NOTE ON WEIGHT SPACES OF IRREDUCIBLE LINEAR REPRESENTATIONS

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Let L denote a finite dimensional, simple Lie algebra over an algebraically closed field F of characteristic zero. It is well known that every weight space of an irreducible representation (ρ, V) admitting a highest weight function is finite dimensional. In a previous paper [2], we have established the existence of a wide class of irreducible representations which admit a one-dimensional weight space but no highest weight function. In this paper we show that the weight spaces of all such representations are finite dimensional. More precisely, we prove:

THEOREM 1. <u>If</u> (ρ, V) <u>is an irreducible representation of</u> L <u>which admits a finite dimensional weight space</u>, <u>then every weight space of</u> (ρ, V) <u>is finite dimensional</u>.

Let U denote the universal enveloping algebra of L; then each representation (irreducible representation) (ρ, V) of L can be uniquely extended to a representation (resp. irreducible representation) of U which we shall again denote by (ρ, V). A map χ from the centre Z of U to the field of scalars F is called a <u>character</u> [1] of (ρ, V) if

$$\rho(z)v = \chi(z)v \qquad (\forall z \in Z)(\forall v \in V).$$

Using Theorem 1 we prove:

THEOREM 2. <u>If</u> (ρ, V) <u>is an irreducible representation of</u> U <u>which admits a finite dimensional weight space</u>, <u>then</u> (ρ, V) <u>admits a character</u>.

1. <u>Weight spaces</u>. Let $\{Y_\beta, H_\alpha, X_\beta \mid \alpha \in \Delta, \beta \in \Gamma^+\}$ denote the Cartan basis of L where Δ and Γ^+ denote the simple and positive roots of L with respect to a fixed Cartan subalgebra $\mathcal{H}$ of L. Then by the Poincaré - Birkhoff - Witt Theorem, U admits a basis $\mathcal{B}$ consisting of all elements of the form:

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(1) $$\prod_{\beta\in\Gamma^+} Y_\beta^{m(\beta)} \prod_{\alpha\in\Delta} H_\alpha^{k(\alpha)} \cdot \prod_{\beta\in\Gamma^+} X_\beta^{n(\beta)}$$

where the exponents $m(\beta)$, $k(\alpha)$ and $n(\beta)$ are non-negative integers and the products Π each preserve a fixed order over their respective index sets. Let U_ξ denote the linear subspace of U generated by the set of all elements of $\mathcal{B}$ for which

$$\sum_{\beta\in\Gamma^+} (m(\beta) - n(\beta))\,\beta = \xi .$$

Clearly the underlying space of U is equal to the direct sum of all subspaces U_ξ where ξ ranges over all linear integral combinations of the simple roots of L.

For any element $\lambda \in \mathcal{H}^*$, the dual space of the Cartan subalgebra, we define

$$V_\lambda = \{v \in V \mid \rho(H)\, v = \lambda(H)v \quad (\forall H \in \mathcal{H})\}$$

The linear functional λ is called a <u>weight function</u> and V_λ is called the corresponding <u>weight space</u> of (ρ, V) if and only if $V_\lambda \neq \{0\}$. The following lemma connects the weight spaces V_λ of (ρ, V) and the subspaces U_ξ of U.

LEMMA 1. <u>If</u> (ρ, V) <u>is an irreducible representation of</u> U, V_λ <u>is a weight space of</u> (ρ, V), <u>and</u> v_o <u>is a non-zero element of</u> V_λ <u>then for any weight space</u> V_γ <u>of</u> (ρ, V) <u>we have</u>

$$\rho(U_{\gamma-\lambda})\, v_o = V_\gamma .$$

<u>Proof</u>. Using the properties of the Cartan basis of L for each $H \in \mathcal{H}$ and each $u \in U_\xi$ we have $[H, u] = \xi(H)u$. From this observation it follows that $\rho(U_\xi)v_o \subseteq V_{\lambda+\xi}$. On the other hand since (ρ, V) is irreducible we have

$$V = \dot{\sum_\xi} \rho(U_\xi)v_o \subseteq \dot{\sum_\xi} V_{\lambda+\xi} = V .$$

Therefore, for each ξ, $\rho(U_\xi)v_o = V_{\lambda+\xi}$.

Now let $\{E_1, E_2, \ldots, E_m\}$ be a fixed order on the Cartan basis of L. Then we may associate with each $Y \in \mathcal{B}$ an ordered m-tuple $(Y_{(1)}, \ldots, Y_{(m)})$ where $Y_{(i)}$ = the exponent of E_i in Y. We define a partial order on $\mathcal{B}$ setting $X \leq Y$ if and only if $X_{(i)} \leq Y_{(i)}$ for

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$i = 1, 2, \ldots, m$. Moreover we set $X < Y$ if and only if $X \leq Y$ and $X \neq Y$. For each linear integral combination ξ of simple roots of L we define an ξ-minimal element to be an element $Y \in U_\xi \cap \mathcal{B}$ such that for each element $Y' \in U_\xi \cap \mathcal{B}$ with $1 \neq Y' \leq Y$ we have $Y' = Y$.

LEMMA 2. There exists only a finite number of ξ-minimal elements for each linear integral combination ξ of simple roots of L.

Proof. (This proof was communicated to me by I. Bouwer and represents a considerable simplication of my original proof.)

It clearly suffices to show that there exists a constant K such that for each ξ-minimal element Y, $Y_{(i)} \leq K$ for $i = 1, 2, \ldots, m$.

Select any ξ-minimal element X and define

$$K_1 = \max \{X_{(i)} \mid i = 1, 2, \ldots, m\}.$$

It follows from the definition of ξ-minimality that each ξ-minimal element Y has at least one component less than or equal to K_1. Inductively assume that we have already defined an integer K_r such that each ξ-minimal element Y has at least r components less than or equal to K_r. We now define K_{r+1} as follows:

Let $I^{(r)} = \{(n_1, \ldots, n_r) \mid n_i \text{ integer with } 1 \leq n_1 < n_2 < \ldots < n_r \leq m\}$ and $J^{(r)} = \{(m_1, m_2, \ldots, m_r) \mid m_i \text{ integer with } 0 < m_i \leq K_r \text{ for } i = 1, 2, \ldots, m\}$. For each pair $(\underline{n}, \underline{m}) \in I^{(r)} \times J^{(r)}$ define

$$P(\underline{n}, \underline{m}) = \{Y \mid Y \text{ is } \xi\text{-minimal and } Y_{(n_i)} = m_i \text{ for } i = 1, 2, \ldots, r\}.$$

If $P(\underline{n}, \underline{m}) = \phi$ ignore it. If, however, $P(\underline{n}, \underline{m}) \neq \phi$ choose $X \in P(\underline{n}, \underline{m})$ and define $K(\underline{n}, \underline{m}) = \max \{X_{(i)} \mid i = 1, 2, \ldots, m\}$. Since the set $I^{(r)} \times J^{(r)}$ is finite we define $K_{r+1} = \max [\{K(\underline{n}, \underline{m}) \mid (\underline{n}, \underline{m}) \in I^{(r)} \times J^{(r)}\} \cup \{K_r\}]$.

We now claim that for each ξ-minimal element Y there are at least $r + 1$ components of Y which are less than or equal to K_{r+1}. Assume to the contrary that Y is ξ-minimal and has exactly r components say $(n_1, n_2, \ldots, n_r)$ such that $Y_{(n_i)} \leq K_{r+1}$. By definition of K_r we may assume that $Y_{(n_i)} \leq K_r \leq K_{r+1}$ for $i = 1, 2, \ldots, r$. Then by definition of K_{r+1} there exists an ξ-minimal element Y' such that $Y'_{(n_i)} = Y_{(n_i)}$ for $n_i \in \{n_1, \ldots, n_r\}$ and $Y'_{(i)} \leq K_{r+1}$ for all other components. This contradiction proves our claim.

By continuing this construction to the m^{th} step we obtain the required bound on the components of ξ-minimal elements.

A straightforward proof using induction on the number of factors shows that $U_{\xi_1} U_{\xi_2} \subseteq U_{\xi_1 + \xi_2}$. In particular $U_o U_o \subseteq U_o$ --- i.e. U_o is a subalgebra of U which is called the cycle subalgebra of U. With this in mind we have the following immediate corollary of Lemma 2.

COROLLARY. Each subspace U_ξ of U is a finitely generated U_o-module. In fact the ξ-minimal elements form a generating set of U_ξ qua U_o-module.

Using the above lemmas we can now prove the first theorem.

THEOREM 1. If (ρ, V) is an irreducible representation of L admitting a finite dimensional weight space, then every weight space of (ρ, V) is finite dimensional.

Proof. Let V_λ denote a finite dimensional weight space of (ρ, V). Then by Lemma 1, if v_o is a non-zero element of V_λ we have $\rho(U_o)v_o = V_\lambda$. Since V_λ is finite dimensional there exists elements $e_1, e_2, \ldots, e_n \in U_o$ such that $\{\rho(e_i)v_o \mid i = 1, 2, \ldots, n\}$ forms a basis for V_λ.

If V_γ denotes a second weight space of (ρ, V) then by Lemma 1 we have

$$V_\gamma = \rho(U_{\gamma-\lambda})v_o .$$

Let $\{u_1, u_2, \ldots, u_k\}$ denote the set of $(\gamma-\lambda)$-minimal elements. Then applying Lemmas 1 and 2 it is clear that $\rho(U_{\gamma-\lambda})v_o$ and hence V_γ is generated by

$$\{\rho(u_i e_j)v_o \mid i = 1, 2, \ldots, k; \quad j = 1, 2, \ldots, n\}$$

as a vector space over F. Therefore V_γ is a finite dimensional vector space over F.

2. Characters. As an application of the results of § 1 we have

THEOREM 2. *Every irreducible representation* (ρ, V) *of* L; *which admits a finite dimensional weight space admits a character.*

Proof. Let (ρ, V) denote an irreducible representation of L then each weight space V_λ of (ρ, V) induces an irreducible representation $(\rho_\lambda, V_\lambda)$ of U_o where $\rho_\lambda(c)v = \rho(c)v$ $(\forall c \in U_o)$ $(\forall v \in V_\lambda)$. Applying Theorem 1 every weight space of (ρ, V) is finite dimensional. Thus we may apply Schur's Lemma to observe that for each element x of the centre of U_o, $\rho_\lambda(x)$ is a scalar multiple of the identity map on the weight space V_λ.

If Z denotes the centre of U, then it is clear that Z is a subset of the centre of U_o. Thus for each weight function λ of (ρ, V) we define a map $\chi_\lambda : Z \to F$ by the condition that

$$\rho_\lambda(z) = \chi_\lambda(z) 1_{V_\lambda} .$$

Since (ρ, V) is an irreducible representation, V is equal to the direct sum of its weight spaces. To complete the proof we need only show that for any two weight functions λ and γ of (ρ, V) we have $\chi_\lambda(z) = \chi_\gamma(z)$ for all $z \in Z$.

Select two non-zero vectors $v \in V_\lambda$ and $w \in V_\gamma$. Since (ρ, V) is irreducible, there exists an element $u \in U$ such that $\rho(u)v = w$. Then for any $z \in Z$ we have

$$\chi_\gamma(z)w = \rho(z)w = \rho(z)\,\rho(u)\,v = \rho(u)\,\rho(z)\,v$$
$$= \chi_\lambda(z)\,\rho(u)v = \chi_\lambda(z)w.$$

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