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Eigenfunction Decay For the Neumann Laplacian on Horn-Like Domains

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Abstract. The growth properties at infinity for eigenfunctions corresponding to embedded eigenvalues of the Neumann Laplacian on horn-like domains are studied. For domains that pinch at polynomial rate, it is shown that the eigenfunctions vanish at infinity faster than the reciprocal of any polynomial. For a class of domains that pinch at an exponential rate, weaker, L^2 bounds are proven. A corollary is that eigenvalues can accumulate only at zero or infinity.

1 Introduction

Let Ω be a connected planar domain which has the following form:

$$\Omega = \{(x, y), x \geq 0, h(x) - f(x) < y < h(x) + f(x)\} \cup \kappa,$$

where κ is a domain with compact closure. In this paper we assume that $f(x)$ is a positive function such that $f(x) \downarrow 0$ as $x \to \infty$. Such domains we shall label "horn-like". In what follows we also assume Ω obeys the segment condition, *i.e.* $\partial\Omega$ has a locally finite open covering $\{O_i\}$ and corresponding nonzero vectors $\{y_i\}$ so that for $t \in (0, 1)$, $x + ty_i \in \Omega$ if $x \in \overline{\Omega} \cap O_i$ (see [15], p. 256 for more on segment condition).

It is well known that the Dirichlet Laplacian for such domains has discrete spectrum. The decay properties of Dirichlet-eigenfunctions have been the object of considerable study recently [1, 2, 3, 4, 6], and both lower and upper bounds on the eigenfunctions in a neighbourhood of infinity have been obtained.

The spectral properties of the Neumann Laplacian are less well understood. For $h(x) = 0$, and under certain rapid decay hypotheses on f and its derivatives (*e.g.* $f(x) = e^{-x^{\alpha}}$, $\alpha > 1$), the spectrum of Neumann Laplacian has been shown to be discrete ([12, 7, 5]). On the other hand, for $f(x)$ of polynomial decay, or $f(x) = e^{-x^{\alpha}}$, $\alpha \in (0, 1)$, it has been shown that the essential spectrum consists of the interval $[0, \infty)$ and that the embedded eigenvalues can accumulate only at zero or infinity ([7, 8, 9, 11, 12, 13]). In the case that the essential spectrum is $[0, \infty)$, it was observed that for domains symmetric about the x-axis, there exist infinitely many embedded eigenvalues [7].

In this note we prove upper bounds for the eigenfunctions corresponding to embedded eigenvalues of Neumann Laplacian in horn-like domains. The methods used for the Dirichlet Laplacian appear not to apply in the case of Neumann boundary conditions. In

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fact lower bounds on the eigenfunctions are not generally possible, since the eigenfunctions observed for symmetric domains in [7] vanish on the x-axis.

At the end of this section we will pose two sets of conditions on the pair (f, h), which we will label polynomial type decay and exponential type decay.

Theorem 1 *Suppose f, h are either of polynomial or of exponential type decay. Suppose*

$$\Delta\tilde{u} \equiv -\frac{\partial^2\tilde{u}}{\partial x^2} - \frac{\partial^2\tilde{u}}{\partial y^2} = E\tilde{u}$$

for $E > 0$, with $\partial\tilde{u}/\partial\eta|_{\partial\Omega} = 0$ and $\tilde{u}$ a unit vector in $L^2(\Omega, dx\,dy)$. Then for any positive integer N, there exists a positive number C such that

$$\|x^N\tilde{u}\|_{L^2(\Omega, dx\,dy)} < C.$$

We remark in passing that for fixed N, the constant C in Theorem 1 can be made uniform for $E \in [\delta, M]$ for $\delta > 0, M < \infty$. Consequently, we can use a compactness argument as in [10] to prove

Corollary 1 *Under the hypotheses of Theorem 1, the eigenvalues of Δ can accumulate only at 0 or infinity.*

As was previously noted, this result was proven by other methods in [7, 13, 8, 10]. However, our methods will treat some domains not treated in those above, such as $h(x) = \ln x$, $f(x) = x^\epsilon$, $\epsilon \in (0, 2)$.

For polynomial type decay, Theorem 1 strengthens to a pointwise estimate:

Theorem 2 *Suppose f, h are of polynomial type decay. Suppose $\Delta\tilde{u} = E\tilde{u}$ for $E > 0$, with $\partial\tilde{u}/\partial\eta|_{\partial\Omega} = 0$ and $\tilde{u}$ a unit vector in $L^2(\Omega, dx\,dy)$. Then for any positive integer N, there exists a constant K such that*

$$|\tilde{u}(x, y)| \leq Kx^{-N}.$$

We remark that the methods of this paper can be generalised to domains of higher dimensions and to manifolds.

The analysis in our proof is an adaptation of the methods in [10]. We use a change of coordinates followed by a unitary transformation to show that the equation $\Delta\tilde{u} = E\tilde{u}$ implies the following equation holds on the semi-infinite strip $\{(r, s), r > 0, s \in (-1, 1)\}$:

$$Hu = Eu,$$

with

$$H = -\frac{\partial}{\partial r}\alpha^{-2}\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\beta^{-2}\frac{\partial}{\partial s} + V, \quad r > 1. \tag{1}$$

Here α, β, and V are functions on the strip that are determined by f, g. For $r > 1$ the associated boundary conditions will be

$$\frac{(\alpha\beta)_s}{2\alpha\beta}u + u_s|_{s=\pm 1} = 0. \tag{2}$$

An important role will be played by a family of Sturm-Liouville problems parametrised by r: $v_{ss}+\lambda v = 0$, $s \in (-1, 1)$ with boundary conditions given by Equation 2 for any fixed r. Let P be the orthogonal projection to the eigenspace associated to the smallest eigenvalue.

We analyse separately the terms Pu and $(I-P)u$. We apply the operator P to the equation $Hu = Eu$. After commuting P to the right, we obtain a family of ordinary differential equations for Pu, parametrised by s. Study of these differential equations yields bounds on Pu. On the other hand, we will show that $f^{-2}D_s^2(I-P)u$ is in L^2. Since, for large r, D_s^2 is strictly negative on the range of $(I-P)$, this will show that $f^{-1}(I-P)u$ is in L^2. This will prove a preliminary decay estimate on u, which can then be strengthened by a bootstrapping argument from [10].

We now define "polynomial-type decay" and "exponential-type decay". Denote the jth order derivative of f by $f^{(j)}$, the jth power of f by f^j, and similarly for h. We say f, h are of "polynomial type decay" if

A1: $f > 0$,
A2: $C_1 x^{-\epsilon} \le f(x) \le C_2 x^{-\gamma}$, for some $\gamma, \epsilon, C_1, C_2 > 0$, and $\gamma < \min(\epsilon, 1/2)$.
A3: $f^{(1)} = O(x^{-1})$, $f^{(j)} = O(x^{-2})$, $j \ge 2$.
A4: $h^{(1)} = O(x^{-1})$, $h^{(j)} = O(x^{-2})$, $j \ge 2$.
A5: $(h^{(1)} f^{(1)}/f)^{(j)} = O(x^{-2})$, $j \ge 0$.
A6: $(h^{(1)} f^{(1)}/f^2)^{(j)} = o(1)$, $(h^{(2)}/f)^{(j)} = o(1)$, $j \ge 0$
A7: $f^{(1)}/f = O(x^{-1})$; $(f^{(1)}/f)^{(j)} = O(x^{-2})$, $j \ge 1$.

We remark that if the assumption $f = O(x^{-\gamma})$ is weakened to $f = o(1)$, then somewhat weaker analogues of Theorems 1, 2 can be proven.

We say f, h are of "exponential type decay" if conditions A1, A4, A5 hold, and conditions A2, A3, A6, A7 are replaced by:

A2′: $f = O(x^{-1})$,
A3′: $f^{(j)} = O(x^{-2})$, $j \ge 2$
A6′: $(h^{(1)} f^{(1)}/f^2)^{(j)} = O(x^{-1})$, $(h^{(2)}/f)^{(j)} = O(x^{-1})$, $j \ge 2$
A7′: $(f^{(1)}/f)^{(j)} = O(x^{-1/2-\gamma-j/2})$, $j \ge 0$, for some $\gamma \in (0, 1/2)$.

Although the conditions for polynomial type decay or exponential type decay are restrictive, they are satisfied by a large number of examples including

1. $h(x) = 0$, $f(x) = x^{-\epsilon}$, $\epsilon > 0$,
2. $h(x) = 0$, $f(x) = \exp(-x^{\alpha})$, $\alpha \in (0, 1/2)$
3. $h(x) = \log(x)$, $f(x) = x^{-\epsilon}$, $\epsilon \in (0, 2)$,
4. $h(x) = x^{-2}\sin x$, $f(x) = x^{-\epsilon}$, $\epsilon \in (0, 2)$,
5. $h(x) = \sin\big(\log(x)\big)$, $f(x) = x^{-\epsilon}$, $\epsilon \in (0, 2)$.

2 Proof

In what follows, we will use interchangeably the notations $\partial x/\partial r$ and x_r, etc. Also, it will often be useful to denote $\partial^j/\partial r^j$ by D_r^j, etc.

We begin by defining a change of coordinates $(x, y) \to (r, s)$. The properties listed below for this change of coordinates are proven in [9]; we merely cite them here. Let

$$s = \frac{y - h(x)}{f(x)}. \tag{3}$$

The coordinate $r = r(x, y)$ will be defined so that the level curves of r are orthogonal to the level curves of s. Thus the level curves of r will be the flow lines for the vector field ∇s. We define $r(x, y)$ geometrically as follows: given (x, y), let $r(x, y)$ be the x-coordinate of the line $s = 0$ intersected with the flow line of $-\nabla s$ starting from (x, y).

The mapping $(x, y) \to (r, s)$ defines a diffeomorphism from the end in Ω to a semi-infinite strip which we can assume to be $\{(r, s) : r > 0, s \in (-1, 1)\}$. Furthermore, $x \sim r$ in the sense that as $x \to \infty$, $r(x, y) - x \to 0$ uniformly in y.

The Euclidean metric in coordinates r, s can be written $\alpha^2 dr^2 + \beta^2 ds^2$, with α, β satisfying the following estimates:

Lemma 1 *Let γ be the positive constant determined by A2 or A7′. Then:*

i) $\alpha^{-2} - 1 = O(r^{-2})$,
ii) $\beta^{-2} - f(r)^{-2} = O\big(r^{-2} f(r)^{-2}\big)$,
iii) $\alpha_r/\alpha^3 = O(r^{-2})$,
iv) $\beta_s/\beta^3 = O\big((f^{(1)}/f)^2\big)$.
v) $V = O(r^{-1-2\gamma})$, $V_r = O(r^{-3/2-2\gamma})$, $V_s = O(r^{-2})$, $V_{rr} = O(r^{-2})$, *with* $\epsilon > 0$
vi) For $i + k \geq 2$ we have the estimates

$$(f^{-1}D_s)^i D_r^k \alpha = O(r^{-2}), \quad (f^{-1}D_s)^i D_r^k \beta = O(r^{-2}), \quad (f^{-1}D_s)^i D_r^k V = O(r^{-2}).$$

vii) $D_r^j\big(\frac{(\alpha\beta)_s}{\alpha\beta}\big) = O(r^{-1-2\gamma} f^2)$, *for* $j \geq 0$.

Note that the measure associated to the metric is $\alpha\beta\, dr\, ds$. If ω is a positive, smooth function defined by

$$\omega^{-1} = \begin{cases} 1 & \text{on } \{r < 0\} \cup \kappa \\ \alpha\beta & \text{for } r > 1 \end{cases}$$

then one can define a unitary transformation from $L^2(\Omega, dx\, dy)$ to $L^2(\Omega, \omega dx\, dy)$ by $Uv = (\omega)^{-1/2} v$. The main reason this transformation is useful is that for $r > 1$,

$$\omega dx\, dy = dr\, ds.$$

Henceforth we will denote $L^2(\Omega, \omega dx\, dy)$ by L^2.

We define the operator H on L^2 by $H = U \Delta U^{-1}$. A straightforward calculation shows that the differential expression for H is given by Equation 1, and that for $r > 1$ Neumann boundary conditions are transformed to those given in Equation 2. Suppose $\Delta \tilde{u} = E\tilde{u}$ for $\tilde{u} \in L^2(\Omega, dx\, dy)$. Letting $u = U\tilde{u}$, we have $Hu = Eu$, with $u \in L^2$.

Let χ be a cutoff function that vanishes in the complement of $\{r > 0\}$ which is identically 1 for $r > 1$.

Lemma 2 *For $i + j \leq 2$, the operator*

$$\chi\big(f^{-1}(r)D_s\big)^j D_r^i (H+1)^{-\frac{i+j}{2}} : L^2 \to L^2$$

extends to a bounded operator.

The proof is found in [8], Lemma 3.

We associate to Equation 1 the Sturm-Liouville problem

$$u_{ss}(s) + \lambda_j u(s) = 0, \quad s \in (-1,1)$$

with boundary conditions given by Equation 2. Here $\lambda_0 < \lambda_1 < \cdots$.

By Lemma 1, it follows that

$$\lim_{r\to\infty} \frac{(\alpha\beta)_s}{2\alpha\beta} = 0.$$

It thus follows that

$$\lim_{r\to\infty} \lambda_0 = 0, \quad \lim_{r\to\infty} \lambda_1 = \pi^2/4. \tag{4}$$

A more careful analysis of λ_0 given in ([8], Lemma 7), together with Lemma 1 part vii of this paper, proves

$$\lambda_0^{(j)} = O\big(r^{-1-2\gamma} f(r)^2\big), \quad j = 0,1,2. \tag{5}$$

Let v_0 be the normalised eigenvector corresponding to λ_0 in the associated Sturm-Liouville problem. It is straightforward to show that for fixed r and for $\lambda_0 \geq 0$,

$$v_0(s) = \left(1 + \frac{\sin 2\sqrt{\lambda_0}}{2\sqrt{\lambda_0}}\right)^{-1/2} \cos(\sqrt{\lambda_0}s). \tag{6}$$

For fixed $r > 1$, let P be the orthogonal projection of $L^2\big((-1,1), ds\big)$ onto v_0. Thus,

$$Pu = v_0 \int_{-1}^{1} u(s)v_0(s)\, ds. \tag{7}$$

The operator χP naturally defines a bounded operator $L^2(\Omega, \omega\, dx\, dy)$.

Since $\chi D_s^2 Pu = -\chi\lambda_0 Pu$, it follows by Equation 5 that the operator $\chi D_s^2 P$ extends to be bounded on $L^2(\Omega, \omega dx\, dy)$, satisfying the estimate

$$\|\chi D_s^2 Pu\|_{L^2} \leq C\| f(r)^2 r^{-1-2\gamma}\chi u\|_{L^2}. \tag{8}$$

Lemma 3 *The following operators extend to bounded operators on L^2:*

A: $r^2\chi[D_r^j, P](H+1)^{(j-1)/2}$, $j = 1,2$,
B: $\chi[H,P](H+1)^{-1}$.

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For the proof of this lemma, the reader is referred [8], particularly Lemma 8.

The key steps in proving Theorem 1 are the three lemmas that follow:

Lemma 4 *Suppose $Hu = Eu$, with $\|u\|_{L^2} = 1$ and $E < M$, M a positive constant. Then*

$$-D_r^2(P\chi u) = E(P\chi u) + w,$$

where w satisfies the estimate

$$\|r^{1+2\gamma}\chi w\|_{L^2} \leq C,$$

with C a positive constant dependent on M but independent of E and u.

Proof The proof goes largely along the lines of Lemma 2 in [10]. By Equation 1, the left hand side of the equation $\chi PHu = E\chi Pu$ can be rewritten

$$-\chi P\alpha^{-2}D_r^2u - \chi P(\alpha^{-2})_rD_ru - \chi P\beta^{-2}D_s^2u - \chi P(\beta^{-2})_sD_su + \chi PVu. \tag{9}$$

We analyse the first term in Equation 9. We have

$$\begin{aligned} &-\chi P\alpha^{-2}D_r^2u \\ &= -\chi[P,\alpha^{-2}]D_r^2u + \chi\alpha^{-2}[P,D_r^2]u + \chi(\alpha^{-2}-1)D_r^2Pu + [\chi,D_r^2]Pu + D_r^2(P\chi u). \end{aligned} \tag{10}$$

We incorporate the first four terms of the right hand side into w as follows. For the first of these terms,

$$\chi[P,\alpha^{-2}]D_r^2u = (E+1)\chi[P,\alpha^{-2}]D_r^2(H+1)^{-1}u.$$

By estimates proven in [10], the operator $r^2\chi[P,\alpha^{-2}]$ is bounded on L^2, while by Lemma 2 of this paper, $D_r^2(H+1)^{-1}$ is bounded on L^2. The second, third, and fourth terms in Equation 10, as well as the remaining terms in Equation 9, can similarly be treated, using Lemmas 1, 2, 3A, the compactness of the support of the derivatives of χ, and Equation 5.

Lemma 5 *Suppose $Hu = Eu$, with $\|u\|_{L^2} = 1$ and with $E < M$, M some positive number. Then there exists $\delta > 0$ such that*

$$\|r^\delta\chi Pu\| \leq \frac{C'}{\sqrt{E}},$$

with constant C' independent of u, E, and s.

Although the proof can be found in [10], we include it here for completeness.

Proof We begin with the result of the previous lemma:

$$-D_r^2(\chi Pu) = E(\chi Pu) + w, \tag{11}$$

with

$$\|r^{1+2\gamma}\chi w\|_{L^2} < C.$$

Fixing s, we view Equation 11 as a family of ordinary differential equations parametrised by s. Using the technique of variation of parameters, we obtain:

(12)

$$\chi Pu$$
$$= -\cos(\sqrt{E}r)\Big(\int_{t=0}^{r} \frac{w}{\sqrt{E}}\sin(\sqrt{E}t)\,dt + C_1\Big) + \sin(\sqrt{E}r)\Big(\int_{t=0}^{r} \frac{w}{\sqrt{E}}\cos(\sqrt{E}t)\,dt + C_2\Big),$$

with C_1, C_2 constants which possibly depend on s. We use this equation to obtain bounds on χPu. Since $\|r^{1+2\gamma}\chi w\|_{L^2} < C$, it follows that

$$\int_{r=0}^{\infty}\int_{s=-1}^{1} |w(r,s)|\,ds\,dr < \infty.$$

It follows that $w(s,r) \in L^1\big((0,\infty), dr\big)$ for almost all s. This, along with the fact that $Pu \in L^2$, implies that Equation 12 can be written as

$$\chi Pu = \frac{\cos(\sqrt{E}r)}{\sqrt{E}}\int_{t=r}^{\infty} w\sin(\sqrt{E}t)\,dt - \frac{\sin(\sqrt{E}r)}{\sqrt{E}}\int_{t=r}^{\infty} w\cos(\sqrt{E}t)\,dt, \tag{13}$$

almost everywhere with respect to s. We estimate the first integral on the right hand side. Fix $\epsilon > 0$, and choose r sufficiently large that $\chi(r) = 1$. Then

$$\begin{aligned}\Big|\int_{t=r}^{\infty} w\sin(\sqrt{E}t)\,dt\Big| &\le \int_{t=r}^{\infty} |w|\,dt\\ &\le r^{-1/2-2\gamma+\epsilon}\int_{t=r}^{\infty} t^{1+2\gamma}t^{-1/2-\epsilon}|w|\,dt\\ &\le r^{-1/2-2\gamma+\epsilon}\Big(\int_{t=r}^{\infty} t^{-1-2\epsilon}\,dt\Big)^{1/2}\Big(\int_{t=r}^{\infty} t^{2+4\gamma}\chi^2|w|^2\,dt\Big)^{1/2}\\ &\le r^{-1/2-2\gamma+\epsilon}r^{-\epsilon}/(2\epsilon)^{1/2}\Big(\int_{t=0}^{\infty} t^{2+4\gamma}\chi^2|w|^2\,dt\Big)^{1/2}.\end{aligned}$$

Clearly, the second integral on the right hand side of Equation 13 obeys the same estimate, and thus

$$|\chi Pu| \le \frac{C}{\sqrt{E}}r^{-1/2-2\gamma}\Big(\int_{t=0}^{\infty} t^{2+4\gamma}\chi^2|w|^2\,dt\Big)^{1/2},$$

this estimate holding for almost all s. Note that C is independent of u, E and s (provided s is not in a set of measure 0). Hence

$$\begin{aligned}\int_{s=-1}^{1} |\chi Pu(r)|^2\,ds &\le \frac{C}{E}\int_{s=-1}^{1} r^{-1-4\gamma}\Big(\int_{t=0}^{\infty} t^{2+4\gamma}\chi^2|w|^2\,dt\Big)\,ds\\ &= \frac{C}{E}r^{-1-4\gamma}\|\chi r^{1+2\gamma}w\|_{L^2}^2\\ &\le \frac{C}{E}r^{-1-4\gamma}. \qquad (14)\end{aligned}$$

Choosing $0 < \delta < 4\gamma$, the lemma follows.

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Lemma 6 *Suppose $Hu = Eu$, with $\|u\| = 1$ and $E < M$ for M a positive constant. Then*

$$\|\chi f^{-1}(I-P)u\|_{L^2} \leq C,$$

with C a positive constant depending on M but independent of E, u.

Proof In what follows, C will denote various constants independent of u and E, provided $\|u\| = 1$ and $E < M$. First we show that $\|\chi f^{-2} D_s^2 (I-P)u\| \leq C$. We have

$$\begin{aligned}\chi f^{-2} D_s^2 (I-P)u &= \chi f^{-2} D_s^2 (H+1)^{-1}(H+1)(I-P)u \\ &= (E+1)\chi f^{-2} D_s^2 (H+1)^{-1}(I-P)u \\ &\quad - (E+1)\chi f^{-2} D_s^2 (H+1)^{-1}[H,P](H+1)^{-1}u.\end{aligned}$$

By Lemma 2,

$$\|\chi f^{-2} D_s^2 (H+1)^{-1}(I-P)u\| \leq C.$$

By Lemmas 2 and 3B, the second term on the right hand side is also bounded by a constant C.

Let λ_2 be the second smallest eigenvalue of the associated Sturm-Liouville problem. By Equation 4, there exists M_0 such that $r > M_0$ implies $\lambda_1 > 1$. In what follows, we label $u_1 = (I-P)u$. Thus

$$\begin{aligned}\infty &> \int_0^\infty \int_{-1}^1 f^{-2}(-D_s^2)\chi u_1 \chi u_1 \\ &\geq \int_0^\infty \int_{-1}^1 f^{-2}\lambda_2 |\chi u_1|^2 \\ &\geq \int_0^{M_0} \int_{-1}^1 f^{-2}\lambda_2 |\chi u_1|^2 + \int_{M_0}^\infty \int_{-1}^1 f^{-2}|\chi u_1|^2 \\ &\geq \|f^{-1}\chi u_1\|_{L^2}^2 - C.\end{aligned}$$

The lemma now follows.

We now complete the proofs of Theorems 1 and 2. Note that by A2 or A2′, $f^{-1}(r) \geq Cr^{\delta'}$ for sufficiently small $\delta' > 0$. Hence, we combine Lemmas 5 and 6 to obtain $\|r^{\delta''}\chi u\| < C$ for $\delta'' = \min(\delta, \delta')$. We can now use a bootstrapping argument to prove that $\|r^N \chi u\| < C$ for any integer N. The reader is referred to ([10], Lemma 4) for details.

An elliptic regularity argument(the reader is referred to [10], end of Section 3) can then be used to prove the pointwise estimate

$$r^N u(r,s) < C$$

with C independent of s and u provided $\|u\|_{L^2} = 1$.

Finally, to obtain estimates in the variables x, y, recall that by definition, $\tilde{u} \equiv U^{-1}u = (\alpha\beta)^{-1/2}u$. Because U is a unitary multiplication operator and $x \sim r$ as $r \to \infty$,

$$\begin{aligned}\|\chi x^N \tilde{u}\|_{L^2(\Omega, dx\,dy)} &= \|\chi x^N U^{-1} u\|_{L^2(\Omega, dx\,dy)} \\ &\leq C\|U^{-1}(\chi r^N u)\|_{L^2(\Omega, dx\,dy)} \\ &\leq C\|\chi r^N u\|_{L^2}.\end{aligned}$$

Theorem 1 follows.

Theorem 2 follows from the pointwise estimates in r, s, the estimates $\alpha \sim 1$ and $\beta \sim f(r)$ as $r \to \infty$, and the hypothesis that $f(x) \geq Cx^{-\epsilon}$.

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