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Tensor Square of the Minimal Representation of $O(p, q)$

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Abstract. In this paper, we study the tensor product $\pi = \sigma^{\min} \otimes \sigma^{\min}$ of the minimal representation $\sigma^{\min}$ of $O(p, q)$ with itself, and decompose π into a direct integral of irreducible representations. The decomposition is given in terms of the Plancherel measure on a certain real hyperbolic space.

1 Introduction

Let G be a real reductive group. The problem of decomposing a given unitary representation π of G into a direct sum (or direct integral) of irreducible representations is one of the fundamental questions of abstract harmonic analysis. An interesting special case of this problem arises for $\pi = \sigma_1 \otimes \sigma_2$ (or, more generally, $\pi = \sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_k$), where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are unitary irreducible representations of G. Though in general this problem is far from being solved, in some special cases the tensor product decomposition was calculated, and the corresponding Plancherel measure written down explicitly. Among others, the following cases are known:

- $G = Sp(2n, \mathbb{R})$, $\pi = \omega^{\otimes k}$, where ω is the oscillator (Weil) representation of G. As shown in [KV], the decomposition of π gives Howe duality for the compact case, *i.e.,* the duality correspondence between the unitary representations of $O(k)$ and the highest weight representations of $Sp(2n, \mathbb{R})$. More generally, one can study $\pi = \omega^{\otimes p} \otimes \overline{\omega}^{\otimes q}$ and recover the duality correspondence for the pair $(O(p, q), Sp(2n, \mathbb{R}))$.
- $G = Sp(2n, \mathbb{R})$, σ_1 and σ_2 – holomorphic discrete series representations [Re].
- $G = SU(2, 2)$, σ_1 – any holomorphic representation, σ_2 – antiholomorphic [ØZ].
- $G = U(p, q)$, σ_1 is a minimal representation of G, $\sigma_2 = \overline{\sigma}_1$ [Z].
- G= conformal group of a real Jordan algebra N, σ_1 and σ_2 – certain unipotent representations of G [DS1, DS2])

The results of [Z, DS2] demonstrate that even for σ_1 and σ_2 minimal, the spectrum of $\pi = \sigma_1 \otimes \sigma_2$ is already quite nontrivial. In particular, the spectrum may contain a discrete part, and the representations arising in the decomposition will provide some new examples of unipotent irreducible representations of G.

In this note, we use an explicit realization of the minimal representation $\sigma^{\min}$ of $G = O(p+1, q+1)$, given recently by Kobayashi and Ørsted [KØ1], to offer a simple approach to decomposing its tensor square $\pi = \sigma^{\min} \otimes \sigma^{\min}$. As was observed by

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many authors, minimal representations are surprisingly "rich objects", with numerous application to physics (*cf.* [KPW]) and number theory, and the structure of π is also quite interesting. We will use the approach of [DS1] (with some arguments appropriately modified, and some simplifications) to show (Theorem 5.1) that the measure for the direct integral decomposition of π can be given in terms of a Plancherel measure for the real hyperbolic space (hyperboloid) $SO_0(p, q-1)/ SO_0(p-1, q-1)$. These rank 1 reductive symmetric spaces were studied by many authors, and the analysis on them is well understood, *e.g.*, [M, Ro].

2 Minimal Representation $\sigma^{\min}$

Let $G = O(p + 1, q + 1)$, $p + q \in 2\mathbb{N}$ (*i.e.*, p and q have the same parity), $p \geq q$. With some modifications, the discussion below applies to the case $q = 1$, $G = O(p + 1, 2)$, but since the results for this case were obtained in [DS1], we will impose the condition $q \geq 2$.

The group G can be viewed as the conformal group of a non-Euclidean Jordan algebra $\overline{N} = \mathbb{R}^{p,q}$. Here $\overline{N} = \mathbb{R} \oplus \mathbb{R}^{p,q-1}$ endowed with a Jordan multiplication

$$(\lambda, u)(\mu, v) = (\lambda\mu + B(u, v), \lambda v + \mu u),$$

where B is a bilinear form of signature $(p, q - 1)$. Then $e = (1, 0)$ is the unit of this algebra, and $e = c + c'$, where $c = (\frac{1}{2}, w)$, $c' = (\frac{1}{2}, w)$ and $B(w, w) = \frac{1}{4}$. Note that c and c' are orthogonal indecomposable idempotents in $\overline{N}$ (*cf.* [FK, p. 63]).

Note that $P = MN$, $M = \mathbb{R}^* \times O(p, q)$ is a maximal parabolic subgroup of G. The group M acts on the nilradical N and also on the opposite nilradical $\overline{N}$. If we endow $\overline{N}$ with the Jordan algebra structure as above, M is the structure group of this Jordan algebra. The orbits of the action of M on $\overline{N}$ (besides the zero orbit) are $C = M \cdot c$ (the light cone in $\mathbb{R}^{p,q}$) and the open orbit $M \cdot e$. Of course the orbit $M \cdot c'$ also gives us the same cone C, but it will be convenient to use the notation $M \cdot c' = C'$.

Another parabolic subgroup of G is $P' = M'A'N'$, where $M' = SL(2) \times O(p - 1, q - 1)$, $A' = \mathbb{R}^*$ and the unipotent radical N' is the Heisenberg group of dimension $2(p + q) - 3$;

Let $\sigma^{\min}$ be the minimal representation of G. The condition on the parity of p and q ensures that $\sigma^{\min}$ exists. Among the several models for the minimal representation given in [KØ1, KØ2], there is a realization of $\sigma^{\min}$ on $\mathcal{L}^2(C, d\mu)$. Here $d\mu$ is an M-equivariant measure on C, which transforms by a certain positive character δ of M.

Fix a Killing form (,) on the Lie algebra $\mathfrak{g}$ of G. Since N and $\overline{N}$ are abelian, we identify them with their Lie algebras ($\mathfrak{n}$ and $\overline{\mathfrak{n}}$, respectively), and the form (,) gives a non-degenerate pairing between $N \simeq \mathfrak{n}$ and $\overline{N} \simeq \overline{\mathfrak{n}}$. We write $\langle y, x\rangle = \exp(2\pi i(y, x))$ for $y \in \overline{N}$ and $x \in N$. Representing an element in C as $r \cdot c$ ($r \in M$), we can write for $f \in \mathcal{L}^2(C, d\mu)$

$$\sigma^{\min}(l)f(r \cdot c) = \delta^{-1/2}(l)f((l^{-1}r) \cdot c), l \in M,$$

$$\sigma^{\min}(n)f(r \cdot c) = \langle r \cdot c, n\rangle f(r \cdot c), n \in N.$$

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This defines a unitary irreducible representation of $P = MN$ on $\mathcal{L}^2(C, d\mu)$, and this representation extends to the unitary representation of G, which is precisely the minimal representation [KØ2].

3 Von Neumann Algebra $\mathcal{VN}(\pi, G)$

From now on, we denote by $\mathcal{VN}(\kappa, Z)$ the von Neumann algebra of a unitary representation κ of the group Z. Here Z is some subgroup of G (usually κ is in fact a representation of some larger subgroup of G, restricted to Z). We remark that all the groups arising in this paper are type I groups, so all direct integral decompositions in this and the following sections are unique.

Let ρ_t ($t \neq 0$) be the unitary irreducible representation of the Heisenberg group N', corresponding to the character $\chi_t(a) = \exp(2\pi i a t)$ on $\mathbb{R}$ (the center of N'). It is known that a minimal representation is irreducible when restricted to any parabolic subgroup. In particular, restriction of $\sigma^{\min}$ on $P' = M'A'N'$ is irreducible as well. On the other hand, it is easy to understand all irreducible representations of P' via Mackey theory. Since the factors $SL(2)$ and $O(p-1, q-1)$ in M' form a dual pair inside $Sp(2p+2q-4, \mathbb{R})$, we can use the oscillator representation ω_1 to extend ρ_1 to an irreducible representation $\widetilde{\rho}_1$ of $M'N'$. Then for any unitary irreducible representation κ of M', we consider $\operatorname{Ind}_{M'N'}^{P'}(\kappa \otimes \widetilde{\rho}_1)$. This representation is unitary and irreducible.

The minimal representation, according to the explicit construction of [KØ1], corresponds to $\kappa = 1$, and we obtain

$$\operatorname{Ind}_{M'N'}^{P'}(1 \otimes \widetilde{\rho}_1) \simeq \sigma^{\min}|_{P'}.$$

Restricting it back to $M'N'$, one gets a direct integral decomposition

$$\sigma^{\min}|_{M'N'} = \int^{\oplus} \widetilde{\rho}_t \, dt,$$

where a direct integral is taken over $\mathbb{R}^{\times} = \mathbb{R}\backslash\{0\}$, and $\widetilde{\rho}_t$ is obtained by extending a representation ρ_t of N' to $M'N'$ via the corresponding oscillator representation ω_t.

This decomposition allows us to write down a similar direct integral decomposition of $\pi|_{M'N'}$, where $\pi = \sigma^{\min} \otimes \sigma^{\min}$. It is well known that for $t + t' \neq 0$, $\rho_t \otimes \rho_{t'} \simeq \rho_{t+t'} \otimes \mathbf{1}$, where $\mathbf{1}$ is a multiple of the trivial representation of N'. Also, $\omega_t \otimes \omega_{t'} \simeq \omega_{\pm} \otimes \omega_{\pm}$, where $\omega_{\pm}$ are the two nonisomorphic oscillator representations (ω_+ and ω_-) of $Sp(2p+2q-4, \mathbb{R})$, restricted to M'. Hence

$$\widetilde{\rho}_t \otimes \widetilde{\rho}_{t'} \simeq \widetilde{\rho}_{t+t'} \otimes \widetilde{\omega}_{\pm},$$

where $\widetilde{\omega}_{\pm}|_{N'}$ acts trivially on $L^2(\mathbb{R}^{p+q-2})$, and $\widetilde{\omega}_{\pm}|_{M'}$ acts by ω_+ or ω_-, depending on the signs of t, t' and $t + t'$.

Collecting the terms with $t + t' = s$ in a double integral for the tensor product $\sigma^{\min} \otimes \sigma^{\min}$, obtain a direct integral over $\mathbb{R}^{\times}$:

$$\pi|_{M'N'} \simeq \int^{\oplus} \vartheta_s \otimes \widetilde{\rho}_s \, ds,$$

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where $\vartheta_s|_{M'}$ is a multiple of $\omega_- \oplus \omega_+$ (it has Gelfand–Kirillov dimension $p+q-1$, compared with $p+q-2$ for $\omega_- \oplus \omega_+$).

The spectrum of $\omega_+|_{M'}$ is given by the Howe duality correspondence for the pair $(SL(2), O(p-1,q-1))$, and it is well known that this spectrum is simple. Moreover, no representation of $O(p-1,q-1)$ enters $\omega_+|_{M'}$ more than once. Combining this with the fact that unitary irreducible representations of $SL(2)$ remain irreducible when restricted to the Borel subgroup B of $SL(2)$, we see that $\mathcal{VN}(\omega_+, M') = \mathcal{VN}(\omega_+, Q)$, where $Q = B \times O(p-1,q-1)$ is a subgroup of M'. Similarly, $\mathcal{VN}(\omega_- \oplus \omega_+, M') = \mathcal{VN}(\omega_- \oplus \omega_+, Q)$.

Finally, we can summarize the results of the preceding discussion in the following

Lemma 3.1 *Let π be the tensor square of the minimal representation of G. Then $\pi|_{M'N'} \simeq \int^{\oplus} \vartheta_s \otimes \widetilde{\rho}_s \, ds$, and for each $s \in \mathbb{R}^{\times}$ one has $\mathcal{VN}(\vartheta_s, M') = \mathcal{VN}(\vartheta_s, Q)$.*

We now study a restriction of π on the subgroup QN' of G. In proving the theorem below, we use the approach of [L].

Theorem 3.2 $\mathcal{VN}(\pi, G) = \mathcal{VN}(\pi, P)$.

Proof It suffices to check that $\mathcal{VN}(\pi, M') \subseteq \mathcal{VN}(\pi, QN')$. Indeed, a subgroup QN' of P' is also contained in P, and P and M' together generate G (since P is a maximal subgroup).

First of all, ρ_s ($s \neq 0$) is a unitary irreducible representation of the Heisenberg group N', hence its von Neumann algebra $\mathcal{VN}(\rho_s, N')$ is simply the algebra of bounded operators on $\mathcal{L}^2(\mathbb{R}^{p+q-2})$. Therefore, the algebra $\mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, QN')$ contains a set $\mathcal{B}$ of all operators of the form $\vartheta_s(x) \otimes u$, where $x \in Q$ and u is an arbitrary bounded operator on $\mathcal{L}^2(\mathbb{R}^{p+q-2})$.

Consider a von Neumann algebra $\mathcal{V}$ generated by $\mathcal{B}$. By Lemma 3.1 $\mathcal{VN}(\vartheta_s, M') = \mathcal{VN}(\vartheta_s, Q)$ and we see that, in particular, $\mathcal{V}$ contains all operators of the form $\vartheta_s(m) \otimes \widetilde{\rho}_s(m)$, $m \in M'$. Since these operators generate the algebra $\mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, M')$, we obtain an inclusion

$$\mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, M') \subseteq \mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, QN').$$

Then

$$\mathcal{VN}(\pi, M') \subseteq \int^{\oplus} \mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, M') \, ds \subseteq \int^{\oplus} \mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, QN') \, ds.$$

Since all representations $\widetilde{\rho}_s$ are nonisomorphic,

$$\int^{\oplus} \mathcal{VN}(\vartheta_s \otimes \widetilde{\rho}_s, QN') \, ds = \mathcal{VN}(\pi, QN')$$

and $\mathcal{VN}(\pi, M') \subseteq \mathcal{VN}(\pi, QN')$, as claimed. ∎

Statements about von Neumann algebras similar to that of Theorem 3.2 are typical when one deals with low-rank representations (in the sense of [L]). Though π is not low-rank as a representation of P, Theorem 3.2 will allow us to derive the conclusions about the decomposition of π by considering its restriction $\pi|_P$ and studying the spectrum of this restriction.

4 Decomposition of $\pi|_P$

We start by taking two copies of $\sigma^{\min}$ and realizing the first one on $\mathcal{L}^2(C, d\mu)$ and the second one on $\mathcal{L}^2(C', d\mu)$, where $C' = M \cdot c'$. To distinguish these copies, we denote them by σ and σ', respectively. Then for $r \in M$, $l \in M$, $n \in N$, $f \in \mathcal{L}^2(C, d\mu)$ and $f' \in \mathcal{L}^2(C', d\mu)$ one has

$$\sigma(l)f(r \cdot c) = \delta^{-1/2}(l)f((l^{-1}r) \cdot c),$$

$$\sigma'(l)f'(r \cdot c') = \delta^{-1/2}(l)f'((l^{-1}r) \cdot c'),$$

$$\sigma(n)f(r \cdot c) = \langle r \cdot c, n\rangle f(r \cdot c),$$

$$\sigma'(n)f'(r \cdot c') = \langle r \cdot c', n\rangle f'(r \cdot c').$$

Set $\mathbf{C} = C \times C'$ and $\mathbf{c} = (c, c') \in \mathbf{C}$. We now realize $\pi \simeq \sigma \otimes \sigma'$ on $\mathcal{L}^2(\mathbf{C}, d\nu)$, where $d\nu$ is the M-equivariant product measure on $C \times C'$. Since $c + c' = e$ (where $e = (1, 0)$ is a unit in a Jordan algebra $\overline{N}$), for $\phi \in \mathcal{L}^2(\mathbf{C}, d\nu)$ we obtain:

$$\pi(l)\phi(r \cdot \mathbf{c}) = \delta^{-1}(l)\phi((l^{-1}r) \cdot \mathbf{c}),$$

$$\pi(n)\phi(r \cdot \mathbf{c}) = \langle r \cdot e, n\rangle \phi(r \cdot \mathbf{c}).$$

Let S and H be the subgroups of $M = \mathbb{R}^* \times O(p, q)$ stabilizing $e \in \overline{N}$ and $\mathbf{c} \in \overline{N} \times \overline{N}$, respectively. Then, $S \simeq SO(p, q-1)$ and H is a symmetric subgroup of S.

Denote by χ the unitary character of N defined by the element $e \in \overline{N}$. The proof of the following lemma is similar to the argument in [DS1, p.18].

Lemma 4.1 $\pi|_P \simeq \operatorname{Ind}_{HN}^{P}(1 \otimes \chi)$.

Proof Denote the unitarily induced representation in the right-hand side by τ. Then τ acts on the space $\mathcal{W}$, where $\mathcal{W}$ consists of the square-summable functions on $P = MN$ satisfying the equivariance relation:

$$\mathcal{W} = \{f \colon P \to \mathbb{C} \mid f(xhn) = \chi(n)^{-1} f(x), x \in P, h \in H, n \in N\},$$

and the action of τ on $\mathcal{W}$ is then given by

$$\tau(l)f(rn) = \delta^{-1}(l)f(l^{-1}rn),$$

$$\tau(n')f(rn) = f(n'^{-1}r\,n)$$

for $rn \in MN$, $l \in M$, $n' \in N$. The factor $\delta^{-1}(l)$ in the first formula arises since we perform *unitary* induction, and the M-equivariant measure on $M/H = C \times C'$ transforms by the character δ^2 of M.

Now the map $F\colon \mathcal{L}^2(\mathbf{C}, d\nu) \to \mathcal{W}$, given by

$$[F\phi](r\,n) = \chi(n)^{-1}\phi(r\cdot \mathbf{c})$$

is well defined and gives an isometry between these two spaces. It remains to check that F also intertwines the actions of π and τ, and this can be verified by direct calculation.

For $n' \in N$ we get

$$\begin{aligned}
[(\tau(n')F)\phi](rn) &= [F\phi](n'^{-1}r\,n) = [F\phi](r(r^{-1}n'^{-1}rn))\\
&= \chi(r^{-1}n'^{-1}r)^{-1}\chi(n)^{-1}\phi(r\cdot\mathbf{c})\\
&= \chi(n)^{-1}\langle r\cdot e, n'\rangle\phi(r\cdot\mathbf{c}) = [(F\pi(n'))\phi](r\cdot\mathbf{c}).
\end{aligned}$$

The verification of the identity $[(\tau(l)F)\phi](rn) = [(F\pi(l))\phi](r\cdot\mathbf{c})$ is even more straightforward, and we conclude that F gives us the desired intertwining operator. ■

Since the character χ is by definition S-invariant, we can perform induction in stages, and write

$$\pi|_P \simeq \operatorname{Ind}_{HN}^P(1\otimes\chi) \simeq \operatorname{Ind}_{SN}^P((\operatorname{Ind}_H^S 1)\otimes\chi).$$

Recall that S is simply a stabilizer of $e \in \mathbb{R}^{p,q}$ in M, and H stabilizes both idempotents c and c' (and, of course, $e = c + c'$ as well). Then H is a symmetric subgroup of S, and the quotient space S/H can be identified with the rank one reductive symmetric space $X = SO_0(p, q-1)/\,SO_0(p-1, q-1)$. For example, if $G = SO(3,3)$, then $X = SO_0(2,1)/\,SO_0(1,1)$ is the hyperboloid of one sheet in $\mathbb{R}^3$. In general, all spaces X are real hyperbolic spaces [F].

Let $d\eta$ be the Plancherel measure for the hyperbolic space X, *i.e.,*

$$\operatorname{Ind}_H^S 1 \simeq L^2(X) \simeq \int^{\oplus} \kappa\, d\eta(\kappa).$$

Here κ ranges over H-spherical representations of S, and every representation enters into this decomposition with multiplicity (at most) one [M, Theorem 30.3]. Put

$$\widetilde{\kappa} = \operatorname{Ind}_{SN}^P(\kappa\otimes\chi).$$

We arrive at the following

Proposition 4.2 *A direct integral* $\int^{\oplus}\widetilde{\kappa}\,d\eta(\kappa)$ *(where* $d\eta(\kappa)$ *is the Plancherel measure for the hyperbolic space* X*) gives a decomposition of* $\pi|_P$ *into unitary irreducible representations of* P*. All these representations are nonisomorphic, and the decomposition is multiplicity-free.*

Proof The existence of the decomposition follows from Lemma 4.1 and the discussion above. It remains to check that the representations $\widetilde{\kappa}$ are irreducible and pairwise nonisomorphic for different κ. All this follows immediately from standard Mackey theory. ■

5 Decomposition of π

We now combine the results of Theorem 3.2 and Proposition 4.2 to express the decomposition of π in terms of the Plancherel measure $d\eta$ on X.

Theorem 5.1 *Let κ be an H-spherical representation of S. For almost every (with respect to the measure $d\eta$) representation κ, the representation $\widetilde{\kappa} = \mathrm{Ind}_{SN}^{P}(\kappa \otimes \chi)$ extends uniquely to the unitary irreducible representation $\theta(\kappa)$ of G, and*

$$\pi \simeq \int^{\oplus} \theta(\kappa)\, d\eta(\kappa). \tag{1}$$

Proof Let $\pi = \int^{\oplus} \gamma\, d\lambda(\gamma)$ be the decomposition of π into irreducibles. By Theorem 3.2, $\mathcal{VN}(\pi, G) = \mathcal{VN}(\pi, P)$. This is possible only when the following is true for (almost) any γ in the decomposition:

- $\gamma|_P$ is irreducible;
- if $\gamma|_P \simeq \gamma'|_P$, then $\gamma \simeq \gamma'$.

Therefore, the Plancherel measure $d\lambda$ in the decomposition of π coincides with the measure $d\eta$ for $\pi|_P$. In other words, almost every $\widetilde{\kappa}$ extends to a unitary representation of G, and the P-decomposition obtained in Proposition 4.2 lifts to G-decomposition (1), as claimed. ■

One of the interesting features of the decomposition (1) is the existence of the discrete part of the spectrum. It is known that for $p \neq q$ the quasiregular representation of $SO_0(p, q-1)$ on $L^2(X)$ contains the representations of the relative discrete series [M]. It follows from Theorem 5.1, that *any* such representation κ (the qualifier "almost" can be dropped in this case), can be lifted to a representation $\theta(\kappa)$ of G. Then the representation $\theta(\kappa)$ enters formula (1) as a discrete summand. One can compare this fact with the results of [Z], where a similar discrete spectrum was discovered in the case of $\sigma \otimes \overline{\sigma}$, σ – minimal (holomorphic) representation of $U(p, q)$. It would be interesting to study these representations $\theta(\kappa)$ and understand their place in the unitary dual of G.

Remark 1 One can also consider the restriction of π to the parabolic subgroup $P'' = [\mathrm{GL}(q+1) \times O(p-q)] \times N''$, where N'' is a nilpotent subgroup with center $ZN'' = \Lambda^2\mathbb{R}^{q+1}$ (N'' is abelian if $p = q$). It follows from [L] that for $q \geq 3$ all representations $\theta(\kappa)$ arising in (1) are rank 2 representations ($\sigma^{\min}$ itself is rank 1, and π has rank 2), and they can be obtained by considering Howe duality for the pair $(Sp(4, \mathbb{R}), G)$. The discrete spectrum for this stable range dual pair is described in [A].

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