigation of skein homology are manifolds whose 0-th level is well known. The most tractable of these are cylinders over surfaces, but even there the homology modules are extremely complicated. The homology of the three sphere, for example, seems to contain as much information as the space of all finite type link invariants [4]. At best, we can show that the algebra structure on the skein module of a cylinder induces a graded product on homology.

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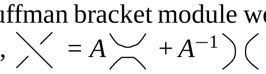
2**. The Kauffman Bracket Skein Module.** Generally, skein modules are constructed as follows. Pick a coefficient ring and form the free module spanned by isotopy classes of links with some sort of decoration. Decorations include orientation of the components and framings. You then impose relations, which usually come in two sorts: skein relations and framing relations. For example, to define the Kauffman bracket module we use isotopy classes of framed links and impose skein relations, 
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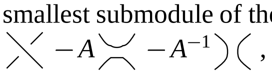
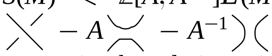
and the framing relations $L \cup \bigcirc = -(A^2 + A^{-2})L$. You should think of the diagrams above as representing pieces of flat annuli. The first relation means that outside of a small embedded ball in M the three links are identical, while inside they appear as shown. The second one means there is a trivial component of the link that can be separated from the rest of the link by a ball.

Let $\mathbb{Z}[A, A^{-1}]$ be the ring of Laurent polynomials with integer coefficients. Let M be a 3-manifold, and let $\mathcal{L}'(M)$ be the set of isotopy classes of framed links in M. A framed link is an embedding of a disjoint union of annuli in M. Let $S'(M)$ be the smallest submodule of the free module $\mathbb{Z}[A, A^{-1}]\mathcal{L}'(M)$ containing all sums of the form , and $L \cup \bigcirc + (A^2 + A^{-2})L$. The Kauffman bracket skein module, $K(M)$, is the quotient $\mathbb{Z}[A, A^{-1}]\mathcal{L}'(M)/S'(M)$.

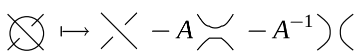
One can define $K(M)$ beginning with a slightly different basis. Let $\mathcal{L}(M)$ denote the set of framed links with no trivial components, including the empty link. Let $S(M) < \mathbb{Z}[A, A^{-1}]\mathcal{L}(M)$ be the submodule generated by all sums of the form , with the stipulation that you use $L \cup \bigcirc = -(A^2 + A^{-2})L$ to rewrite the relation without trivial components.

3**. Kauffman Bracket Homology.** A crossing ball for a framed link is an embedding of the pair (B^3, D^2) so that inside the crossing ball the link looks like  with D^2 lying in the page. A framed link with n crossing balls is one where the balls are ordered and disjoint. We number the crossing balls from 1 to n, corresponding to their ordering. Two such objects are equivalent if there is an ambient isotopy of the links that carries crossing ball to crossing ball in an order preserving fashion. The set of isotopy classes, excluding links with trivial components, will be denoted $\mathcal{L}^n(M)$.

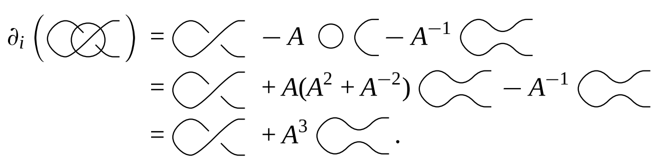
The i-th ball operator

$$\partial_i \colon \mathbb{Z}[A, A^{-1}]\mathcal{L}^n(M) \longrightarrow \mathbb{Z}[A, A^{-1}]\mathcal{L}^{n-1}(M)$$

is defined locally at the i-th crossing ball by



along with any necessary applications of $L \cup \bigcirc = -(A^2 + A^{-2})L$. For example,



The remaining crossing balls inherit the original ordering. For $n \geq 1$, define $\partial \colon \mathbb{Z}[A, A^{-1}]\mathcal{L}^n(M) \longrightarrow \mathbb{Z}[A, A^{-1}]\mathcal{L}^{n-1}(M)$ to be the alternating sum $\sum(-1)^i \partial_i$. The boundary operator on framed links without crossing balls is the zero map. The proof that $\partial \circ \partial = 0$ is essentially the same as for singular homology. Hence, we have a chain complex, $\left(\mathbb{Z}[A, A^{-1}]\mathcal{L}^n(M), \partial\right)$.

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The cycles are

$$Z_n(M) = \ker\left\{\partial : \mathbb{Z}[A, A^{-1}]L^n(M) \longrightarrow \mathbb{Z}[A, A^{-1}]L^{n-1}(M)\right\},$$

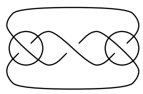
the boundaries are

$$B_n(M) = \operatorname{im}\left\{\partial : \mathbb{Z}[A, A^{-1}]L^{n+1}(M) \longrightarrow \mathbb{Z}[A, A^{-1}]L^n(M)\right\},$$

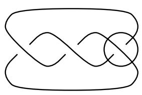
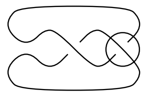
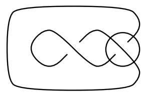
and the n-th Kauffman bracket homology of a manifold is

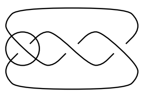
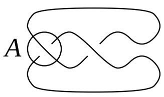
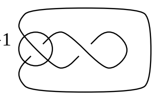
$$K_n(M) = Z_n(M)\big/B_n(M).$$

For an example of a 2-cycle, consider the element

$T =$  $\in L^2(S^3)$

with the crossing balls numbered from left to right. The ball operators are

$\partial_1(T) =$  $- A$  $- A^{-1}$  , and

$\partial_2(T) =$  $- A$  $- A^{-1}$  .

A rotation of each diagram by 180° shows that $\partial_1(T) = \partial_2(T)$, so T is a 2-cycle.

Now suppose that F is a compact oriented surface. We will explain how the Kauffman bracket homology of $F \times I$ admits a graded product. Let S, T be framed links with n and m crossing balls in $F \times I$. Since $F \times I = F \times \left[0, \frac{1}{2}\right] \cup F \times \left[\frac{1}{2}, 1\right]$, and each piece is homeomorphic to $F \times I$ via rescaling the last coordinate, we can form $S \cdot T$ by embedding S in $F \times \left[0, \frac{1}{2}\right]$ and T in $F \times \left[\frac{1}{2}, 1\right]$. Order the crossing balls using those of S followed by those of T. This operation is well defined up to isotopy, and it extends linearly to a map from $L^n(F \times I) \otimes L^m(F \times I)$ to $L^{(n+m)}(F \times I)$. Notice that $\partial_{n+m}(S \cdot T) = \partial_n(S) \cdot T + (-1)^n S \cdot \partial_m(T)$. Hence, the product induces a graded product on the skein homology of $F \times I$.

In $D^2 \times I$ this construction gives a well defined product in the skein homology of the 3-sphere. It can also be used to build cycles in an arbitrary 3-manifold M; given an n-cycle and and m-cycle, there is an $(n + m)$-cycle formed from their disjoint union with crossing balls ordered as above.

4. **Examples and Problems.** The following technique for showing that a chain is not a boundary is due to Chuck Livingston, improving on our original device. Let ζ be a primitive sixth root of unity. Notice that this implies that $\zeta + \zeta^{-1} = 1$. Since ζ^2 is a third root of unity, $\zeta^2 + \zeta^{-2} = -1$. We define a map

$$\epsilon \colon \mathbb{Z}[A, A^{-1}]L^n \longrightarrow \mathbb{Z}[\zeta]$$

as follows. Each $L \in L^n$ has $\epsilon(L) = 1$. Set $\epsilon(A) = \zeta$, $\epsilon(A^{-1}) = \zeta^{-1}$, and extend linearly.

LEMMA 1. $B_n \subseteq \ker(\epsilon)$.

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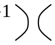
PROOF. Given $L \in \mathcal{L}^{n+1}(M)$,

$$\epsilon\big(\partial_i(L)\big) = \epsilon\Big($$  $$- A$$  $$- A^{-1}$$  $$\Big) = 0,$$

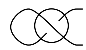
provided there are no trivial components. However, if a diagram has a trivial component,

$$\epsilon(L \cup$$  $$) = \epsilon\big(-(A^2 + A^{-2})L\big) = \epsilon(L),$$

so the computation above is still valid. ∎

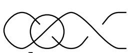
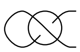
The example from the last section has $\epsilon(\mathcal{T}) \neq 0$. Hence, $\mathcal{T}$ is not a boundary and $K_2(S^3) \neq 0$. The construction generalizes easily.

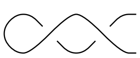
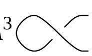
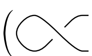
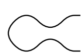
PROPOSITION 1. *Suppose that $L \in \mathcal{L}^{2n}(M)$ and there is an ambient isotopy of the underlying link which cyclically permutes the crossing balls. Then $K_{2n}(M) \neq 0$.*

Any framed knot in M may be endowed with $2n$ crossing balls of the form  . There is an isotopy of M, fixed outside a regular neighborhood of the knot, which permutes the crossing balls, so every $K_{2n}(M) \neq 0$. Another example is provided by embedding a $(2, 2n)$ torus link into a regular neighborhood of any knot in M.

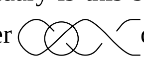
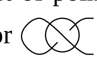
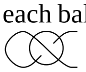
Framing also creates nontrivial homology—immediately in $K_1(M)$, and via a clever state sum argument suggested by the referee for higher levels.

PROPOSITION 2. *For any 3-manifold M and positive integer n, $K_n(M) \neq 0$.*

PROOF. We begin by proving that $K_1(M)$ is nonzero. Consider the one chain  $- A^3$  , with the usual understanding that the two links are identical except as shown. Its boundary is

 $$+ A^3$$  $$- A^3 \Big($$  $$+ A^{-3}$$  $$\Big) = 0,$$

so it's a cycle. It's not a boundary for the usual reasons.

The general case is essentially the same but with more terms. Remove n disjoint 3-balls from M and mark each of the newly created boundary spheres with two points. Embed a framed tangle in the punctured M so that its boundary is this set of points. There are 2^n ways of completing this tangle by inserting either  or  in each ball. Each choice is weighted by the coefficient $-A^3$ raised to the number of times  appears. The sum over all choices is an n-cycle on which ϵ is non-zero. ∎

The constructions performed here could be carried out using any of the relations for a general skein module [1]. We have restricted our attention to a particular module and have only touched upon its possible homology structure. Many open questions remain.

First of all, what is $K_n(S^3)$, and how exactly does it relate to finite type invariants? There are two major impediments to working this example: we can construct only some cycles, and we have little or no idea what boundaries look like. In other words, how do we determine that we have enough cycles, and how do we tell them apart? The latter question is likely to be answered by link invariants, and these are likely to be of finite type.

Our understanding of $K_n(F \times I)$ is dependent on knowledge of the original skein module. Specifically, the product on the homology ring comes directly from the well known product on K_0. What other facets of K_0 translate to K_n? In particular, the skein module is free and finitely generated (as an algebra); is its homology?

One of the original motivations for skein homology was to expand what is perhaps the most puzzling phenomenon in the Kauffman bracket skein module, torsion. What effect does torsion in K_0 have on K_n? Do torsion elements generate distinct homology classes in each level? If so, can one use this to prove that torsion in K_0 implies infinite rank?

There are other avenues as well, such as the relationship between the Kauffman bracket skein module and the $SL_2(\mathbb{C})$-characters of the fundamental group. One would like to know how this is expressed in higher homology modules. Also, all of the above questions can be asked for other skein modules.

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Department of Mathematics
Boise State University
Boise, ID 83725
USA
email: bullock@math.idbsu.edu
kania@math.idbsu.edu

Department of Mathematics
University of Iowa
Iowa City, IA 52245
USA
email: frohman@math.uiowa.edu

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