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THE SET OF JULIA POINTS FOR FUNCTIONS OMITTING TWO VALUES

BY
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Let f be a function defined in the unit disk $D(|z|<1)$. For each point $e^{i\theta}$ on the unit circle $C(|z|=1)$ and each subset S of D, we denote by $C_S(f, e^{i\theta})$ the cluster set of f at $e^{i\theta}$ relative to s, i.e.

$$C_S(f, e^{i\theta})=\bigcap_{j=1}^{\infty} \overline{f(S\cap N(e^{i\theta}, j))},$$

where $N(e^{i\theta}, j)=\{z\in D:|z-e^{i\theta}|<1/j\}$.

By a Stolz angle $\Delta(e^{i\theta})$ at $e^{i\theta}$, we mean a triangle Δ with one vertex at $e^{i\theta}$ and $\Delta\subset D$. A point $e^{i\theta}$ on C is called a Plessner point of f, if $C_{\Delta(e^{i\theta})}(f, e^{i\theta})=W$ for each possible choice of $\Delta(e^{i\theta})$, where W denotes the Riemann sphere. We denote the set of all Plessner points of f by $I(f)$.

In [5], Peter Lappan has proved the following two theorems.

THEOREM A. *If f is a meromorphic function in D, then $I(f)$ is a G_δ subset of C.*

THEOREM B. *If E is a G_δ subset of C, then there exists a holomorphic function f in D for which $I(f)=E$.*

On the other hand, in [1] E. F. Collingwood and G. Piranian have introduced the notion of Julia point. A point $e^{i\theta}$ is called a Julia point of f, provided in each Stolz angle $\Delta(e^{i\theta})$ the function f assumes all values on the Riemann sphere except possibly two. We denote the set of all Julia points of f by $J(f)$. P. Colwell [2] has proved that both Theorems A and B are true for $J(f)$ instead of $I(f)$. As far as Julia point is concerned, we may ask whether there can actually be constructed a function f such that f omits two values and yet $J(f)=E$. For this, we claim the following

THEOREM C. *If E is a G_δ subset of C, and if $w_1, w_2\in W$, then there exists a meromorphic function f in D which omits w_1, w_2, and $J(f)=E$.*

Proof. Without loss of generality, we may assume that

$$E=\bigcap_{n=1}^{\infty} E_n, E_1\supset E_2\supset\cdots\supset E_n\supset\cdots,$$

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and

$$E_n = \bigcup_{j=1}^{\infty} I_{n,j}, \quad \text{where } I_{n,j} \text{ is an open arc on } C,\ j=1,2,\ldots.$$

For each $I_{n,j}$, we form a circular triangle $G_{n,j}$ in D by joining the endpoints of $I_{n,j}$ with two segments which are equal length and orthogonal to each other in D. We set

$$G_n = \bigcup_{j=1}^{\infty} G_{n,j}, \qquad F_n = \overline{D - G_n}, \qquad D_n = \left\{z : |z| \le 1 - \frac{1}{n}\right\},$$

and

$$H_n = D_n \cup F_n, \qquad n = 1, 2, \ldots.$$

It is geometrically clear that we can construct two sequences $\{z_k\}$ and $\{z'_k\}$ of points in D such that

(1) $$z_k, z'_k \notin H_n, \quad \text{for} > k \quad k(n).$$

(2) for any $e^{i\theta} \in E$, any Stolz angle $\Delta(e^{i\theta})$ contains infinitely many of the pairs z_k, z'_k, and

(3) the hyperbolic distance $\rho(z_k, z'_k) \to 0$ is $k \to \infty$,

Clearly each H_n is a compact set for which the complement $W - H_n$ is connected and therefore by a theorem of S. N. Mergelyan [6, Theorem 1.4], there is a polynomial $P_n(z)$ such that

(4) $$|P_n(z)| < \frac{1}{2^n}, \quad \text{for} \quad z \in H_n.$$

By virtue of J. L. Walsh's lemma [7, p. 310] and the property (1), those polynomials $P_n(z)$ can be chosen to interpolate at the points z_k and z'_k in the following manner

(5) $$\sum_{j=1}^{n} P_j(z_k) = 0, \quad \text{for} \quad z_k \in H_{n+1} \backslash H_n$$

and

(6) $$\sum_{j=1}^{n} P_j(z'_k) = 10, \quad \text{for} \quad z'_k \in H_{n+1} \backslash H_n.$$

Let $g(z) = \sum_{n=1}^{\infty} P_n(z)$, then by (4) we can see g is holomorphic in D. Moreover, from (4), we find that no point of the complement $C - E$ is a Plessner point. It remains to prove that each point $e^{i\varepsilon} \in E$ is a Julia point. By the properties (2) and (3) and a result in [3, Theorem 1], it suffices to prove that there exists a real number δ such that

(7) $$\mathscr{X}(g(z_k), g(z'_k)) \ge \delta > 0 \quad \text{for sufficiently large } k,$$

where $\mathscr{X}(a, b)$ is the chodal distance between a and b.

Now, from (4) and (5), we have for sufficiently large k, $|g(z_k)| < 1$ while (4) and (6) give $|g(z'_k)| > 9$. This yields (7) and therefore $J(g) = E$.

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Finally, if $w_1=\infty$ and w_2 is finite, set

$$f(z)=e^{g(z)}+w_2,$$

while if both w_1 and w_2 are finite, set

$$f(z)=(w_1-w_2e^{g(z)})/(1-e^{g(z)}).$$

It is obvious that f omits w_1, w_2, and $J(f)=E$. This completes the proof.

Remarks

1. If we call a point $e^{i\theta}$ a strong Julia point provided in each Stolz angle $\Delta(e^{i\theta})$, the function f assumes all values on the Riemann sphere, and if we denote by $J^*(f)$ the set of all strong Julia points, then for meromorphic functions the above theorem is still true for $J^*(f)$ instead of $J(f)$. Simply replace f by $g\circ f$ where g is any non-constant elliptic function.

2. Since a Julia point is also a Plessner point, Theorem B follows immediately from the above theorem.

3. Our theorem also improves the following theorem of Lappan and Piranian [4]:

For each number $\alpha(0\le\alpha\le 2\pi)$, there exists a holomorphic function in D whose set of Plessner points is dense on C and has measure α.

4. Our theorem is sharp in the sense that it is impossible to improve f omitting more than two values.

References

1. E. F. Collingwood and G. Piranian, *Tsuji functions with segments of Julia.* Math. Z. **84** (1964), 246–253.
2. P. Colwell, *Julia points of functions meromorphic in a disc.* Bull. London Math. Soc. **4** (1972), 327–329.
3. P. M. Gauthier, *A criterion for normalcy*, Nagoya Math. J. **32** (1968), 277–282.
4. P. Lappan and G. Piranian, *Holomorphic functions with dense sets of Plessner points*, Proc. Amer. Math. Soc. **21**, no. **3** (1969), 555–556.
5. P. Lappan, *A characterization of Plessner points*, Bull. London Math. Soc. **2** (1970), 60–62.
6. S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, [Uspehi Mat. Nauk (N.S.) **7**, no. **2** (**48**) (1952), 31–122]. Amer. Math. Soc. Transl. 101 (1954), 99 pp.
7. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloq. Publ. Vol. **XX** (1960).

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