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Compositio Math. **160** (2024), 915–944.

[doi:10.1112/S0010437X24007061](https://doi.org/10.1112/S0010437X24007061)



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ABSTRACT

In this paper, we present a solution to the problem of the analytic classification of germs of plane curves with several irreducible components. Our algebraic approach follows precursive ideas of Oscar Zariski and as a subproduct allows us to recover some particular cases found in the literature.

1. Introduction

A remarkable step towards the understanding of the local structure of a germ of complex plane curve was the characterization of its topological type. Let \mathcal{C}_1 and \mathcal{C}_2 be germs of analytic reduced plane curves at the origin of \mathbb{C}^2 . We say that \mathcal{C}_1 and \mathcal{C}_2 are topologically equivalent (as embedded germs) if there exist U and V neighborhoods at the origin of \mathbb{C}^2 and a homeomorphism $\Phi : U \rightarrow V$ such that $\Phi(\mathcal{C}_1 \cap U) = \mathcal{C}_2 \cap V$. In this case, Zariski in [Zar71, Zar06] says that \mathcal{C}_1 and \mathcal{C}_2 have the same *topological type* or they are *equisingular*. When Φ is an analytic isomorphism, the curves are called *analytically equivalent*.

The local topology of plane curves has been studied since the first decades of the last century, with important contributions from Brauner, Burau, Zariski, Milnor, and others (see [BK86]). For an irreducible plane curve (branch) the topological type of the curve is equivalent to the topology of the complement of the associated link, that is, the intersection of the curve with a small sphere centered at the origin. This intersection is an iterated torus knot. The topological type of the curve is completely described by the type of the knot which is characterized by pairs of integers called Puiseux pairs. There are many complete discrete invariants that determine and are determined by Puiseux pairs, for instance, the characteristic exponents, the value semigroup of the curve, the multiplicity sequence associated to the canonical resolution, and others. For plane curves with several branches, the local topology can be described by the value semigroup of the curve or, equivalently, by the value semigroup of each branch and the intersection multiplicities of pairs of branches (see [Wal72, Zar71]).

With regard to analytic equivalence, there are many hard problems. For instance, how we can decide if two curves are analytically equivalent? What is the associated moduli space?

Any introduction to the analytic classification subject of plane curves is hardly comparable to the review by Washburn presented in [Was88]. We just cite some contributions to the theme in chronological order.

Received 20 January 2023, accepted in final form 18 October 2023, published online 19 March 2024.

2020 Mathematics Subject Classification 14H20 (primary), 14H10, 32S05, 32S10 (secondary).

Keywords: plane curves, analytic classification, analytic invariants.

The authors were partially supported by grant 2019/07316-0, São Paulo Research Foundation (FAPESP) and the first author was partially supported by CNPq-Brazil Proc. 303638/2020-6.

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In 1965, Ebey [Ebe65] presented normal forms for some classes of irreducible curves given by parameterizations. In a course given at the Centre de Mathématiques de l'École Polytechnique in 1973, Zariski [Zar06] considered an irreducible analytic plane curve \mathcal{C} and denoted by $\mathbb{L} = \mathbb{L}(\mathcal{C})$ the equisingular class of \mathcal{C} , that is, the set of all branches equisingular to \mathcal{C} . The moduli space \mathbb{M} is the quotient space of \mathbb{L} by the analytic equivalence relation. Zariski studied the moduli space for some equisingular classes and gave a formula for the dimension of the generic component \mathbb{M}_g of \mathbb{M} for curves with semigroup $\langle n, m \rangle$ in which $1 < n < m$ and $m \equiv 1 \pmod n$. A few years later, Delorme (in 1978; see [Del78]), considering irreducible curves with semigroup $\langle n, m \rangle$, presented a combinatorial method to obtain the dimension of the generic component of the moduli space. In 1979, Granger presented normal forms for plane curves with nonsingular transversal branches (an ordinary multiple point) and gave a formula to compute the dimension of \mathbb{M}_g (see [Gra79]).

Bruce and Gaffney in 1982 (see [BG82]), classified the simple irreducible plane curves, that is, when the moduli space is a zero-dimensional space. A more general approach for the moduli space was presented by Laudal and Pfister in 1988 (see [LP88]) where, for irreducible plane curves with semigroup $\langle n, m \rangle$, they fixed an analytic invariant (the Tjurina number) in order to describe normal forms given by elements in $\mathbb{C}[X, Y]$. In 1994, Greuel and Pfister (see [GP94]) developed a general method to construct coarse moduli spaces for singularities in the sense of Mumford Geometric Invariant Theory. Kang (in 2000; see [Kan00]) and Câmara and Scárdua (in 2018; see [CS18]), using different methods, classified analytic plane curves defined by weighted homogeneous polynomials. Kolgushkin and Sadykov (in 2001; see [KS01]) obtained normal forms for stably simple reducible curve singularities in complex spaces of any dimension.

In 2011, Hefez and the first author (see [HH11]) presented a solution for the analytic classification of irreducible plane curves as considered by Zariski in [Zar06] (see also [HH21]). Stratifying each equisingularity class by the analytical invariant given by the set of values of Kähler differentials, they presented normal forms for each stratum that allowed them to describe the moduli space. Later, the authors in a joint work with Hefez (in 2015; see [HHR15]), generalized such method to obtain the analytic classification of plane curves with two irreducible components.

Genzmer and Paul (in 2016; see [GP16]), using tools of Foliation theory, described the moduli space for generic plane curves such that every branch admits semigroup $\langle n, m \rangle$ and they presented a method to obtain the normal form for the generic case. Ayuso and Ribón (in 2020; see [AR20]), using holomorphic flows recovered the normal forms for branches presented in [HH11]. In 2022, Genzmer [Gen22] obtained a formula for the dimension of \mathbb{M}_g for any irreducible plane curve.

It is remarkable to observe that for about 50 years the questions proposed by Oscar Zariski in [Zar06], directly or indirectly, motivated these and so many other researchers that we could not possibly mention them all. The aim of this work is to present an answer to the intricate problem of the analytic classification of reduced plane curves in a fixed topological class. The combinatorial issues involved in this problem were probably our biggest challenge and makes this case not a simple generalization of the results about curves with two irreducible components presented in [HHR15].

We consider a topological class fixing the *value semiring* Γ of a reduced plane curve \mathcal{C} . The value semiring is equivalent, as a set, to the classical value semigroup. The main difference is that the value semiring, equipped with the tropical operations, admits a finite minimal set of generators in contrast to the value semigroup (see [CH20a]), and such generators allow us to recover directly the topological data pointed out by Zariski, that is, the value semigroup of each branch and their mutual intersection multiplicities. With our approach we recover the normal forms for the irreducible and two branches cases, presented in [HH11] and [HHR15], but it is not a simple induction step. In fact, the passage $1 \rightarrow 2 \rightarrow r$ branches requires extra efforts and

finer analysis in several situations. For example, understanding the semigroup for two branches as presented by Garcia (see [Gar82]) is not sufficient to describe the semigroup for $r \geq 3$. In fact, Mata (see [Mat87]) introduced new ingredients and non-trivial combinatorial aspects to perform this task. This important object and other invariants defined by a set of values of fractional ideals are described in § 2.

Our strategy is, under the action of a permutation group, to consider the branches of the curve in blocks according to their tangent lines, then we order the blocks and their elements by the multiplicities of the branches. In addition, we identify the appropriated group \mathcal{G} of local diffeomorphisms that preserve such properties. This is done in § 3. In § 4 we obtain the Puiseux block form of a multigerm (Proposition 4.6) that is a convenient parameterization of each component of the multigerm.

Section 5 contains the main results of this work. We introduce the \mathcal{G} -invariant $\Lambda_{\mathcal{G}}$ that corresponds to values of elements in a fractional ideal $\mathcal{I}_{\mathcal{G}}$ of the local ring \mathcal{O} of \mathcal{C} . The elements of $\mathcal{I}_{\mathcal{G}}$ are closely related to elements of the tangent space to the \mathcal{G} -orbit of the multigerm φ associated to the curve \mathcal{C} (Proposition 5.1). In Theorem 5.8 we present a \mathcal{G} -normal form of φ by a reduction process using the set $\Lambda_{\mathcal{G}}$, more specifically by their fibers. Theorem 5.10 takes into account the homothety group action and by this result we can decide whether two curves are analytically equivalent or not as we discuss in the end of the section.

Finally, in § 6 we apply our techniques to recover some known results related with the analytic equivalence of plane curves: the irreducible case, bigerms, and the ordinary multiple point singularity. Concerning the case of plane curves in which each branch admits value semigroup $\langle n, m \rangle$ and mutual intersection multiplicities equal to nm we apply our results to obtain a pre-normal form taking into account the value semiring Γ (Proposition 6.7) and we discuss about the generic component of the corresponding moduli space (Proposition 6.8 and Corollary 6.10).

2. Analytic equivalence and invariants

We denote by $\mathbb{C}\{X, Y\}$ the power series ring with complex coefficients in the variables X and Y , which are absolutely convergent in a neighborhood of the origin in \mathbb{C}^2 and by $\mathcal{M} = \langle X, Y \rangle$ its maximal ideal. Let \mathcal{C} be the germ of an analytic reduced plane curve in $(\mathbb{C}^2, 0)$ defined by a reduced element $f \in \mathcal{M} \setminus \{0\}$, or in other words, $\mathcal{C} = f^{-1}(0) \cap U$, where U is an open neighborhood at the origin 0 in \mathbb{C}^2 .

If $f = f_1 \cdots f_r$ is the decomposition of f into irreducible factors, then each f_i defines a branch, that is, an irreducible analytic plane curve, denoted by \mathcal{C}_i for $i = 1, \dots, r$.

Given a branch \mathcal{C}_i of \mathcal{C} with multiplicity n_i , that is, $f_i \in \mathcal{M}^{n_i} \setminus \mathcal{M}^{n_i+1}$, we can parameterize it by $(a_i t_i^{n_i} + \cdots, b_i t_i^{n_i} + \cdots) \in \mathbb{C}\{t_i\} \times \mathbb{C}\{t_i\}$ with $a_i \neq 0$ or $b_i \neq 0$. In what follows, we consider primitive parameterizations, that is, those that cannot be obtained from others composed with higher powers of t_i . We often identify such parameterization with the smooth map-germ $\varphi_i : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ defined by $t_i \mapsto (x_i, y_i) := (a_i t_i^{n_i} + \cdots, b_i t_i^{n_i} + \cdots)$.

We call

$$[\varphi_1, \dots, \varphi_r] = \begin{bmatrix} x_1 & \cdots & x_i & \cdots & x_r \\ y_1 & \cdots & y_i & \cdots & y_r \end{bmatrix}$$

a multigerm for \mathcal{C} and we denote by \mathcal{P} the set of multigerms of plane curves with r branches.

The analytic equivalence of reduced plane curves with r branches is translated, as we noted in [HHR15], into $\mathcal{S} \times \mathcal{A}$ -equivalence on \mathcal{P} , where \mathcal{S} is the symmetric group on r elements and $\mathcal{A} = \{(\rho_1, \dots, \rho_r, \sigma); \rho_i \in \text{Diff}(\mathbb{C}, 0) \text{ and } \sigma \in \text{Diff}(\mathbb{C}^2, 0), 1 \leq i \leq r\}$ is the group of right-left equivalence, where $\text{Diff}(\mathbb{C}^l, 0)$ denotes the diffeomorphism group of $(\mathbb{C}^l, 0)$.

Given $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}$ and $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ the action of \mathcal{A} on \mathcal{P} is defined as

$$\begin{aligned}
 (\rho_1, \dots, \rho_r, \sigma) \cdot \varphi &:= [\sigma \circ \varphi_1 \circ \rho_1^{-1}, \dots, \sigma \circ \varphi_r \circ \rho_r^{-1}] \\
 &= \begin{bmatrix} \sigma_1(x_1 \circ \rho_1^{-1}, y_1 \circ \rho_1^{-1}) & \cdots & \sigma_1(x_r \circ \rho_r^{-1}, y_r \circ \rho_r^{-1}) \\ \sigma_2(x_1 \circ \rho_1^{-1}, y_1 \circ \rho_1^{-1}) & \cdots & \sigma_2(x_r \circ \rho_r^{-1}, y_r \circ \rho_r^{-1}) \end{bmatrix},
 \end{aligned}$$

where $\sigma = (\sigma_1, \sigma_2)$.

We say that $\varphi, \psi \in \mathcal{P}$ are \mathcal{A} -equivalent, denoted by $\varphi \sim_{\mathcal{A}} \psi$, if and only if they are in the same \mathcal{A} -orbit. Now, the group $\mathcal{S} \times \mathcal{A}$ acts on \mathcal{P} by

$$(\pi, (\rho_1, \dots, \rho_r, \sigma)) \cdot \varphi := \pi((\rho_1, \dots, \rho_r, \sigma) \cdot \varphi) := [\sigma \circ \varphi_{\pi(1)} \circ \rho_{\pi(1)}^{-1}, \dots, \sigma \circ \varphi_{\pi(r)} \circ \rho_{\pi(r)}^{-1}],$$

for all $\pi \in \mathcal{S}$, $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}$ and $\varphi \in \mathcal{P}$.

It is a well known fact that for a plane curve \mathcal{C} defined by $f \in \mathbb{C}\{X, Y\}$, the isomorphism class of its local ring $\mathcal{O} := \mathbb{C}\{X, Y\}/\langle f \rangle$ completely determines the analytic class of \mathcal{C} . The aim of this section is to present discrete \mathcal{A} -invariants related to some fractional ideals of \mathcal{O} .

A parameterization $\varphi_i = (x_i, y_i) \in \mathbb{C}\{t_i\} \times \mathbb{C}\{t_i\}$ of a branch \mathcal{C}_i given by $f_i \in \mathbb{C}\{X, Y\}$ provides the exact sequence

$$\{0\} \rightarrow \langle f_i \rangle \rightarrow \mathbb{C}\{X, Y\} \xrightarrow{\varphi_i^*} \mathbb{C}\{x_i, y_i\} \rightarrow \{0\},$$

where $\varphi_i^*(h) := h(x_i, y_i)$. In what follows, we identify $\mathcal{O}_i := \mathbb{C}\{X, Y\}/\langle f_i \rangle$ with the subalgebra $\mathbb{C}\{x_i, y_i\} \subseteq \mathbb{C}\{t_i\}$. In particular, the integral closure of \mathcal{O}_i in its quotient field $\mathcal{Q}_i = \mathbb{C}((t_i))$ is $\overline{\mathcal{O}}_i = \mathbb{C}\{t_i\}$ and $\mathcal{O}_i = \overline{\mathcal{O}}_i$ if and only if $n_i = 1$.

We consider the natural discrete normalized valuation

$$\begin{aligned}
 \nu_i: \mathcal{Q}_i &\longrightarrow \overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\} \\
 \frac{p}{q} &\longmapsto \nu_i\left(\frac{p}{q}\right) := \text{ord}_{t_i}(p) - \text{ord}_{t_i}(q),
 \end{aligned}$$

where $p, q \in \mathcal{O}_i, q \neq 0$ and $\nu_i(0) = \infty$. The set

$$\Gamma_i = \{\nu_i(p); p \in \mathcal{O}_i\} = \{\nu_i(h) := \nu_i(\varphi_i^*(h)); h \in \mathbb{C}\{X, Y\}\}$$

is a submonoid of $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $S_i = \Gamma_i \cap \mathbb{N}$ is the classical *value semigroup* of the branch \mathcal{C}_i , which is an \mathcal{A} -invariant and a complete topological invariant.

The conductor ideal of \mathcal{O}_i in $\overline{\mathcal{O}}_i$ is $(\mathcal{O}_i : \overline{\mathcal{O}}_i) = \{h \in \mathcal{O}_i; h\overline{\mathcal{O}}_i \subseteq \mathcal{O}_i\} = \langle t_i^{\mu_i} \rangle$, where μ_i , called conductor of Γ_i , satisfies $\mu_i - 1 \notin \Gamma_i$ and $\mu_i + \overline{\mathbb{N}} \subset \Gamma_i$. In this case, the conductor of Γ_i coincides with the Milnor number of \mathcal{C}_i , that is, $\mu_i = \dim_{\mathbb{C}}(\mathbb{C}\{X, Y\}/\langle (f_i)_X, (f_i)_Y \rangle)$ where $(f_i)_X$ and $(f_i)_Y$ indicate the derivatives of f_i with respect to X and Y , respectively.

The previous concepts can be extended for a reduced curve \mathcal{C} with r branches defined by $f = f_1 \cdots f_r$.

Considering $I = \{1, \dots, r\}$ and the monomorphism

$$\begin{aligned}
 \mathcal{O} &\rightarrow \bigoplus_{i \in I} \mathcal{O}_i \\
 p &\mapsto (p_1, \dots, p_r),
 \end{aligned}$$

where p_i denotes the class of $p \in \mathcal{O}$ in \mathcal{O}_i , it is possible to verify that the total ring of fractions of \mathcal{O} is $\mathcal{Q} = \bigoplus_{i \in I} \mathcal{Q}_i$ and the integral closure of \mathcal{O} in \mathcal{Q} is $\overline{\mathcal{O}} = \bigoplus_{i \in I} \mathbb{C}\{t_i\}$. As before, if $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ is a multigerms associated to \mathcal{C} , then we identify \mathcal{O} with the subalgebra

$\{\varphi^*(h) := (\varphi_1^*(h), \dots, \varphi_r^*(h)), h \in \mathbb{C}\{X, Y\}\} \subseteq \bigoplus_{i \in I} \mathcal{O}_i$ and we set

$$\Gamma = \{\nu(h) := (\nu_1(h), \dots, \nu_r(h)); h \in \mathbb{C}\{X, Y\}\} \subset \Gamma_1 \times \dots \times \Gamma_r.$$

The conductor ideal of \mathcal{O} in $\overline{\mathcal{O}}$ is

$$(\mathcal{O} : \overline{\mathcal{O}}) = \bigoplus_{i \in I} \left((\mathcal{O}_i : \overline{\mathcal{O}}_i) \cdot \prod_{\substack{j \in I \\ j \neq i}} \varphi_j^*(f_j) \right) = \bigoplus_{i \in I} \langle t_i^{\kappa_i} \rangle,$$

where $\kappa_i = \mu_i + \sum_{\substack{j \in I \\ j \neq i}} \nu_j(f_j)$. The element $\kappa = (\kappa_1, \dots, \kappa_r)$, called the conductor of Γ , satisfies $\kappa + \overline{\mathbb{N}}^r \subset \Gamma$ and $\kappa - e_i \notin \Gamma$ for any element e_i in the canonical basis of \mathbb{Q}^r (see [Gor52]).

Zariski in [Zar71] showed that (up to permutation of the branches) the topological class of \mathcal{C} is totally characterized by $\Gamma_i \cap \mathbb{N}$ and the intersection multiplicity $\nu_i(f_j)$ ($= \nu_j(f_i)$) of \mathcal{C}_i and \mathcal{C}_j for $i, j \in I$ with $i \neq j$. On the other hand, Waldi (see [Wal72]) proved that $\Gamma \cap \mathbb{N}^r$ is also a complete topological invariant and Mata, in [Mat87], showed that Γ determines and it is determined by Γ_i and $\nu_i(f_j)$ for all $i, j \in I$, $i \neq j$, connecting the results of Waldi and Zariski.

In contrast to Γ_i , the (additive) semigroup Γ is not finitely generated, but equipped with the tropical operations, the set $(\Gamma, \inf, +)$ is a finitely generated semiring¹ (see [CH20a]), where $\inf\{(\alpha_1, \dots, \alpha_r), (\beta_1, \dots, \beta_r)\} = (\min\{\alpha_1, \beta_1\}, \dots, \min\{\alpha_r, \beta_r\})$, that we call the *value semiring* associated to \mathcal{C} . More precisely, there exists a minimal set of generators $\{v_j, j = 1, \dots, g\} \subset \Gamma$ such that any $\gamma \in \Gamma$ can be written as

$$\gamma = \inf \left\{ \sum_{j=1}^g a_{1j} v_j, \dots, \sum_{j=1}^g a_{rj} v_j \right\}, \tag{1}$$

where $a_{kj} \in \mathbb{N}$, $k \in I$, $1 \leq j \leq g$. The elements in $\{v_j, j = 1, \dots, g\} \cap \mathbb{N}^r$ are precisely the valuation of branches that achieve maximal contact with some \mathcal{C}_i for $i \in I$ and the subset of generators with some coordinate equal to ∞ is precisely $\{\nu(f_i), i \in I\}$ (see [Mat87, CH20a]).

In a more general situation, given any (regular) fractional ideal $\mathcal{I} \subseteq \mathcal{Q} = \bigoplus_{i \in I} \mathcal{Q}_i$ of \mathcal{O} we can consider the set of values

$$\nu(\mathcal{I}) = \{\nu(z) := (\nu_1(z_1), \dots, \nu_r(z_r)); z = (z_1, \dots, z_r) \in \mathcal{I} \subseteq \mathcal{Q}\} \subseteq \overline{\mathbb{Z}}^r.$$

The set $(\nu(\mathcal{I}), \inf)$ is a Γ -semimodule, that is, $\Gamma + \nu(\mathcal{I}) \subseteq \nu(\mathcal{I})$. Moreover, there exists (unique) $(\zeta_1, \dots, \zeta_r) =: \inf(\nu(\mathcal{I})) \in \nu(\mathcal{I})$ such that $\zeta_i \leq \alpha_i$ for every $(\alpha_1, \dots, \alpha_r) \in \nu(\mathcal{I})$ and $\nu(\mathcal{I})$ admits a ‘conductor’ $\varrho \in \nu(\mathcal{I})$, that is, $\varrho + \overline{\mathbb{N}}^r \subseteq \nu(\mathcal{I})$ and $\varrho - e_i \notin \nu(\mathcal{I})$ for all e_i in the canonical \mathbb{Q} -basis of \mathbb{Q}^r (see [GH20] for more properties concerning values set of fractional ideals of \mathcal{O}).

The next definition generalizes the corresponding concept introduced by Garcia in [Gar82] and Mata in [Mat87] for $\Gamma \cap \mathbb{N}^r$.

DEFINITION 2.1. Let \mathcal{I} be a fractional ideal of \mathcal{O} with set of values $\Delta = \nu(\mathcal{I})$. Given $J \subseteq I$ the J -fiber of $\gamma = (\gamma_1, \dots, \gamma_r) \in \overline{\mathbb{N}}^r$ with respect to Δ is the set

$$F_J^\Delta(\gamma) = \{(\delta_1, \dots, \delta_r) \in \Delta; \delta_j = \gamma_j \text{ for every } j \in J \text{ and } \delta_i > \gamma_i \text{ for all } i \in I \setminus J\}.$$

We say that $\gamma \in \overline{\mathbb{N}}^r$ is

- (i) a *maximal* element of Δ , if $F_I^\Delta(\gamma) \neq \emptyset$ (i.e. $\gamma \in \Delta$) and $F_{\{i\}}^\Delta(\gamma) = \emptyset$ for all $i \in I$;
- (ii) a *relative maximal* element of Δ , if γ is maximal and $F_J^\Delta(\gamma) \neq \emptyset$, for all $J \subseteq I$ with $\sharp J \geq 2$;
- (iii) an *absolute maximal* element of Δ , if γ is maximal and $F_J^\Delta(\gamma) = \emptyset$, for all $J \subsetneq I$, $J \neq \emptyset$.

¹ The set $(\Gamma, \inf, +)$ is a semiring since (Γ, \inf) and $(\Gamma, +)$ are monoids with identity elements $\underline{\infty} = (\infty, \dots, \infty)$ and $\underline{0} = (0, \dots, 0)$ respectively; $\inf\{\alpha + \beta, \alpha + \gamma\} = \alpha + \inf\{\beta, \gamma\}$ and $\underline{\infty} + \alpha = \underline{\infty}$ for every $\alpha, \beta, \gamma \in \Gamma$.

Remark 2.2. Let \mathcal{I} be a (regular) fractional ideal of \mathcal{O} such that $\text{inf}(\nu(\mathcal{I})) = (\zeta_1, \dots, \zeta_r)$ and $\varrho = (\varrho_1, \dots, \varrho_r)$ is the conductor of $\nu(\mathcal{I})$. In [CH20b, Theorem 16], the authors presented an algorithm to compute a finite set of generators of $\nu(\mathcal{I})$ (as Γ -semimodule) and showed that $\nu(\mathcal{I})$ is characterized by its elements in the box $R = [\zeta_1, \varrho_1] \times \dots \times [\zeta_r, \varrho_r]$. In particular, we have that all maximal elements of $\nu(\mathcal{I})$ belong to R .

For $\mathcal{I} = \mathcal{O}$ we obtain $\nu(\mathcal{O}) = \Gamma$ and Γ is determined by its elements in $[0, \kappa_1] \times \dots \times [0, \kappa_r]$.

Other \mathcal{A} -invariants that play an important role in this work are related with set of orders of differential 1-forms.

Let $\Omega^1 = \mathbb{C}\{X, Y\} dX + \mathbb{C}\{X, Y\} dY$ be the $\mathbb{C}\{X, Y\}$ -module of differential 1-forms. If $\varphi_i = (x_i, y_i) \in \mathbb{C}\{t_i\} \times \mathbb{C}\{t_i\}$ is a parameterization of a plane branch \mathcal{C}_i defined by f_i , we extend the homomorphism φ_i^* to Ω^1 in the following way.

Given $\omega = a(X, Y) dX + b(X, Y) dY \in \Omega^1$, we define

$$\varphi_i^*(\omega) := t_i \cdot (\varphi_i^*(a) \cdot x'_i + \varphi_i^*(b) \cdot y'_i) \in \mathcal{Q}_i,$$

where x'_i and y'_i denote, respectively, the derivative of $x_i, y_i \in \mathbb{C}\{t_i\}$ with respect to t_i .

Note that the kernel of φ_i^* is

$$\{\omega \in \Omega^1; \varphi_i^*(\omega) = 0\} = \left\{ \omega \in \Omega^1; \frac{\omega \wedge df_i}{dX \wedge dY} \in \langle f_i \rangle \right\} = f_i \cdot \Omega(\log \mathcal{C}_i),$$

where $\Omega(\log \mathcal{C}_i)$ is the module of logarithmic differential forms along \mathcal{C}_i which is the dual module of the logarithmic vector field $Der(-\log \mathcal{C}_i)$ along \mathcal{C}_i (see [Sai80]).

In addition, if $\mathcal{F}_i := f_i \cdot \Omega^1 + \mathbb{C}\{X, Y\} \cdot df_i$, then the Kähler differential module of \mathcal{O}_i is $\Omega_i \approx \Omega^1/\mathcal{F}_i$ and its torsion submodule is given by $\mathcal{T}_i \approx (f_i \cdot \Omega(\log \mathcal{C}_i))/\mathcal{F}_i$. Moreover,

$$\frac{\Omega_i}{\mathcal{T}_i} \approx \frac{\Omega^1}{f_i \cdot \Omega(\log \mathcal{C}_i)} \approx \varphi_i^*(\Omega^1).$$

The set $\varphi_i^*(\Omega^1) \subset \mathcal{Q}_i$ is a fractional ideal of \mathcal{O}_i and

$$\Lambda_i = \{\nu_i(\omega) := \nu_i(\varphi_i^*(\omega)); \omega \in \Omega^1\}$$

is an \mathcal{A} -invariant of \mathcal{C}_i . As $\nu_i(dh) = \nu_i(h)$ for any $h \in \mathcal{M}$ we have $\Gamma_i \setminus \{0\} \subseteq \Lambda_i$. The set Λ_i is one of the main tools considered in [HH11] in order to classify plane branches up to analytical equivalence.

Similarly for a multigerm $\varphi = [\varphi_1, \dots, \varphi_r]$ we get the \mathcal{A} -invariant

$$\Lambda = \nu(\varphi^*(\Omega^1)) = \{\nu(\omega) := (\nu_1(\omega), \dots, \nu_r(\omega)); \omega \in \Omega^1\} \subset \Lambda_1 \times \dots \times \Lambda_r \subset \overline{\mathbb{N}}^r. \tag{2}$$

Remark that $\varphi^*(\Omega^1) \approx \Omega^1/f \cdot \Omega(\log \mathcal{C})$ is a fractional ideal of \mathcal{O} with $\Gamma \setminus \{0\} \subseteq \Lambda$ and conductor $\varrho = (\varrho_1, \dots, \varrho_r)$ satisfying $\varrho_i \leq \kappa_i$ for $i = 1, \dots, r$.

In particular, by Remark 2.2, the set Λ is characterized by its points in the box $[0, \varrho_1] \times \dots \times [0, \varrho_r] \subseteq [0, \kappa_1] \times \dots \times [0, \kappa_r]$. Consequently, for each topological class determined by the semiring Γ there is a finite number of possible Λ sets.

The set Λ is related to the Tjurina number of \mathcal{C} (see [BGHH20]), the set $\nu(J(f)) = \{\nu(h); h \in J(f) := \langle f, f_X, f_Y \rangle \subseteq \mathbb{C}\{X, Y\}\}$ and the values of residues of elements in $f \cdot \Omega(\log \mathcal{C})$. More explicitly, if $\omega \in f \cdot \Omega(\log \mathcal{C})$, then there exist $\eta \in \Omega^1, g, h \in \mathbb{C}\{X, Y\}, g \notin \bigcup_{i=1}^r \langle f_i \rangle$ such that $g \cdot \omega = h \cdot df + f \cdot \eta$. The class $\text{res}(\omega)$ of h/g in \mathcal{Q} is called the residue of ω and we put $\text{Res}(f) = \{\text{res}(\omega), \omega \in f \cdot \Omega(\log \mathcal{C})\}$ (see [Sai80]). In [Pol18], Pol showed that $\nu(J(f)) = \Lambda + \kappa - (1, \dots, 1)$ and $\lambda \in \Lambda$ if and only if $-\lambda \notin \nu(\text{Res}(f))$.

Using the value semiring Γ or more specifically the subset $\nu(\mathcal{M}^2)$ of values of \mathcal{M}^2 we can obtain a distinguished presentation for a multigerm as we show in §4. We consider particular

fractional ideals of \mathcal{O} determined by differential 1-forms that are related to the tangent space to the orbit of a multigerms according to the action of a Lie group.

3. Group actions

Let $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ such that $\varphi_i(t_i) = (a_i t_i^{n_i} + \dots, b_i t_i^{n_i} + \dots)$ with $a_i \neq 0$ or $b_i \neq 0$. Given $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}$ with $\rho_i^{-1}(t_i) = u_i t_i + \dots$ and $\sigma(X, Y) = (\alpha X + \beta Y + \dots, \gamma X + \delta Y + \dots)$ such that $u_i, \alpha\delta - \beta\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ we have

$$\psi_i(t_i) := \sigma \circ \varphi_i \circ \rho_i^{-1}(t_i) = ((\alpha a_i + \beta b_i)u_i^{n_i} t_i^{n_i} + \dots, (\gamma a_i + \delta b_i)u_i^{n_i} t_i^{n_i} + \dots).$$

Denoting by $j^k \xi$ the k -jet of a map-germ ξ at the origin we put $j^k \varphi := [j^k \varphi_1, \dots, j^k \varphi_r]$.

The invariance of the multiplicity of the branches gives us a one-to-one correspondence between $j^1 \sigma$ and a Möbius transformation on $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ given by $T(\theta) = (\gamma + \delta\theta)/(\alpha + \beta\theta)$ such that if $\theta_i := b_i/a_i \in \overline{\mathbb{C}}$ is the slope of the tangent line $a_i Y - b_i X = 0$ of \mathcal{C}_i at the origin, then $T(\theta_i)$ is the slope of the tangent line to the curve corresponding to ψ_i for $1 \leq i \leq r$. In this way, up to change of coordinates, we can consider the tangent lines of the branches in a such way that up to three chosen directions are fixed.

If all branches have the same tangent line at the origin with slope θ , considering $\gamma = -\delta\theta$, then $T(\theta) = 0$. If the branches have exactly two tangent lines with distinct slopes θ_1 and θ_2 , then taking $\gamma = -\delta\theta_1$ and $\alpha = -\beta\theta_2$ we obtain $T(\theta_1) = 0$ and $T(\theta_2) = \infty$. On the other hand, if there exist at least three branches with distinct tangent lines of slopes θ_1, θ_2 , and θ_3 , we can consider the unique Möbius transformation T such that $T(\theta_1) = 0, T(\theta_2) = \infty$, and $T(\theta_3) = 1$.

By the above description and an $\mathcal{S} \times \mathcal{A}$ -action, we may adjust the components of $\varphi \in \mathcal{P}$ in a particular way.

DEFINITION 3.1. Given $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ we say that φ is in *block form* if there are sets $B_1 = \{\varphi_{k_1} = \varphi_1, \dots, \varphi_{k_2-1}\}, B_2 = \{\varphi_{k_2}, \dots, \varphi_{k_3-1}\}, \dots, B_s = \{\varphi_{k_s}, \dots, \varphi_r\}$ satisfying the following.

- (i) The tangent line of every branch in B_i has the same slope θ_i and, if $i \neq j$ then $\theta_i \neq \theta_j$ for $1 \leq i, j \leq s$. In addition, $\theta_1 = 0, \theta_2 = \infty$ and $\theta_3 = 1$.
- (ii) If φ_j and φ_k are elements of B_i with $j < k$, then $n_j \leq n_k$.
- (iii) We have $n_{k_1} \leq n_{k_2} \leq \dots \leq n_{k_s}$.

A set B_i as above is called a *block* of φ .

It is clear that any element in \mathcal{P} is $\mathcal{S} \times \mathcal{A}$ -equivalent to a multigerms in block form.

From now on we consider \mathcal{A} -action on block form multigerms in \mathcal{P} and without loss of generality we can suppose $j^{n_i} \varphi_i = (a_i t_i^{n_i}, b_i t_i^{n_i})$ such that

$$\begin{aligned} a_i &= 1, & b_i &= 0 & \text{if } \varphi_i \in B_1, \\ a_i &= 0, & b_i &= 1 & \text{if } \varphi_i \in B_2, \\ a_i &= 1, & b_i &= 1 & \text{if } \varphi_i \in B_3 \text{ and} \\ a_i &= 1, & b_i &= \theta_j & \text{if } \varphi_i \in B_j \text{ for } j > 3. \end{aligned}$$

The next lemma is an immediate consequence of the above explanation and gives us the subgroups of \mathcal{A} that preserve the tangent cone of the curve \mathcal{C} and the properties of the block form.

In what follows, we consider the subgroups of \mathcal{A} :

$$\begin{aligned} \mathcal{H} &= \{(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}; \rho_i = u_i t_i, \sigma = (\alpha X, \delta Y) \text{ and } u_i, \alpha, \delta \in \mathbb{C}^*\}, \\ \mathcal{H}' &= \{(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}; \rho_i = u_i t_i, \sigma = (\alpha X, \alpha Y) \text{ and } u_i, \alpha \in \mathbb{C}^*\}, \\ \tilde{\mathcal{A}}_1 &= \{(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}; j^1 \rho_i = t_i, j^1 \sigma = (X + \beta Y, Y) \text{ and } \beta \in \mathbb{C}\}, \\ \mathcal{A}_1 &= \{(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{A}; j^1 \rho_i = t_i \text{ and } j^1 \sigma = (X, Y)\}. \end{aligned}$$

LEMMA 3.2. *The subgroups of \mathcal{A} that preserve the elements of \mathcal{P} as in Definition 3.1, according to number s of blocks, are as follows.*

s	Subgroup of \mathcal{A}
1	$\mathcal{H} \circ \tilde{\mathcal{A}}_1$
2	$\mathcal{H} \circ \mathcal{A}_1$
≥ 3	$\mathcal{H}' \circ \mathcal{A}_1$

Proof. The description follows from the correspondence between the Möbius transformation $T(\theta) = (\gamma + \delta\theta)/(\alpha + \beta\theta)$ and $j^1\sigma = (\alpha X + \beta Y, \gamma X + \delta Y)$ with $\sigma \in \text{Diff}(\mathbb{C}^2, 0)$.

For $s = 1$, all branches have tangent line with slope 0 and to preserve it we must consider $\gamma = 0$. Thus, the corresponding element in \mathcal{A} is a composition of elements of \mathcal{H} and $\tilde{\mathcal{A}}_1$.

If $\varphi \in \mathcal{P}$ has just two blocks with $\theta_1 = 0$ and $\theta_2 = \infty$, then we must consider $T(\theta) = \delta\theta/\alpha$ to preserve the slopes of the tangent lines and consequently the corresponding elements in \mathcal{A} are obtained as a composition of elements of \mathcal{H} and \mathcal{A}_1 .

On the other hand, if $s \geq 3$ the branches of the first three blocks have tangent line with slope 0, ∞ , and 1, respectively, and the only Möbius transformation that preserve them is the identity ($\gamma = \beta = 0$ and $\alpha = \delta$). Consequently, the associated diffeomorphisms belong to $\mathcal{H}' \circ \mathcal{A}_1$. \square

The standard strategy to solve the analytic equivalence problem is to find a representative element in each orbit (a normal form), for instance, with a few number of parameters, in such a way that it is manageable to decide whether two normal forms are equivalent or not.

Note that the action of any subgroup of $\mathcal{S} \times \mathcal{H}$ on an element $\varphi \in \mathcal{P}$ does not introduce or eliminate terms in the multigerms components. Thus, to obtain equivalent multigerms in the block form with short parameterizations it is natural to consider the \mathcal{A}_1 -action or $\tilde{\mathcal{A}}_1$ -action according to the number s of blocks described in the previous lemma.

In a more general situation, concerning a Lie group action \mathfrak{G} on an affine space \mathbb{A} , the *complete transversal theorem* (CTT; see [BKP97]) provides us a method for obtaining representative elements in the orbit $\mathfrak{G}(v)$ of $v \in \mathbb{A}$, if we have the description of the tangent space $T\mathfrak{G}(v)$ to orbit $\mathfrak{G}(v)$ at v . More precisely, the following version will be useful for our purposes.

THEOREM 3.3 (Complete transversal theorem). *Let \mathfrak{G} be a Lie group acting on an affine space \mathbb{A} with underlying vector space V and let W be a subspace of V . Suppose that $T\mathfrak{G}(v+w) = T\mathfrak{G}(v)$ for all $v \in \mathbb{A}$ and $w \in W$. If $v \in \mathbb{A}$ and $W \subseteq T\mathfrak{G}(v)$, then $\mathfrak{G}(v+w) = \mathfrak{G}(v)$, for all $w \in W$, that is, the \mathfrak{G} -orbits of $v+w$ and v coincide.*

Let us apply the above theorem in our context. To simplify the notation let \mathcal{G} be one of the groups \mathcal{A}_1 or $\tilde{\mathcal{A}}_1$ and $I = \{1, \dots, r\}$.

The description of $T\mathcal{A}_1(\varphi)$ for $\varphi \in \mathcal{P}$ is classical (see, for instance, [Wal81]) and we do not need any effort to extend it to the group $\tilde{\mathcal{A}}_1$. Explicitly, we get

$$TG(\varphi) = \left\{ \begin{bmatrix} x'_1 \cdot \epsilon_1 + \varphi_1^*(\eta_1) & \dots & x'_r \cdot \epsilon_r + \varphi_r^*(\eta_1) \\ y'_1 \cdot \epsilon_1 + \varphi_1^*(\eta_2) & \dots & y'_r \cdot \epsilon_r + \varphi_r^*(\eta_2) \end{bmatrix}; \epsilon_i \in \langle t_i^2 \rangle, i \in I \right\}, \tag{3}$$

where $\begin{cases} \eta_1, \eta_2 \in \mathcal{M}^2, & \text{if } \mathcal{G} = \mathcal{A}_1, \\ \eta_1 \in \langle X^2, Y \rangle, \eta_2 \in \mathcal{M}^2, & \text{if } \mathcal{G} = \tilde{\mathcal{A}}_1. \end{cases}$

Denoting by B^k the set of k -jets of elements of a set B , we know that \mathcal{G}^k is a unipotent Lie group that acts on \mathcal{P}^k in the natural way:

$$j^k(\rho_1, \dots, \rho_r, \sigma) \cdot j^k\varphi := [j^k(\sigma \circ \varphi_1 \circ \rho_1^{-1}), \dots, j^k(\sigma \circ \varphi_r \circ \rho_r^{-1})].$$

The corresponding tangent space $T\mathcal{G}^k(j^k\varphi)$ is the set of k -jets of elements in $TG(\varphi)$.

If H^k denotes the \mathbb{C} -vector space

$$\left\{ \begin{bmatrix} c_1 t_1^k & \dots & c_r t_r^k \\ d_1 t_1^k & \dots & d_r t_r^k \end{bmatrix}; c_i, d_i \in \mathbb{C} \text{ with } c_i = d_i = 0 \text{ if } k \leq n_i, i \in I \right\} \tag{4}$$

then, we can prove, similarly as in Proposition 2 of [HHR15], that $T\mathcal{G}^k(j^k\varphi + \zeta) = T\mathcal{G}^k(j^k\varphi)$ for any $j^k\varphi \in \mathcal{P}^k$ and $\zeta \in H^k$.

In this way, $\mathfrak{G} = \mathcal{G}^k$, $\mathbb{A} = \mathcal{P}^k$, and $W \subseteq H^k \cap T\mathcal{G}^k(j^k\varphi)$ fulfill the hypothesis of Theorem 3.3, that is, for any $w \in W$ we have that $j^k\varphi + w$ is \mathcal{G}^k -equivalent to $j^k\varphi$ and, consequently, there exists $\psi \in \mathcal{P}$ which is \mathcal{A} -equivalent to φ with $j^k\psi = j^k\varphi + w$.

In the next two sections we recognize elements in $H^k \cap T\mathcal{G}^k(j^k\varphi)$ using \mathcal{G} -invariants.

4. Puiseux block form

In this section, we present a convenient parameterization for the components of the multigerms $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ preserving the block form, as in Definition 3.1. In addition, we exhibit some elimination criteria for terms of φ .

For simplicity, we denote by E_{ji} for $j = 1, 2, i \in I$, an element of the canonical basis of the complex matrices of order $2 \times r$ and \mathcal{G} denotes one of the groups \mathcal{A}_1 or $\tilde{\mathcal{A}}_1$.

Given a multigerms $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ with $\varphi_i(t_i) = (x_i(t_i), y_i(t_i))$, an element

$$\begin{bmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{bmatrix} := \sum_{i=1}^r (u_i E_{1i} + v_i E_{2i})$$

with $u_i, v_i \in \mathbb{C}\{t_i\}$ belongs to $TG(\varphi)$ if and only if there exist ϵ_i for $i = 1, \dots, r$ and $\eta = (\eta_1, \eta_2)$ satisfying (3) such that

$$\begin{cases} u_i = x'_i \cdot \epsilon_i + \varphi_i^*(\eta_1), \\ v_i = y'_i \cdot \epsilon_i + \varphi_i^*(\eta_2). \end{cases} \tag{5}$$

As we remarked in §3 describing $H^k \cap T\mathcal{G}^k(j^k\varphi)$ we can apply Theorem 3.3 in order to obtain representative elements in a same \mathcal{G}^k -orbit.

Note that our analysis, up to this point, has taken into account a few \mathcal{A} -invariants, just the multiplicities of the branches and the behavior of the tangent cone. It is time to consider finer invariants as defined in §2.

Let $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ be a multigerms of plane curve in block form, with local ring \mathcal{O} . The (fractional) ideal $\varphi^*(\mathcal{M}^2)$ of \mathcal{O} plays a relevant role and provides us an elimination criterion for terms in φ .

LEMMA 4.1. *With the standard notation, if $k = \gamma + \sum_{\substack{j \in I \\ j \neq i}} \nu_i(f_j) \in \nu_i(\mathcal{M}^2)$ with $\gamma \in \Gamma_i$, then there exists $\psi = [\psi_1, \dots, \psi_r]$ \mathcal{G} -equivalent to φ such that $j^k \varphi_j = j^k \psi_j$ for all $j \in I \setminus \{i\}$ and $j^k \psi_i = j^{k-1} \varphi_i$.*

Proof. By hypothesis, there exists an element $h \in \mathbb{C}\{X, Y\}$ with $h \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j \in \mathcal{M}^2$ and $j^k(\varphi_i^*(h \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j)) = t_i^k$.

Taking $\epsilon_j = 0$ for $j \in I$ and $\eta_l = \alpha_l h \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j$ for $l = 1, 2$ in (5), we get

$$\alpha_1 t_i^k E_{1i} + \alpha_2 t_i^k E_{2i} = \begin{bmatrix} 0 & \cdots & \alpha_1 t_i^k & \cdots & 0 \\ 0 & \cdots & \alpha_2 t_i^k & \cdots & 0 \end{bmatrix} \in H^k \cap T\mathcal{G}^k(j^k \varphi).$$

By Theorem 3.3, we obtain $\psi \in \mathcal{G}(\varphi)$ with $j^k \psi = j^k \varphi + \alpha_1 t_i^k E_{1i} + \alpha_2 t_i^k E_{2i}$ and for a convenient choice of $\alpha_1, \alpha_2 \in \mathbb{C}$ we have $j^k \psi_j = j^k \varphi_j$ for all $j \in I \setminus \{i\}$ and $j^k \psi_i = j^{k-1} \varphi_i$. \square

Remark 4.2. Note that the description of tangent vector $\alpha_1 t_i^k E_{1i} + \alpha_2 t_i^k E_{2i} \in H^k \cap T\mathcal{G}^k(j^k \varphi)$, $k = \gamma + \sum_{\substack{j \in I \\ j \neq i}} \nu_i(f_j) \in \nu_i(\mathcal{M}^2)$ with $\gamma \in \Gamma_i$ in the previous lemma gives us a clue to obtain elements $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{G}$ such that $(\rho_1, \dots, \rho_r, \sigma) \cdot \varphi = \psi$ with $j^k \psi_j = j^k \varphi_j$ for all $j \in I \setminus \{i\}$ and $j^k \psi_i = j^{k-1} \varphi_i$. In fact, with the above notation, it is sufficient to consider $\rho_j(t_j) = t_j$ for $j \in I$ and $\sigma(X, Y) = (X - \alpha_1 \cdot h \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j, Y - \alpha_2 \cdot h \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j)$ with appropriate $\alpha_1, \alpha_2 \in \mathbb{C}$.

As a consequence of the above lemma we obtain an estimate for the finite determinacy of a given $\varphi \in \mathcal{P}$.

Denoting $(\mathcal{O} : \overline{\mathcal{O}})^c$ the contraction of $(\mathcal{O} : \overline{\mathcal{O}})$ by φ^* we have the following result.

PROPOSITION 4.3. *If d_i is the conductor of the set $\nu_i(\mathcal{M}^2 \cap (\mathcal{O} : \overline{\mathcal{O}})^c)$ for $i = 1, \dots, r$, then $\varphi \underset{\mathcal{G}}{\sim} [j^{d_1-1} \varphi_1, \dots, j^{d_r-1} \varphi_r]$.*

Proof. For each $i \in I$ we put $\varphi_i - j^{d_i-1} \varphi_i = (t_i^{d_i} u_{i1}, t_i^{d_i} u_{i2})$ with $u_{il} \in \mathbb{C}\{t_i\}$ for $l = 1, 2$. As d_i is the conductor of $\nu_i(\mathcal{M}^2 \cap (\mathcal{O} : \overline{\mathcal{O}})^c)$ there exists $h_{il} \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j \in \mathcal{M}^2 \cap (\mathcal{O} : \overline{\mathcal{O}})^c$ such that $\varphi_i^*(h_{il} \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j) = t_i^{d_i} u_{il}$ for $l = 1, 2$.

Now, taking $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{G}$, where $\rho_j(t_j) = t_j$ for $j \in I$ and

$$\sigma(X, Y) = \left(X - \sum_{i \in I} h_{i1} \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j, Y - \sum_{i \in I} h_{i2} \cdot \prod_{\substack{j \in I \\ j \neq i}} f_j \right),$$

we obtain $\psi = (\rho_1, \dots, \rho_r, \sigma) \cdot \varphi = [j^{d_1-1} \varphi_1, \dots, j^{d_r-1} \varphi_r]$. \square

Since $\kappa = (\kappa_1, \dots, \kappa_r)$ is the conductor of the semiring Γ , the above-mentioned integer d_i satisfies $d_i \geq \kappa_i$ with equality if $(\mathcal{O} : \overline{\mathcal{O}})^c \subseteq \mathcal{M}^2$. More precisely, we have the following.

PROPOSITION 4.4. *Let $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ be a block form multigerms and s the number of blocks. For each $i \in I$, the integer d_i in the last proposition satisfies*

$$d_i = \begin{cases} \kappa_i + 2 & \text{if } r = 1 \text{ and } n_i \leq 2; \\ \kappa_i + 1 & \text{if } r = s = 2 \text{ and } n_1 = n_2 = 1; \\ \kappa_i & \text{otherwise.} \end{cases}$$

Proof. Let us consider the cases $r = 1$, $r = 2$, and $r \geq 3$ separately.

Case $r = 1$. Let φ_1 be a parameterization of a branch with multiplicity n_1 . As we remarked in §2, κ_1 is the Milnor number μ_1 .

If $n_1 = 1$, then $\mu_1 = 0$ and $\mathcal{O} = \overline{\mathcal{O}}$. Thus, $d_1 = \min \nu_i(\mathcal{M}^2 \cap (\mathcal{O} : \overline{\mathcal{O}})^c) = 2 = \mu_1 + 2 = \kappa_1 + 2$. For $n_1 > 1$ we have $\overline{\mathcal{O}} \neq \mathcal{O}$ and $(\mathcal{O} : \overline{\mathcal{O}})^c \subseteq \mathcal{M}$. Setting $m = \min(\Gamma \setminus n_1\mathbb{N})$, we have

$$\nu_1(\mathcal{M} \setminus \mathcal{M}^2) = \left\{ n_1, 2n_1, \dots, \left\lceil \frac{m}{n_1} \right\rceil n_1, m \right\}.$$

If $n_1 = 2$, then $\Gamma = \langle 2, m \rangle = 2\mathbb{N} + m\mathbb{N}$ and $\mu_1 = m - 1$, which implies that $\nu_1((\mathcal{O} : \overline{\mathcal{O}})^c) = \{\gamma \in \mathbb{N}; \gamma \geq m - 1\}$. Thus, the conductor d_1 of $\nu_1(\mathcal{M}^2 \cap (\mathcal{O} : \overline{\mathcal{O}})^c)$ satisfies $d_1 = m + 1 = \mu_1 + 2$. On the other hand, for $n_1 > 2$ we have $\mu_1 > m$, $(\mathcal{O} : \overline{\mathcal{O}})^c \subseteq \mathcal{M}^2$ and consequently $d_1 = \kappa_1$.

Case $r = 2$. Consider a plane curve \mathcal{C} given by $f = f_1 \cdot f_2$ with corresponding multigerms $[\varphi_1, \varphi_2]$.

As $(\mathcal{O} : \overline{\mathcal{O}}) = (\mathcal{O}_1 : \overline{\mathcal{O}}_1) \cdot \varphi_1^*(f_2) \oplus (\mathcal{O}_2 : \overline{\mathcal{O}}_2) \cdot \varphi_2^*(f_1)$ and $f_1, f_2 \in \mathcal{M}$, it follows that if $n_i > 1$ for some $i \in \{1, 2\}$, then $(\mathcal{O} : \overline{\mathcal{O}})^c \subseteq \mathcal{M}^2$ and $d_i = \kappa_i$.

Let us consider $n_1 = n_2 = 1$. In this case we have $\kappa = (\kappa_1, \kappa_2) = (\nu_1(f_2), \nu_2(f_1))$. If $s = 1$, then $\varphi_1, \varphi_2 \in B_1$ and $\nu_1(f_2) = \nu_2(f_1) > 1$. Hence, $(\mathcal{O} : \overline{\mathcal{O}})^c \subseteq \mathcal{M}^2$ and $d_i = \kappa_i$ for $i = 1, 2$. If $s = 2$, then $\varphi_1 \in B_1$ and $\varphi_2 \in B_2$. Thus, $\nu_1(f_2) = \nu_2(f_1) = 1$ and $d_i = \min \nu_i(\mathcal{M}^2 \cap (\mathcal{O} : \overline{\mathcal{O}})^c) = 2 = \kappa_i + 1$ for $i = 1, 2$.

Case $r \geq 3$. Now, we consider a plane curve \mathcal{C} given by $f = f_1 \cdots f_r$ with $f_i \in \mathcal{M}$.

As $(\mathcal{O} : \overline{\mathcal{O}}) = \bigoplus_{i \in I} (\mathcal{O}_i : \overline{\mathcal{O}}_i) \cdot \prod_{\substack{j \in I \\ j \neq i}} \varphi_j^*(f_j)$, we get $(\mathcal{O} : \overline{\mathcal{O}})^c \subseteq \mathcal{M}^2$ and $d_i = \kappa_i$ for $i \in I$. \square

Another direct application of Theorem 3.3 is obtained as a consequence of the following lemma.

LEMMA 4.5. *Let $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ be a block form multigerms associated to a curve \mathcal{C} and $\theta_i = b_i/a_i \in \overline{\mathbb{C}}$ the slope of the tangent line to \mathcal{C}_i at the origin. For each $i \in I$, if $k > n_i$, then*

$$t_i^k(a_i E_{1i} + b_i E_{2i}) \in H^k \cap TG^k(j^k \varphi).$$

Proof. As $\varphi_i(t_i) = (a_i t_i^{n_i} + \dots, b_i t_i^{n_i} + \dots)$ and $k > n_i$, taking $\epsilon_i = (1/n_i)t_i^{k-n_i+1} \in \langle t_i^2 \rangle$, $\epsilon_l = 0$ for $l \in I \setminus \{i\}$, and $\eta_1 = \eta_2 = 0$ in (5), we obtain

$$t_i^k(a_i E_{1i} + b_i E_{2i}) = \begin{bmatrix} 0 & \dots & a_i t_i^k & \dots & 0 \\ 0 & \dots & b_i t_i^k & \dots & 0 \end{bmatrix} \in H^k \cap TG^k(j^k \varphi). \quad \square$$

By the above lemma and Theorem 3.3, for every $k > n_i$ there exists a block form multigerms $\psi \in \mathcal{G}(\varphi)$ with $j^k \psi = j^k \varphi + \alpha t_i^k(a_i E_{1i} + b_i E_{2i})$ for any $\alpha \in \mathbb{C}$. Thus, if $\varphi_i = (x_i, y_i) \in B_1$, that is, $a_i = 1$ and $b_i = 0$, then we can choose α in such way that $j^k \psi_i = (j^{k-1} x_i, j^k y_i)$, that is, we eliminate the k -order term of the first component of φ_i . On the other hand, for $\varphi_i \in B_2$ there exists α such that $j^k \psi_i = (j^k x_i, j^{k-1} y_i)$. Finally, for $\varphi_i \in B_j$ with $j > 2$, as $a_i \cdot b_i \neq 0$, we can choose α to obtain $j^k \psi_i = (j^{k-1} x_i, j^k y_i)$ or $j^k \psi_i = (j^k x_i, j^{k-1} y_i)$.

Similarly to Remark 4.2, as consequence of the previous lemma, we can exhibit an element of \mathcal{G} to perform the proposed action. In fact, with the above notation, it is sufficient to consider $\rho_i^{-1}(t_i) = t_i + \alpha \epsilon_i$ with $\alpha \in \mathbb{C}$, $\epsilon_i \in \langle t_i^2 \rangle$ given as in the proof of Lemma 4.5, $\epsilon_j = 0$ for $j \in I \setminus \{i\}$ and $\sigma(X, Y) = (X, Y)$.

Thus, we recover the classical Puiseux expansion for plane curves.

PROPOSITION 4.6. Any multigerms $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ is \mathcal{G} -equivalent to a block form multigerms $\psi = [\psi_1, \dots, \psi_r]$ with

$$\psi_i(t_i) = \begin{cases} \left(t_i^{n_i}, \theta_l t_i^{n_i} + \sum_{j>n_i} a_{ij} t_i^j \right) & \text{for } \psi_i \in B_l, l \neq 2; \\ \left(\sum_{j>n_i} a_{ij} t_i^j, t_i^{n_i} \right) & \text{for } \psi_i \in B_2; \end{cases} \tag{6}$$

with $\theta_1 = 0, \theta_3 = 1$, and $\theta_k \neq \theta_l$ for $1 \leq k, l \leq r$, and $k \neq l$.

Proof. We may suppose that φ is given by a block form multigerms. For each $i \in I$ and for every $k > n_i$ we consider elements in \mathcal{G} to apply the process described after Lemma 4.5. If $\varphi_i = (x_i, y_i) \in B_l$ with $l \geq 3$, then we choose $j^k \psi_i = (j^{k-1} x_i, j^k y_i)$.

Repeating this process for all k satisfying $n_i < k < d_i$ we obtain $j^{d_i} \psi_i = (t_i^{n_i}, j^{d_i} y_i)$ for $\psi_i \notin B_2$ and $j^{d_i} \psi_i = (j^{d_i} x_i, t_i^{n_i})$ for $\psi_i \in B_2$. The result follows from Proposition 4.3. \square

We call the multigerms as highlighted in the above result the *Puiseux block form* of φ .

From the above results, we obtain the well-known representative in the analytical class for the particular cases of Proposition 4.4. More explicitly, if φ is a Puiseux block form, then:

$$\begin{aligned} \varphi &\underset{\mathcal{A}}{\sim} (t, 0) \quad \text{if } r = n_1 = 1; \\ \varphi &\underset{\mathcal{A}}{\sim} (t^2, t^m) \quad \text{if } r = 1, n_1 = 2 \text{ and } m = \min(\Gamma \setminus 2\mathbb{N}); \\ \varphi &\underset{\mathcal{A}}{\sim} [\psi_1, \psi_2]; \quad \psi_1 = (t_1, 0), \quad \psi_2 = (0, t_2) \quad \text{if } r = s = 2, n_1 = n_2 = 1. \end{aligned}$$

Note that if $r = 1, n_1 = 2$, and $m = \min(\Gamma \setminus 2\mathbb{N})$ the previous results provide us with the result $\varphi \underset{\mathcal{A}}{\sim} (t^2, at^m)$ with $a \neq 0$. However, considering $\rho(t) = t$ and $\sigma(X, Y) = (X, a^{-1}Y)$ we obtain the above equivalence.

In what follows, we consider the non-exceptional cases and consequently we can suppose that the finite determinacy order for each φ_i of a Puiseux block form multigerms is κ_i .

5. \mathcal{G} -Normal forms and analytic equivalence

In § 4, we apply the CTT (Theorem 3.3) to reduce a multigerms $\varphi \in \mathcal{P}$ to its Puiseux block form taking into account only a few topological invariants. Although such forms are a shorter way of presenting a multigerms, it is not easy to decide whether two Puiseux block form multigerms correspond to analytically equivalent plane curves.

Our goal in this section is to partition a topological class into a finite number of strata so that in each stratum we have a constant \mathcal{A} -invariant and every element belonging to it admits a particular Puiseux block form, that we call a *normal form*, which will allow us to distinguish them from the analytical equivalence viewpoint.

As before, according to Lemma 3.2, \mathcal{G} denotes \mathcal{A}_1 if the Puiseux block form of $\varphi \in \mathcal{P}$ has at least two blocks and $\tilde{\mathcal{A}}_1$ if there is a single block.

Recall that as a consequence of Lemma 4.5, $\{\sum_{\varphi_i \notin B_2} \mathbb{C} \cdot t_i^k E_{1i} + \sum_{\varphi_i \in B_2} \mathbb{C} \cdot t_i^k E_{2i}\} \cap H^k \subseteq T\mathcal{G}^k(j^k \varphi)$ for every k . Thus, in order to obtain a Puiseux block form multigerms with a smaller number of terms, it is sufficient to describe elements in the set $D^k := \{\sum_{\varphi_i \notin B_2} \mathbb{C} \cdot t_i^k E_{2i} + \sum_{\varphi_i \in B_2} \mathbb{C} \cdot t_i^k E_{1i}\} \cap H^k$ that belong to $T\mathcal{G}^k(j^k \varphi)$. In fact, if $\zeta \in D^k \cap T\mathcal{G}^k(j^k \varphi)$, then φ is \mathcal{G} -equivalent to a Puiseux block form multigerms such that its k -jet is $j^k \varphi + \zeta$, that is, there

exists a multigerms ψ , \mathcal{A} -equivalent to φ , with $j^{k-1}\psi = j^{k-1}\varphi$ and, by adjusting coefficients in ζ , the element $j^k\psi$ has a smaller number of non-zero terms than $j^k\varphi$.

The key to obtain our normal form is the connection between $D^k \cap TG^k(j^k\varphi)$ and the set of values of a particular fractional ideal as we describe in the following.

Considering $\varphi \in \mathcal{P}$ and

$$\Omega_{\mathcal{A}_1} = \{\eta_2 dX - \eta_1 dY; \eta_1, \eta_2 \in \mathcal{M}^2\}, \quad \Omega_{\tilde{\mathcal{A}}_1} = \{\eta_2 dX - \eta_1 dY; \eta_1 \in \langle X^2, Y \rangle \text{ and } \eta_2 \in \mathcal{M}^2\},$$

we get the fractional ideal

$$\mathcal{I}_{\mathcal{G}} = \left\{ \left(\frac{\varphi_1^*(\omega)}{n_1 t_1^{n_1}}, \dots, \frac{\varphi_r^*(\omega)}{n_r t_r^{n_r}} \right); \omega \in \Omega_{\mathcal{G}} \right\}$$

of \mathcal{O} according to the group \mathcal{G} .

We define $\Lambda_{\mathcal{G}} := \{\nu(w); w \in \mathcal{I}_{\mathcal{G}}\} \subset \Lambda - n \subseteq \overline{\mathbb{N}}^r$, where Λ is the \mathcal{A} -invariant defined in (2) and $n := (n_1, \dots, n_r)$ with n_i the multiplicity of the branch φ_i .

Note that the conductor $\varrho' = (\varrho'_1, \dots, \varrho'_r)$ of $\Lambda_{\mathcal{G}}$ satisfies $\varrho'_i \leq \varrho_i - n_i < \kappa_i$, where $(\varrho_1, \dots, \varrho_r)$ and $(\kappa_1, \dots, \kappa_r)$ are the conductors of Λ and Γ , respectively. As a consequence of Remark 2.2, for each fixed value semiring Γ there are only finitely many \mathcal{A} -invariants $\Lambda_{\mathcal{G}}$ that can be computed by the algorithm presented in Theorem 16 of [CH20b].

In what follows, we consider the set $\Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$ of all Puiseux block form multigerms in \mathcal{P} , as given in (6), with semiring Γ and $\Lambda_{\mathcal{G}}$ fixed.

PROPOSITION 5.1. *Given $\varphi = [\varphi_1, \dots, \varphi_r] \in \Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$ and $k \in \mathbb{N}^*$ there exists $w = (w_1, \dots, w_r) \in \mathcal{I}_{\mathcal{G}}$ with $j^k w_i = \alpha_i t_i^k$, $\alpha \in \mathbb{C}$ if and only if $\sum_{\varphi_i \notin B_2} \alpha_i t_i^k E_{2i} - \sum_{\varphi_i \in B_2} \alpha_i t_i^k E_{1i} \in TG^k(j^k\varphi)$.*

Proof. Let $w = (w_1, \dots, w_r) \in \mathcal{I}_{\mathcal{G}}$ with $j^k w_i = j^k(\varphi_i^*(\omega)/n_i t_i^{n_i}) = \alpha_i t_i^k$ for $i \in I$ and some $\omega = \eta_2 dX - \eta_1 dY \in \Omega_{\mathcal{G}}$.

As $\epsilon_i = -(\varphi_i^*(\eta_1)/n_i t_i^{n_i-1}) \in \langle t_i^2 \rangle$ for $\varphi_i \notin B_2$, $\epsilon_i = -(\varphi_i^*(\eta_2)/n_i t_i^{n_i-1}) \in \langle t_i^2 \rangle$ for $\varphi_i \in B_2$, η_1 and η_2 satisfying (5) we have the desired element in $TG^k(j^k\varphi)$.

Conversely, if $\sum_{\varphi_i \notin B_2} \alpha_i t_i^k E_{2i} - \sum_{\varphi_i \in B_2} \alpha_i t_i^k E_{1i} \in TG^k(j^k\varphi)$, then there exist $\epsilon_i \in \langle t_i^2 \rangle$, $i \in I$, η_1 , and η_2 as given in (3) such that

$$\begin{cases} x'_i \cdot \epsilon_i + \varphi_i^*(\eta_1) = P_i \\ y'_i \cdot \epsilon_i + \varphi_i^*(\eta_2) = \alpha_i t_i^k + Q_i \end{cases} \quad \text{if } \varphi_i \notin B_2, \quad \begin{cases} x'_i \cdot \epsilon_i + \varphi_i^*(\eta_1) = -\alpha_i t_i^k + P_i \\ y'_i \cdot \epsilon_i + \varphi_i^*(\eta_2) = Q_i \end{cases} \quad \text{if } \varphi_i \in B_2, \quad (7)$$

with $\nu_i(P_i) > k$ and $\nu_i(Q_i) > k$ for $i \in I$. Considering $\omega = \eta_2 dX - \eta_1 dY \in \Omega_{\mathcal{G}}$ we provide the element $w \in \mathcal{I}_{\mathcal{G}}$ satisfying the proposition. \square

Note that if $k \leq n_i$, then, by the definition of $\mathcal{I}_{\mathcal{G}}$, α_i should be zero in (7).

In order to simplify the notation, from now on given a non-empty subset $J \subseteq I$, we denote the J -fiber of $\underline{k} = (k, \dots, k) \in \mathbb{N}^r$ with respect to $\Lambda_{\mathcal{G}}$ by $F_J(\underline{k})$. The following corollary is immediate.

COROLLARY 5.2. *We have $F_J(\underline{k}) \neq \emptyset$ if and only if there exist $\alpha_i \neq 0$ for every $i \in J$ and $\alpha_i = 0$ for every $i \in I \setminus J$ such that $\sum_{\varphi_i \notin B_2} \alpha_i t_i^k E_{2i} - \sum_{\varphi_i \in B_2} \alpha_i t_i^k E_{1i} \in TG^k(j^k\varphi)$.*

The next proposition describes subspaces of $D^k \cap TG^k(j^k\varphi)$ in order to choose terms which can be eliminated by \mathcal{G} -action. Such subspaces are related to the fibers of \underline{k} with respect to $\Lambda_{\mathcal{G}}$.

PROPOSITION 5.3. *Given $k \in \mathbb{N}^*$ there exists a d -dimensional subspace in $D^k \cap TG^k(j^k\varphi)$ if and only if there exists $L = \{l_1, \dots, l_d\} \subseteq I$ satisfying the following condition:*

$$\text{for each } l \in L, \text{ there exists } J_l \subseteq I \text{ with } l \in J_l \setminus \bigcup_{i \in L \setminus \{l\}} J_i, \text{ and } F_{J_l}(\underline{k}) \neq \emptyset. \quad (\star)$$

Proof. We show that there exist $v_1, \dots, v_d \in D^k \cap T\mathcal{G}^k(j^k\varphi)$ linearly independent if and only if there exists $L = \{l_1, \dots, l_d\} \subseteq I$ satisfying (\star) .

Let $\{v_i := \sum_{\varphi_j \notin B_2} \alpha_{ij} t_j^k E_{2j} + \sum_{\varphi_j \in B_2} \alpha_{ij} t_j^k E_{1j} \in D^k \cap T\mathcal{G}^k(j^k\varphi); 1 \leq i \leq d\}$ be a set of linearly independent vectors. Then there exists $L = \{l_1, \dots, l_d\} \subseteq I$ with $[\alpha_{il_j}]_{1 \leq i, j \leq d} \in GL_d(\mathbb{C})$.

In this way, for each $1 \leq i \leq d$ there exists $w_i = \sum_{\varphi_j \notin B_2} \beta_{ij} t_j^k E_{2j} + \sum_{\varphi_j \in B_2} \beta_{ij} t_j^k E_{1j} \in \text{span}\{v_1, \dots, v_d\} \subseteq D^k \cap T\mathcal{G}^k(j^k\varphi)$ with $\beta_{il_i} = 1$ and $\beta_{ij} = 0$ for $j \in L \setminus \{l_i\}$. Taking the set $J_{l_i} = \{j \in I; \beta_{ij} \neq 0\}$, the previous corollary ensures the condition (\star) .

On the other hand, suppose that there is $L = \{l_1, \dots, l_d\} \subseteq I$ satisfying the condition (\star) . Thus, by Corollary 5.2, for each $i = 1, \dots, d$, there exists $v_i = \sum_{\varphi_j \notin B_2} \alpha_{ij} t_j^k E_{2j} + \sum_{\varphi_j \in B_2} \alpha_{ij} t_j^k E_{1j} \in D^k \cap T\mathcal{G}^k(j^k\varphi)$ such that $\alpha_{ij} \neq 0$ if $j \in J_{l_i}$ and $\alpha_{ij} = 0$ for $j \in I \setminus J_{l_i}$. As the matrix $[\alpha_{il_j}]_{1 \leq i, j \leq d}$ is invertible, it follows that v_1, \dots, v_d are linearly independent. \square

Remark 5.4. Note that, by the previous result, if $L = \{l_1, \dots, l_d\} \subseteq I$ satisfies the condition (\star) , then there exist $v_1, \dots, v_d \in D^k \cap T\mathcal{G}^k(j^k\varphi)$ linearly independent with $v_i = \sum_{\varphi_j \notin B_2} \alpha_{ij} t_j^k E_{2j} + \sum_{\varphi_j \in B_2} \alpha_{ij} t_j^k E_{1j}$ such that $\alpha_{ij} \neq 0$ if $j \in J_{l_i}$ and $\alpha_{ij} = 0$ for $j \in I \setminus J_{l_i}$. In particular, for any $b_{l_1}, \dots, b_{l_d} \in \mathbb{C}$ there exists $v \in \text{span}\{v_1, \dots, v_d\}$ with $v = \sum_{\varphi_j \notin B_2} b_j t_j^k E_{2j} + \sum_{\varphi_j \in B_2} b_j t_j^k E_{1j} \in D^k \cap T\mathcal{G}^k(j^k\varphi)$ or, equivalently, by Proposition 5.1, there exists $w = (w_1, \dots, w_r) \in \mathcal{I}_{\mathcal{G}}$ with $j^k w_j = b_j t_j^k$ for any $b_{l_1}, \dots, b_{l_d} \in \mathbb{C}$.

As an immediate consequence we can establish an elimination criterion based on the set $\Lambda_{\mathcal{G}}$.

COROLLARY 5.5. *Let $k \in \mathbb{N}^*$, $\varphi \in \mathcal{P}$ as in (6), and $L = \{l_1, \dots, l_d\} \subseteq I$ satisfying the condition (\star) . Then there exists $\psi \in \mathcal{P}$ such that $\psi \underset{\mathcal{G}}{\sim} \varphi$, $j^{k-1}\psi = j^{k-1}\varphi$, and $j^k \psi_l = j^{k-1} \varphi_l$ for all $l \in L$.*

Proof. Since φ is given as in (6) we have that $j^k\varphi = j^{k-1}\varphi + w$, with $w = \sum_{\varphi_j \notin B_2} \beta_j t_j^k E_{2j} + \sum_{\varphi_j \in B_2} \beta_j t_j^k E_{1j} \in D^k$. By the above remark, there exists $v = \sum_{\varphi_j \notin B_2} b_j t_j^k E_{2j} + \sum_{\varphi_j \in B_2} b_j t_j^k E_{1j} \in D^k \cap T\mathcal{G}^k(j^k\varphi)$ with $b_l = -\beta_l$ for all $l \in L$. By Theorem 3.3, it follows that φ is \mathcal{G} -equivalent to some $\psi \in \mathcal{P}$ with $j^k\psi = j^k\varphi + v = j^{k-1}\varphi + w + v$, that is, $j^{k-1}\psi = j^{k-1}\varphi$ and $j^k \psi_l = j^{k-1} \varphi_l$ for all $l \in L$. \square

Given $k > 0$ the parameter elimination method established in the above result depends upon a set L satisfying the condition (\star) in Proposition 5.3. The aim is to eliminate terms of order k from the greater number of branches prioritizing the first ones. We can formalize this by defining the injective map $S : \{L; L \subseteq I\} \rightarrow \mathbb{N}^r$ given by $S(L) = (z_1, \dots, z_r)$, where

$$z_i = \begin{cases} 1 & \text{if } i \in L, \\ 0 & \text{if } i \notin L. \end{cases}$$

DEFINITION 5.6. For each fixed k , we consider the set of indices $L_k \subseteq \{1, \dots, r\}$ such that $S(L_k) = \max_{GLex} \{S(L); L \text{ satisfying } (\star) \text{ in Proposition 5.3}\}$, where \max_{GLex} means the maximum with respect to graded lexicographic order.

Remark 5.7. Let ϱ be the conductor of $\Lambda_{\mathcal{G}}$ and $\underline{k} \in \varrho + \mathbb{N}^r$. We have that $F_{\{i\}}(\underline{k}) \neq \emptyset$ for all $i \in I$, consequently $L_k = I$.

If \underline{k} is an absolute maximal of $\Lambda_{\mathcal{G}}$, then $F_I(\underline{k}) \neq \emptyset$ and $F_J(\underline{k}) = \emptyset$ for all $J \subsetneq I$. In this case, $L_k = \{1\}$.

If \underline{k} is a relative maximal of $\Lambda_{\mathcal{G}}$, then $F_{\{i\}}(\underline{k}) = \emptyset$ for all $i \in I$ and $F_J(\underline{k}) \neq \emptyset$ for all $J \subseteq I$ with $\#J > 1$. Note that for any $i_0 \in I$, the set $L = I \setminus \{i_0\}$ and $J_l = \{l, i_0\}$ for all $l \in L$ satisfy the condition (\star) . Then, in this case we have $L_k = I \setminus \{r\}$.

Now we can establish one of the main results of this paper:

THEOREM 5.8 (\mathcal{G} -normal form). *With the above notation any $\varphi = [\varphi_1, \dots, \varphi_r] \in \Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$ is \mathcal{G} -equivalent to $\psi = [\psi_1, \dots, \psi_r] \in \mathcal{P}$ such that*

$$\psi_i(t_i) = \begin{cases} \left(t_i^{n_i}, \theta_l t_i^{n_i} + \sum_{\substack{j > n_i \\ i \notin L_j}} a_{ij} t_i^j \right) & \text{for } \psi_i \in B_l, l \neq 2; \\ \left(\sum_{\substack{j > n_i \\ i \notin L_j}} a_{ij} t_i^j, t_i^{n_i} \right) & \text{for } \psi_i \in B_2, \end{cases} \tag{8}$$

where L_j is given in Definition 5.6, $\theta_1 = 0$, $\theta_3 = 1$, and $\theta_k \neq \theta_l$ for $1 \leq k, l \leq r$ with $k \neq l$. Moreover, if $\psi' = [\psi'_1, \dots, \psi'_r] \in \mathcal{P}$ is another multigerms \mathcal{G} -equivalent to φ as (8), then $\psi' = \psi$, that is, the \mathcal{G} -normal form of φ is unique.

Proof. By Proposition 4.6 we can consider $\varphi \in \mathcal{P}$ in its Puiseux block form. For all $k > 0$ taking L_k as defined previously and applying the Corollary 5.5 we obtain a \mathcal{G} -normal form ψ as (8). Remark that by Proposition 4.3 it is sufficient to consider $k < \max\{d_i; i \in I\}$ where d_i is given in Proposition 4.4.

Now suppose $\psi' = [\psi'_1, \dots, \psi'_r] \in \mathcal{P}$ as (8), \mathcal{G} -equivalent to φ and $\psi' \neq \psi$. Let us consider $k = \min\{l; j^l \psi \neq j^l \psi'\}$.

By Proposition 5.3 there exists $\{v_i \in D^k \cap T\mathcal{G}^k(j^k \psi); l_i \in L_k\}$ linearly independent (possibly empty) and subsets $J_{l_i} \subseteq I$ satisfying the condition (\star) with $v_i = \sum_{\psi_j \notin B_2} \alpha_{ij} t_j^k E_{2j} + \sum_{\psi_j \in B_2} \alpha_{ij} t_j^k E_{1j}$ such that $\alpha_{ij} \neq 0$ if $j \in J_{l_i}$ and $\alpha_{ij} = 0$ for $j \in I \setminus J_{l_i}$.

Let $\mathcal{F}^k \subseteq \mathcal{G}^k$ be the subgroup that leave invariant the affine space $N^k = j^{k-1} \psi + V$ where

$$V = \left\{ \sum_{\psi_j \notin B_2} \beta_j t_j^k E_{2j} + \sum_{\psi_j \in B_2} \beta_j t_j^k E_{1j} \in D^k; \beta_j \in \mathbb{C} \text{ and } \beta_l = 0 \text{ for all } l \in L_k \right\}.$$

As the ground field is \mathbb{C} and \mathcal{F}^k is unipotent it follows that \mathcal{F}^k is a connected group. Thus, the orbit $\mathcal{F}^k(j^k \psi) \subseteq N^k \subseteq \mathcal{G}^k(j^k \psi)$ is connected and it contains $j^k \psi$ and $j^k \psi'$. Let $w := \sum_{\psi_j \notin B_2} \gamma_j t_j^k E_{2j} + \sum_{\psi_j \in B_2} \gamma_j t_j^k E_{1j} \in T\mathcal{F}^k(j^k \psi) \subseteq V \subseteq T\mathcal{G}^k(j^k \psi)$ be a non-zero vector and use the notation $J' = \{j; \gamma_j \neq 0\} \subseteq I$. Remark that $l \notin J'$ for all $l \in L_k$.

Choosing $j_0 \in J' =: J'_{j_0}$, taking the element $w_l := v_l - (\alpha_{lj_0} / \gamma_{j_0}) w \in T\mathcal{G}^k(j^k \psi)$ and setting $J'_l = \{j \in J_l \setminus \{j_0\}; \alpha_{lj} \gamma_{j_0} \neq \alpha_{lj_0} \gamma_j\}$ for all $l \in L_k$, it follows by Corollary 5.2 that $F_{J'_l}(\underline{k}) \neq \emptyset$. In addition, for each $l \in L' := L_k \cup \{j_0\}$ we have that $l \in J'_l \setminus \bigcup_{i \in L' \setminus \{l\}} J'_i$ and $F_{J'_l}(\underline{k}) \neq \emptyset$. In this way, $L' \supseteq L_k$ satisfies the condition (\star) in Proposition 5.3, which contradicts the maximality of the set L_k .

Hence, $\psi = \psi'$ and the \mathcal{G} -normal form of φ , as (8), is unique. □

Remark 5.9. Note that the parameters a_{ij} in the \mathcal{G} -normal form (8) are not necessarily free because, in order to guarantee that the multigerms is in $\Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$, some algebraic conditions must be imposed on the coefficients. The conditions imposed to have the semiring Γ are easily determined by the non-zero coefficients of the terms with characteristic exponents and some conditions to get the intersection multiplicity of each pair of branches. On the other hand, the algebraic conditions related with $\Lambda_{\mathcal{G}}$ can be obtained applying the algorithm presented in [CH20b].

As we remarked in § 2, the analytic equivalence of plane curves with r irreducible components can be translated by the $\mathcal{S} \times \mathcal{A}$ -action on \mathcal{P} . Any element in \mathcal{P} is $\mathcal{S} \times \mathcal{A}$ -equivalent to a multigerms

given in a Puiseux block form and the \mathcal{A} -action on such elements splits according to the action of subgroups given in Lemma 3.2, namely $\mathcal{H} \circ \tilde{\mathcal{A}}_1$, $\mathcal{H} \circ \mathcal{A}_1$, or $\mathcal{H}' \circ \mathcal{A}_1$.

In Theorem 5.8 we presented \mathcal{G} -normal forms for Puiseux block form multigerms with $\mathcal{G} = \tilde{\mathcal{A}}_1$ or $\mathcal{G} = \mathcal{A}_1$. In this way, to present \mathcal{A} -normal forms of such multigerms we have to consider the action of \mathcal{H} or \mathcal{H}' , that we call *homothety actions*, on the respective \mathcal{G} -normal forms. Recall that an action by a subgroup of \mathcal{H} does not introduce nor eliminate terms in a multigerm.

Let us consider $\varphi = [\varphi_1, \dots, \varphi_r] \in \mathcal{P}$ given as (8) and the parameter vector

$$p = (a_{1,n_1+1}, \dots, a_{1,k_0}, a_{2,n_2+1}, \dots, a_{2,k_0}, \dots, a_{r,n_r+1}, \dots, a_{r,k_0}), \tag{9}$$

where $k_0 = \min\{z \in \mathbb{N}; z \in \varrho + \mathbb{N}^r\}$ (see Remark 5.7).

Given an element $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{H}$ where $\rho_i(t_i) = u_i t_i$ and $\sigma(X, Y) = (\alpha X, \gamma Y)$ with $\alpha, \gamma, u_i \in \mathbb{C}^*$ for $i \in I$, in order to keep $\psi = \sigma \circ \varphi \circ \rho^{-1}$ as in (8), that is, $j^{n_i}(\varphi_i) = j^{n_i}(\psi_i)$ for all $i \in I$, we must have

$$u_i = \alpha^{-(1/n_i)} \quad \text{if } \varphi_i \notin B_2, \quad u_i = \gamma^{-(1/n_i)} \quad \text{if } \varphi_i \in B_2$$

and $\theta_l \gamma \alpha^{-1} = \theta_l$ for all $\varphi_i \in B_l$ with $l \neq 2$. Remark that $\alpha = \gamma$ for $(\rho_1, \dots, \rho_r, \sigma) \in \mathcal{H}' \subset \mathcal{H}$, that is, if we have $s > 2$ blocks.

The corresponding parameter vector of ψ is $(b_{1,n_1+1}, \dots, b_{1,k_0}, \dots, b_{r,n_r+1}, \dots, b_{r,k_0})$ with

$$b_{ij} = \begin{cases} \gamma \alpha^{-(j/n_i)} a_{ij} & \text{if } \psi_i \in B_1 \\ \alpha \gamma^{-(j/n_i)} a_{ij} & \text{if } \psi_i \in B_2 \end{cases} \quad \text{for } s \leq 2, \quad \text{and} \quad b_{ij} = \alpha^{(n_i-j)/n_i} a_{ij} \quad \text{for all } i \in I, \text{ if } s \geq 3. \tag{10}$$

Note that the parameter vector p is null if and only if

$$\varphi_1 = \psi_1 = (t_1, 0), \quad \varphi_2 = \psi_2 = (0, t_2), \quad \varphi_i = \psi_i = (t_i, \theta_i t_i) \quad \text{for all } 3 \leq i \leq r, \tag{11}$$

with $\theta_i \neq \theta_j$ for $i \neq j$.

On the other hand, if p is a non-zero vector and $a_{i_0 j_0}$ is the first non-zero coordinate, then ψ_i (and φ_i) for all $i < i_0$ is given as (11) and we can choose $\alpha \in \mathbb{C}^*$ in (10) corresponding to $a_{i_0 j_0}$ in order to normalize the corresponding coefficient in ψ , that is, to obtain $b_{i_0 j_0} = 1$.

Eventually (10) allow us to normalize one more coefficient, but for this purpose a finer analysis would be necessary. However, the above description it is sufficient to present the following theorem that gives a solution (up to \mathcal{S} -action) for the analytical equivalence problem for plane curves with r irreducible components.

THEOREM 5.10 (\mathcal{A} -normal form). *Let $\varphi = [\varphi_1, \dots, \varphi_r] \in \Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$ be a multigerm associated to the curve \mathcal{C} . Then φ is \mathcal{A} -equivalent to $\psi = [\psi_1, \dots, \psi_r]$ as (8) with parameter vector $p = (b_{1,n_1+1}, \dots, b_{r,k_0})$ such that the following hold.*

- If $p = (0, \dots, 0)$, then $\psi_1 = (t_1, 0)$, $\psi_2 = (0, t_2)$, $\psi_i = (t_i, \theta_i t_i)$ for all $3 \leq i \leq r$.
- If $p \neq (0, \dots, 0)$, then $b_{i_0 j_0} = 1$ is the first non-zero coordinate of p . In addition, $\psi' \in \Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$ as (8) with parameter vector $p' = (b'_{1,n_1+1}, \dots, b'_{r,k_0})$ and first non-zero coordinate $b'_{i_0 j_0} = 1$ is \mathcal{A} -equivalent to ψ if and only if we have the following conditions according to the number s of blocks.

$s = 1$	There exists $c \in \mathbb{C}^*$ such that $b_{ij} = c^{(j n_{i_0} - j_0 n_i) / n_i n_{i_0}} b'_{ij}$ for all coordinates of p and p' .
$s = 2$	If $\psi_{i_0} \in B_1$ there exists $c \in \mathbb{C}^*$ such that: $b_{ij} = c^{(j n_{i_0} - j_0 n_i) / n_i n_{i_0}} b'_{ij}$ for all coordinates of p and p' with $\psi_i \in B_1$; $b_{ij} = c^{(j j_0 - n_{i_0} n_i) / n_i n_{i_0}} b'_{ij}$ for all coordinates of p and p' with $\psi_i \in B_2$.
$s \geq 3$	If $\psi_{i_0} \in B_2$ there exists $c \in \mathbb{C}^*$ and $b_{ij} = c^{(j n_{i_0} - j_0 n_i) / n_i n_{i_0}} b'_{ij}$ for all entries of p and p' .
$s \geq 3$	There exists $c \in \mathbb{C}^*$, $c^{(j_0 - n_{i_0}) / n_{i_0}} = 1$, and $b_{ij} = c^{(j - n_i) / n_i} b'_{ij}$ for all entries of p and p' .

Proof. By Theorem 5.8 we obtain a unique \mathcal{G} -normal form for φ as (8) and, as described previously, if the parameter vector is non-zero, then the \mathcal{G} -normal form is \mathcal{H} -equivalent (or \mathcal{H}' -equivalent if $s \geq 3$) to $\psi = [\psi_1, \dots, \psi_r]$ as (8) with parameter vector $p = (b_{1,n_1+1}, \dots, b_{r,k_0})$ such that its first non-zero coordinate is $b_{i_0 j_0} = 1$.

Now it is sufficient to describe the parameter vector $p' = (b'_{1,n_1+1}, \dots, b'_{r,k_0})$ of a multigerms $\psi' \in \Sigma_{\Gamma, \Lambda_{\mathcal{G}}}$ as (8) with the first non-zero coordinate $b'_{i_0 j_0} = 1$ in the orbit $\mathcal{H}(\psi)$ if $s \leq 2$ or in $\mathcal{H}'(\psi)$ if $s \geq 3$. This will be done using (10).

If $s = 1$, by (10) we have $\gamma \alpha^{-(j_0/n_{i_0})} = 1$ and $\gamma \alpha^{-(j/n_i)} b_{ij} = b'_{ij}$. Consequently, $\gamma = \alpha^{j_0/n_{i_0}}$ and $b_{ij} = \alpha^{(jn_{i_0} - j_0 n_i)/n_i n_{i_0}} b'_{ij}$ for all coordinates of p and p' .

For multigerms with two blocks it follows, by (10), that $b'_{ij} = \gamma \alpha^{-(j/n_i)} b_{ij}$ if $\psi_i \in B_1$ and $b'_{ij} = \alpha \gamma^{-(j/n_i)} b_{ij}$ if $\psi_i \in B_2$. If $\psi_{i_0} \in B_1$, then $\gamma = \alpha^{j_0/n_{i_0}}$ and $b_{ij} = \alpha^{(jn_{i_0} - j_0 n_i)/n_i n_{i_0}} b'_{ij}$ for coordinates of p and p' with $\psi_i \in B_1$ and $b_{ij} = \alpha^{(jj_0 - n_{i_0} n_i)/n_i n_{i_0}} b'_{ij}$ for all coordinates of p and p' with $\psi_i \in B_2$. If $\psi_{i_0} \in B_2$, then $\alpha = \gamma^{j_0/n_{i_0}}$ and $b_{ij} = \gamma^{(jn_{i_0} - j_0 n_i)/n_i n_{i_0}} b'_{ij}$ for all entries of p and p' .

Finally, for $s \geq 3$ we have $\alpha^{(n_{i_0} - j_0)/n_{i_0}} = 1$ and $\alpha^{(n_i - j)/n_i} b_{ij} = b'_{ij}$ for all $i \in I$ and $n_i + 1 \leq j \leq k_0$. □

The above theorem provides a method to decide whether two plane curves are analytically equivalent. In fact, considering an associated multigerms to a plane curve \mathcal{C} with r irreducible components we compute their \mathcal{A} -invariants Γ and $\Lambda_{\mathcal{G}}$ using the appropriate group \mathcal{G} and the algorithms in [CH20b] for instance. Applying the results in this section, we obtain an \mathcal{A} -normal form $\varphi = [\varphi_1, \dots, \varphi_r]$ for the multigerms of \mathcal{C} .

Given another plane curve \mathcal{C}' with multigerms associated $\psi = [\psi_1, \dots, \psi_r]$, if the \mathcal{A} -invariants Γ^π and $\Lambda_{\mathcal{G}}^\pi$ of $\pi(\psi)$ are distinct to the corresponding ones of φ for every $\pi \in \mathcal{S}$, then \mathcal{C} and \mathcal{C}' are not analytically equivalent. If it is not the case, we consider the subgroup \mathcal{R} of \mathcal{S} such $\Gamma^\pi = \Gamma$ and $\Lambda_{\mathcal{G}}^\pi = \Lambda_{\mathcal{G}}$ for all $\pi \in \mathcal{R}$. The curves \mathcal{C} and \mathcal{C}' are analytically equivalent if and only if there exists $\pi \in \mathcal{R}$ such that φ and an \mathcal{A} -normal form for $\pi(\psi)$ satisfy the corresponding condition in Theorem 5.10.

6. Particular cases

In this final section, we recover some known results regarding the analytic classification of plane curves using Theorems 5.8 and 5.10 and we illustrate how we can apply them in particular topological classes.

6.1 Irreducible case and bigerms

For the irreducible case, i.e. $s = r = 1$, we must consider $\mathcal{G} = \tilde{\mathcal{A}}_1$ and the normal form described in Theorem 5.8 is rewritten as $\varphi = (t^n, \sum_{\substack{j>n \\ L_j=0}} c_j t^j)$.

Note that $L_j = \emptyset$ if and only if $F_{\{1\}}(j) = \emptyset$, that is, $j \notin \Lambda_{\mathcal{G}}$.

If $n = 1$, then $\Lambda_{\mathcal{G}} = \overline{\mathbb{N}} \setminus \{0, 1\}$ and φ is \mathcal{A} -equivalent to $(t, 0)$.

Let us suppose that $n \geq 2$ and $\Gamma \cap \overline{\mathbb{N}} = \langle n, v_1, \dots, v_g \rangle$. Thus, $\varphi = (t^n, c_{v_1} t^{v_1} + \sum_{j \notin \Lambda_{\mathcal{G}}} c_j t^j)$ with $c_{v_1} \neq 0$ and, by Theorem 5.10, we have $\varphi \underset{\mathcal{A}}{\sim} (t^n, t^{v_1} + \sum_{j \notin \Lambda_{\mathcal{G}}} b_j t^j)$.

If $j \in \Lambda_{\mathcal{G}}$ for every $j > v_1$, then $\varphi \underset{\mathcal{A}}{\sim} (t^n, t^{v_1})$. By [Zar66], this is equivalent to claim that $\Lambda \setminus \Gamma = \emptyset$.

If there exists $j \notin \Lambda_G$ for some $j > v_1$, then setting $\lambda = \min\{j > v_1; j \notin \Lambda_G\}$ we have

$$\varphi \underset{\mathcal{A}}{\sim} \left(t^n, t^{v_1} + b_\lambda t^\lambda + \sum_{\lambda < j \notin \Lambda_G} b_j t^j \right) \tag{12}$$

with $b_\lambda \neq 0$ and $\lambda + n = \nu(\varphi^*(\omega_0))$ where $\omega_0 := v_1 Y dX - nX dY$.

Remark 6.1. Note that $\{j > \lambda; j \notin \Lambda_G\} = \{j > \lambda; j \notin \Lambda - n\}$. In fact, by definition we have $\Lambda_G \subset \Lambda - n$. On the other hand, given $\omega = (\alpha_0 + \alpha_1 X + \alpha_2 Y + h_1) dX + (\alpha_3 + \alpha_4 X + h_2) dY \in \Omega^1$ with $\alpha_i \in \mathbb{C}$ and $\omega_1 = h_1 dX + h_2 dY \in \Omega_G$ in order to have $\nu(\varphi^*(\omega)) - n > \lambda$ we must have $\alpha_0 = \alpha_1 = \alpha_3 = 0$ and $n\alpha_2 + v_1\alpha_4 = 0$. In this way, $\omega = (\alpha_2/v_1)\omega_0 + \omega_1$, as $\nu(\varphi^*(\omega_0)) - n = \lambda \neq \nu(\varphi^*(\omega_1)) - n \in \Lambda_G$ (by (12)), we have $\nu(\varphi^*(\omega)) - n = \nu(\varphi^*(\omega_1)) - n \in \Lambda_G$.

By the above remark, if $\Lambda \setminus \Gamma \neq \emptyset$, that is, there exists $\lambda = \min\{j > v_1; j \notin \Lambda_G\}$, then $\varphi \underset{\mathcal{A}}{\sim} (t^n, t^{v_1} + b_\lambda t^\lambda + \sum_{\lambda < j \notin \Lambda - n} b_j t^j)$ with $b_\lambda \neq 0$.

Taking $c \in \mathbb{C}^*$ with $c^{\lambda - v_1} = b_\lambda$, $\rho(t) = ct$ and $\sigma(X, Y) = (c^n X, c^{v_1} Y)$ we obtain $(\rho, \sigma) \cdot \varphi = (t^n, t^{v_1} + t^\lambda + \sum_{\lambda < j \notin \Lambda - n} a_j t^j)$, that is, the normal form presented in Theorem 2.1 in [HH11].

For bigerms, we have two distinct cases according to the number $s \in \{1, 2\}$ of Puiseux blocks. By Theorem 5.8 we get

$$\varphi_1 = \left(t_1^{n_1}, \sum_{\substack{j > n_1 \\ 1 \notin L_j}} a_{1j} t_1^j \right) \quad \text{and} \quad \varphi_2 = \begin{cases} \left(t_2^{n_2}, \sum_{\substack{j > n_2 \\ 2 \notin L_j}} a_{2j} t_2^j \right) & \text{if } s = 1; \\ \left(\sum_{\substack{j > n_2 \\ 2 \notin L_j}} a_{2j} t_2^j, t_2^{n_2} \right) & \text{if } s = 2. \end{cases}$$

Note that $1 \notin L_j$ if and only if $F_{\{1\}}(\underline{j}) = F_{\{1,2\}}(\underline{j}) = \emptyset$, that is, $F_{\{1\}}(\underline{j}) = \emptyset$ and \underline{j} is not a maximal point of Λ_G . On the other hand, $2 \notin L_j$ means that $F_{\{2\}}(\underline{j}) = \emptyset$.

Hence, for curves with two branches, Theorem 5.8 recovers Theorem 6 of [HHR15].

6.2 Singular ordinary point

Let us consider a plane curve \mathcal{C} with a singular ordinary point at the origin of multiplicity $r > 1$, that is, \mathcal{C} has r irreducible regular components given by $f_i \in \mathcal{M} \setminus \mathcal{M}^2$ such that the slopes θ_i of their tangent lines satisfy $\theta_i \neq \theta_j$ for all $i, j \in I, i \neq j$.

Applying Proposition 3.9 of [CH20a] for this case we obtain that the value semiring Γ is minimally generated by

$$v_i = \nu(f_i) = (1, \dots, 1, \infty, 1, \dots, 1); \quad i \in I$$

where the symbol ‘ ∞ ’ occupies the i th coordinate. In particular, any element $\gamma \in \Gamma$ can be written as (1), i.e. $\gamma = \inf \{ \sum_{i \in I} a_{1i} v_i, \dots, \sum_{i \in I} a_{ri} v_i \}$. If $a_{li} \neq 0$ for all $i \in I$, then $\sum_{i \in I} a_{li} v_i = (\infty, \dots, \infty)$ which is irrelevant for the computation of γ . Thus, we can assume $a_{li} = 0$ for every $l \in I$ and some $i \in I$ which depends on l .

Note that $\mu_i = 0$ and $\nu_j(f_i) = 1$ for all $i, j \in I$ with $i \neq j$, then the conductor of Γ is $\kappa = (r - 1, \dots, r - 1)$. In particular, $F_J^\Gamma(\underline{k}) \neq \emptyset$ for every $k \geq r - 1$ and all $\emptyset \neq J \subseteq I$.

By Propositions 4.3 and 4.4, for $r \in \{1, 2, 3\}$ the plane curve \mathcal{C} admits a multigerms φ that is \mathcal{A} -equivalent to $[(t_1, 0)]$, $[(t_1, 0), (0, t_2)]$, or $[(t_1, 0), (0, t_2), (t_3, t_3)]$, respectively.

In what follows we consider $r \geq 4$. Note that the appropriated subgroup to apply Theorem 5.8 is $\mathcal{G} = \mathcal{A}_1$.

Using the generators of the value semiring Γ we can describe the non-empty fibers $F_J^\Gamma(\underline{k})$ for $0 \leq k < r$.

LEMMA 6.2. For a singular ordinary point with multiplicity r the J -fiber of $\underline{k} \in \mathbb{N}^r$ of Γ is non-empty if and only if $\sharp J \geq r - k$. Moreover, for any $J \subseteq I$ with $\sharp J \geq r - k$ we have

$$F_J^\Gamma(\underline{k}) = \{\underline{k} + \alpha; \alpha = (\alpha_1, \dots, \alpha_r) \in \overline{\mathbb{N}}^r \text{ with } \alpha_j = 0 \text{ if } j \in J \text{ and } \alpha_j \neq 0 \text{ if } j \in I \setminus J\}.$$

Proof. As the conductor of Γ is $\kappa = (r - 1, \dots, r - 1)$ it is sufficient to describe the fiber $F_J^\Gamma(\underline{k})$ for every $\underline{k} = (k, \dots, k) \in \mathbb{N}^r$, $0 \leq k \leq r - 1$ and every $J \subseteq I$ with $J \neq \emptyset$.

First, note that if $F_J^\Gamma(\underline{k}) \neq \emptyset$, then $\sharp J \geq r - k$. In fact, let $\gamma \in F_J^\Gamma(\underline{k})$. Since $\gamma \in \Gamma$, there exists $h \in \mathbb{C}\{X, Y\}$, namely of multiplicity n_h , such that $\gamma = \nu(h)$. If $\sharp J < r - k$, as $n_i = 1$, $\theta_i \neq \theta_l$ for all $i, l \in I$ with $i \neq l$, we must have $m_h \leq n_h \leq \nu_j(h) = k < r - \sharp J$, for all $j \in J$, where m_h is the number of distinct tangent lines of h . However, in this way, there exists $i \in I \setminus J$ such that f_i and h do not share the same tangent line. Thus, $\nu_i(h) = n_h \leq k$ and, therefore, $\gamma \notin F_J^\Gamma(\underline{k})$ which is a contradiction.

Now we describe $F_J^\Gamma(\underline{k})$ with $\sharp J \geq r - k$.

For $0 \leq k \leq r - 1$ we take any $J_0 \subseteq I$ with $\sharp J_0 = r - k (\geq 1)$ and let us consider the sets $I \setminus J_0 = \{i_1, \dots, i_k\}$ and $\{\widehat{i_1}, \dots, \widehat{i_m}, \dots, i_k\} := \{i_1, \dots, i_k\} \setminus \{i_m\}$.

Fixing $i_0 \in J_0$ we set

$$\zeta_0 = \sum_{l \in \{i_1, \dots, i_k\}} \nu(f_l), \quad \zeta_m = \sum_{l \in \{i_1, \dots, \widehat{i_m}, \dots, i_k\}} \nu(f_l) + (1 + \alpha_m)\nu(f_{i_0}) \in \overline{\mathbb{N}}^r,$$

with $1 \leq m \leq k$ and $\alpha_1, \dots, \alpha_k \in \mathbb{N}$. Note that the coordinates of ζ_0 and ζ_m , respectively, satisfy

$$\zeta_{0j} = \begin{cases} k & \text{if } j \in J_0, \\ \infty & \text{if } j \in I \setminus J_0, \end{cases} \quad \zeta_{mj} = \begin{cases} \infty & \text{if } j \in \{i_0, i_1, \dots, \widehat{i_m}, \dots, i_k\}, \\ k + \alpha_m & \text{if } j \notin \{i_0, i_1, \dots, \widehat{i_m}, \dots, i_k\}. \end{cases}$$

Thus, $\gamma = (\gamma_1, \dots, \gamma_r) := \inf \{\zeta_0, \zeta_1, \dots, \zeta_k\} \in \Gamma$ is such that $\gamma_j = k$ if $j \in J_0$, $\gamma_j = k + \alpha_j$ with $\alpha_j \in \mathbb{N}$ for $j \in I \setminus J_0$.

Hence, the above construction ensures that for any $J \subseteq I$ with $\sharp J \geq r - k$ we have

$$F_J^\Gamma(\underline{k}) = \{\underline{k} + \alpha; \alpha = (\alpha_1, \dots, \alpha_r) \in \overline{\mathbb{N}}^r \text{ with } \alpha_j = 0 \text{ if } j \in J \text{ and } \alpha_j \neq 0 \text{ if } j \in I \setminus J\}. \quad \square$$

Remark that $\nu(\mathcal{M}^2) - \underline{1} \subseteq \Lambda_{\mathcal{A}_1}$ which implies that its corresponding fibers satisfy $\emptyset \neq F_J^\Gamma(\underline{k} + \underline{1}) - \underline{1} \subseteq F_J^\Gamma(\underline{k})$ for all $k \geq 2$ and $\sharp J \geq r - (k + 1)$.

Taking $L = \{1, \dots, \min\{k + 2, r\}\}$ and $J_i = \{i, k + 3, \dots, r\}$ for $i \in L$ in Proposition 5.3, then Corollary 5.5 allows us to eliminate the term corresponding to t_i^k of the component φ_i for all $i = 1, \dots, \min\{k + 2, r\}$ preserving the analytic class of \mathcal{C} . With this approach we obtain Theorem 1.1 in [GP11].

PROPOSITION 6.3. Any curve with a singular ordinary point with r branches is analytically equivalent to a curve defined by a multigerms $\psi = [\psi_1, \dots, \psi_r]$ with

$$\psi_1 = (t_1, 0), \quad \psi_2 = (0, t_2), \quad \psi_i = \left(t_i, \theta_i t_i + \sum_{j=2}^{i-3} a_{ij} t_i^j \right), \quad i = 3, \dots, r, \quad (13)$$

where $\theta_3 = 1$ and $\theta_i \neq \theta_l$ for $i \neq l$.

For $r \geq 4$, Granger [Gra79] considers the generic parameter vector to obtain \mathcal{A} -normal forms and to compute the dimension of the moduli space. The same result was obtained with other methods by Genzmer and Paul [GP11]. From now on, we consider the generality hypothesis and we apply our techniques with the purpose to compute the dimension of the moduli space.

The generality of the coefficients give us a symmetry on the coordinates of $\Lambda_{\mathcal{A}_1}$. In this way, for $k \geq 2$ the set $L_k \subseteq I$ in Theorem 5.8 is $L_k = \{1, 2, \dots, e(k)\}$ for some $4 \leq e(k) \leq r$. By Remark 5.4, $e(k)$ is the maximum integer such that for any $b_{ik} \in \mathbb{C}$ with $1 \leq i \leq e(k)$ there exists $w \in \mathcal{I}_{\mathcal{A}_1}$ with $j^k w = (b_{1k}t_1^k, \dots, b_{rk}t_r^k)$.

Recall that $w = (t_1^{-1}\varphi_1^*(\omega), \dots, t_r^{-1}\varphi_r^*(\omega)) \in \mathcal{I}_{\mathcal{A}_1}$ and $t_i^{-1}\varphi_i^*(\omega) = \varphi_i^*(p) \cdot x'_i + \varphi_i^*(q) \cdot y'_i \in \mathbb{C}\{t_i\}$ for $i \in I$ where $\omega = p dX + q dY \in \Omega^1$ and $p, q \in \langle X, Y \rangle^2$.

In what follows, we use the notation

$$\omega_h = \left(\sum_{m=0}^h \alpha_{h,m} X^{h-m} Y^m \right) dX + \left(\sum_{m=0}^h \beta_{h,m} X^{h-m} Y^m \right) dY \in \Omega^1 \tag{14}$$

with $\alpha_{h,m}, \beta_{h,m} \in \mathbb{C}$. Note that given $\varphi = [\varphi_1, \dots, \varphi_r]$ as (13) and $\omega \in \Omega^1$, if we want to obtain $\nu_i(\omega) = k$ and consequently to determine $e(k)$, it is sufficient to consider $w = \sum_{h=2}^k w_h$.

LEMMA 6.4. *With the above notation, if $r \geq 4$, we have $e(2) = 4$ and $e(k) = \min\{2k + 1, r\}$ for every $k \geq 3$.*

Proof. Considering a generic multigerms φ given as in (13) if $\omega = \omega_2$, that is $k = 2$, then

$$\begin{aligned} j^2(t_1^{-1}\varphi_1^*(\omega)) &= \alpha_{2,0}t_1^2, & j^2(t_2^{-1}\varphi_2^*(\omega)) &= \beta_{2,2}t_2^2, \\ j^2(t_i^{-1}\varphi_i^*(\omega)) &= (\alpha_{2,0} + (\alpha_{2,1} + \beta_{2,0})\theta_i + (\alpha_{2,2} + \beta_{2,1})\theta_i^2 + \beta_{2,2}\theta_i^3)t_i^2, & \text{for all } 3 \leq i \leq r. \end{aligned}$$

Now, it is immediate to verify that the maximum integer $e(2)$ such that the system $j^2(t_i^{-1}\varphi_i^*(\omega)) = b_{i2}t_i^2$ admits solution for any $b_{i2} \in \mathbb{C}$ with $1 \leq i \leq e(2)$ is precisely $e(2) = 4$.

We show that for any $3 \leq l \leq [(r - 1)/2]$ if $j^l(t_i^{-1}\varphi_i^*(\omega_{l-1} + \omega_l)) = 0$ for every $i \in I$, then $w_{l-1} = 0$.

Note that for w_h as (14) with $h \geq 2$, we get

$$\begin{aligned} j^{h+1}(t_1^{-1}\varphi_1^*(\omega_h)) &= \alpha_{h,0}t_1^h, \\ j^{h+1}(t_2^{-1}\varphi_2^*(\omega_h)) &= \beta_{h,h}t_2^h, \\ j^{h+1}(t_i^{-1}\varphi_i^*(\omega_h)) &= \left(\alpha_{h,0} + \sum_{m=1}^h (\alpha_{h,m} + \beta_{h,m-1})\theta_i^m + \beta_{h,h}\theta_i^{h+1} \right) t_i^h \\ &\quad + \left(\sum_{m=0}^{h-1} ((m+1)\alpha_{h,m+1} + (m+2)\beta_{h,m})\theta_i^m + (h+2)\beta_{h,h}\theta_i^h \right) a_{i2}t_i^{h+1} \end{aligned} \tag{15}$$

for $3 \leq i \leq r$.

The condition $j^l(t_i^{-1}\varphi_i^*(\omega_{l-1} + \omega_l)) = 0$ for every $i \in I$ implies that $\alpha_{l-1,0} = \alpha_{l,0} = \beta_{l-1,l-1} = \beta_{l,l} = 0$ and the coefficient of t^{l-1} in $t_i^{-1}\varphi_i^*(\omega_{l-1} + \omega_l)$ vanishes for every $3 \leq i \leq r$, that is,

$$\begin{pmatrix} \theta_3 & \theta_3^2 & \cdots & \theta_3^l \\ \vdots & \vdots & & \vdots \\ \theta_r & \theta_r^2 & \cdots & \theta_r^l \end{pmatrix} \cdot \begin{pmatrix} \alpha_{l-1,1} + \beta_{l-1,0} \\ \vdots \\ \alpha_{l-1,l-1} + \beta_{l-1,l-2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, $\alpha_{l-1,m} = -\beta_{l-1,m-1}$ for $1 \leq m \leq l - 1$. In addition, as the coefficient of t^l in $t_i^{-1}\varphi_i^*(\omega_{l-1} + \omega_l)$ is null, by (15) we must have

$$\begin{pmatrix} a_{32} & \theta_3 a_{32} & \cdots & \theta_3^{l-2} a_{32} & \theta_3 & \cdots & \theta_3^l \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{r2} & \theta_3 a_{32} & \cdots & \theta_r^{l-2} a_{r2} & \theta_r & \cdots & \theta_r^l \end{pmatrix} \cdot \begin{pmatrix} \beta_{l-1,0} \\ \vdots \\ \beta_{l-1,l-2} \\ \alpha_{l,1} + \beta_{l,0} \\ \vdots \\ \alpha_{l,l} + \beta_{l,l-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{16}$$

As we suppose φ generic, the coefficient matrix has maximal rank $\min\{2l - 1, r - 2\}$ and we get the solution $\beta_{l-1,m} = 0$ for $0 \leq m \leq l - 2$ and $\alpha_{l,m} = -\beta_{l,m-1}$ for $1 \leq m \leq l$. Consequently, $w_{l-1} = 0$.

In order to obtain $j^k(t_i^{-1}\varphi_i^*(\sum_{h=2}^k \omega_h)) = b_{ik}t_i^k$ it is necessary that $j^l(t_i^{-1}\varphi_i^*(\omega_h)) = 0$ for each $h = 2, \dots, k - 1$ and for all $l = h, \dots, k - 1$. From this and the above discussion, it follows by induction that $\omega_h = 0$ for $2 \leq h \leq k - 2$. Thus, it is sufficient to consider $\omega = \omega_{k-1} + \omega_k$.

By a similar computation as in (15) using $l = k$ we get the system

$$\begin{cases} \alpha_{k,0} = b_{1k}, \\ \beta_{k,k} = b_{2k}, \\ \alpha_{k,0} + \sum_{m=0}^{k-2} \beta_{k-1,m} \theta_i^m a_{i2} + \sum_{m=1}^k (\alpha_{k,m} + \beta_{k,m-1}) \theta_i^m + \beta_{k,k} \theta_i^{k+1} = b_{ik} \quad \text{for } 3 \leq i \leq r. \end{cases}$$

Substituting the two first solutions into the other equations, we obtain a system with the same coefficient matrix as (16). Thus, generically we can solve at most $e(k) = 2 + \min\{2k - 1, r - 2\} = \min\{2k + 1, r\}$ such equations. \square

As a consequence, we recover the analytical normal form for regular transversal branches in the generic case, as given in [Gra79, Proposition 2] and [GP11].

PROPOSITION 6.5. *Generically, any plane curve with a singular ordinary point and $r \geq 4$ is analytically equivalent to a curve defined by a multigerms $\psi = [\psi_1, \dots, \psi_r]$ with*

$$\begin{aligned} \psi_1 &= (t_1, 0), & \psi_2 &= (0, t_2), & \psi_3 &= (t_3, t_3), & \psi_4 &= (t_4, \theta_4 t_4), \\ \psi_i &= \left(t_i, \theta_i t_i + a_{i2} t_i^2 + \sum_{k=3}^{[(i-2)/2]} a_{ik} t_i^k \right), & & & & & & 5 \leq i \leq r. \end{aligned}$$

Proof. The previous lemma gives us $L_2 = \{1, 2, 3, 4\}$ and $L_k = \{1, \dots, \min\{2k + 1, r\}\}$ for $k \geq 3$. Thus, by Theorem 5.8, we have the above normal form for $1 \leq i \leq 4$ and

$$\psi_i = \left(t_i, \theta_i t_i + a_{i2} t_i^2 + \sum_{\substack{k \geq 3 \\ i \notin L_k}} a_{ik} t_i^k \right) \quad \text{for } 5 \leq i \leq r.$$

The result follows since that $i \notin L_k$ is equivalent to $k \leq [(i - 2)/2]$ for $k \geq 3$. \square

Note that for $r \leq 3$ the moduli space corresponding is a single point and for $r = 4$ is a one-dimensional space.

For $r \geq 5$ the parameter vector associated to the normal form ψ presented in the previous result is $(a_{5,2}, a_{6,2}, a_{6,3}, \dots, a_{r,2}, \dots, a_{r,[(r-2)/2]})$. Generically, $a_{5,2} \neq 0$ and by Theorem 5.10 we get a multigerms \mathcal{A} -equivalent to ψ with parameter vector $p = (1, a'_{6,2}, a'_{6,3}, \dots, a'_{r,2}, \dots, a'_{r,[(r-2)/2]})$.

In particular, the dimension of the moduli space \mathbb{M}_r for multigerms of curves with singular ordinary point and $r \geq 5$ branches is the number of parameters in p plus $r - 3$ which corresponds to $\#\{\theta_i; 4 \leq i \leq r\}$. If we denote $L_1 = \{1, 2, 3\}$ just to unify the notation, we have that $\#I \setminus L_1 = \#\{\theta_i; 4 \leq i \leq r\}$. Thus,

$$\dim_{\mathbb{C}} \mathbb{M}_r = \sum_{k \geq 1} (\#I \setminus L_k) - 1 = \sum_{k=1}^{\lfloor (r-2)/2 \rfloor} (r - 2k - 1) = \begin{cases} \frac{(r-2)^2}{4} & \text{for } r \text{ even,} \\ \frac{(r-1)(r-3)}{4} & \text{for } r \text{ odd,} \end{cases}$$

as originally obtained by Granger [Gra79].

6.3 Multigerms with $\Gamma_i \cap \mathbb{N} = \langle n, m \rangle$ and $\nu_i(f_i) = nm$

We have just studied a class of multigerms in which each block has just one element. In this subsection, we consider the opposite situation, that is, curves such that we have only one block.

Let \mathcal{C} be a plane curve defined by a reduced element $f = \prod_{i=1}^r f_i \in \mathbb{C}\{X, Y\}$ such that each branch \mathcal{C}_i admits $\Gamma_i \cap \mathbb{N} = \langle n, m \rangle$, $GCD(n, m) = 1 < n < m$, and $\nu_i(f_i) = nm$ for all $i, l \in I$ with $i \neq l$. We present a *pre-normal form* considering just the topological data and we express the dimension of the generic component $\mathbb{M}_r(n, m)$ of the moduli space by means of the sets L_k . For the specific case of $n = 2$ and $m = 3$ we present a closed formula for $\dim_{\mathbb{C}} \mathbb{M}_r(2, 3)$.

The curve \mathcal{C} admits a multigerms with Puiseux block form $\varphi = [\varphi_1, \dots, \varphi_r]$ where

$$\varphi_i = \left(t_i^n, a_{im} t_i^m + \sum_{j>m} a_{ij} t_i^j \right)$$

with $a_{im}^n \neq a_{lm}^n$, $a_{im} \neq 0 \neq a_{lm}$ for every $1 \leq i, l \leq r$, and $i \neq l$. In this situation we have a single block and consequently the corresponding group to consider in Theorem 5.8 is $\mathcal{G} = \tilde{\mathcal{A}}_1$.

The topological class of \mathcal{C} can be characterized by the value semiring Γ that, according to Proposition 3.9 of [CH20a], is $\Gamma = \langle v_1, v_2, \dots, v_{r+2} \rangle$ with

$$v_1 = (n, \dots, n), \quad v_2 = (m, \dots, m), \quad v_{i+2} = (nm, \dots, \infty, \dots, nm) \quad i \in I, \tag{17}$$

where the symbol ‘ ∞ ’ occupies the i th coordinate in v_{i+2} .

The conductor of Γ is $\kappa = (\kappa_1, \dots, \kappa_r)$ with $\kappa_i = rnm - n - m + 1$ for all $i \in I$. In particular, if $k \geq rnm$, then $F_J^\Gamma(\underline{k}) \neq \emptyset$ for every $\emptyset \neq J \subseteq I$ and as $\Gamma \subset \Gamma_1 \times \dots \times \Gamma_r$, if $k \notin \langle n, m \rangle$, then $F_J^\Gamma(\underline{k}) = \emptyset$.

The next lemma characterizes the non-empty fibers of $\underline{k} \in \mathbb{N}^r$ with respect to Γ for $k < rnm$.

LEMMA 6.6. *Given $k \in \langle n, m \rangle$ with $cnm \leq k < (c + 1)nm$ and $0 \leq c \leq r - 1$ we have*

$$F_J^\Gamma(\underline{k}) \neq \emptyset \Leftrightarrow \begin{cases} \#J \geq r - c & \text{if } k - cnm \in \langle n, m \rangle, \\ \#J \geq r - (c - 1) & \text{if } k - cnm \notin \langle n, m \rangle. \end{cases}$$

Proof. Remark that $F_J^\Gamma(\underline{k}) \neq \emptyset$ if and only if there exists $\gamma = (\gamma_1, \dots, \gamma_r) \in \Gamma$ with $\gamma_j = k$ for every $j \in J$ and $\gamma_j > k$ for every $j \in I \setminus J$. By (1), $\gamma \in \Gamma$ can be expressed as

$$\gamma = \inf \left\{ \sum_{s=1}^{r+2} \alpha_{1,s} v_s, \dots, \sum_{s=1}^{r+2} \alpha_{r,s} v_s \right\}, \tag{18}$$

where $v_s = (v_{s1}, \dots, v_{sr})$ is given in (17) for $1 \leq s \leq r + 2$.

For $j \in J$ there exists $l \in I$ such that

$$\gamma_j = k = \sum_{s=1}^{r+2} \alpha_{l,s} v_{sj} = \alpha_{l,1}n + \alpha_{l,2}m + \sum_{s=3}^{r+2} \alpha_{l,s}nm$$

with $\alpha_{l,j+2} = 0$. In particular, $\#J \geq \#\{i \in I; \alpha_{l,i+2} = 0\} = r - \#\{i \in I; \alpha_{l,i+2} \neq 0\}$.

If $k - cnm \notin \langle n, m \rangle$, then $\sum_{s=3}^{r+2} \alpha_{l,s} \leq c - 1$ and $\#J \geq r - (c - 1)$. If $k - cnm \in \langle n, m \rangle$, then $\sum_{s=3}^{r+2} \alpha_{l,s} \leq c$ and $\#J \geq r - c$.

On the other hand, if $k - cnm = \alpha_1n + \alpha_2m \in \langle n, m \rangle$, considering any non-negative integers $\alpha_3, \dots, \alpha_{r+2}$ such that $c = \sum_{s=3}^{r+2} \alpha_s$ we get $\#\{i \in J; \alpha_{i+2} \neq 0\} \leq c$, $\sum_{s=1}^{r+2} \alpha_s v_s \in F_J^\Gamma(\underline{k})$ for $J = \{i \in I; \alpha_{i+2} = 0\}$ and $\#J \geq r - c$. If $k - cnm = g \notin \langle n, m \rangle$, then $k = g + nm + (c - 1)nm = \alpha_1n + \alpha_2m + (c - 1)nm$ and considering $c - 1 = \sum_{s=3}^{r+2} \alpha_s$ we have $\sum_{s=1}^{r+2} \alpha_s v_s \in F_J^\Gamma(\underline{k})$ for $J = \{i \in I; \alpha_{i+2} = 0\}$ and $\#J \geq r - (c - 1)$. \square

Given $k > m$ we have that $\underline{k} \in \nu(\mathcal{M}^2) \subset \Lambda_{\tilde{\mathcal{A}}_1}$. Thus, considering $F_J(\underline{k})$ the J -fiber of \underline{k} with respect to $\Lambda_{\tilde{\mathcal{A}}_1}$ we get $F_J^\Gamma(\underline{k+n}) - \underline{n} \subset F_J(\underline{k})$ and, by the previous lemma, we obtain

$$F_J^\Gamma(\underline{k+n}) - \underline{n} \neq \emptyset \Leftrightarrow \begin{cases} \#J \geq r - c & \text{if } k - cnm \in \langle n, m \rangle - n, \\ \#J \geq r - (c - 1) & \text{if } k - cnm \notin \langle n, m \rangle - n. \end{cases}$$

For each $k > m$ with $cnm - n \leq k < (c + 1)nm - n$ taking

$$L = \{1, \dots, c + 1\}, \quad J_i = \{i, c + 2, c + 3, \dots, r\} \quad \text{for } i \in L \text{ if } k - cnm \in \langle n, m \rangle - n;$$

$$L = \{1, \dots, c\}, \quad J_i = \{i, c + 1, c + 2, \dots, r\} \quad \text{for } i \in L \text{ if } k - cnm \notin \langle n, m \rangle - n$$

in Proposition 5.3 and Corollary 5.5 we obtain the following result.

PROPOSITION 6.7. Any curve with value semiring generated by (17) is analytically equivalent to a curve defined by a multigerms $\varphi = [\varphi_1, \dots, \varphi_r]$ with

$$\varphi_1 = \left(t_1^n, a_{1m}t_1^m + \sum_{\substack{j>m \\ j \notin \langle n, m \rangle - n}}^{nm-n-1} a_{1j}t_1^j \right), \quad \varphi_i = \left(t_i^n, \sum_{j=m}^{(i-1)nm-n-1} a_{ij}t_i^j + \sum_{\substack{j \geq (i-1)nm-n \\ j - (i-1)nm \notin \langle n, m \rangle - n}}^{im-n-1} a_{ij}t_i^j \right),$$

for $2 \leq i \leq r$.

As in this topological class the value semiring Γ is totally determined by the m -jet of the multigerms φ , we can proceed with the study of $\Lambda_{\tilde{\mathcal{A}}_1}$ for the generic case and consequently we obtain information about the generic component of the moduli space $\mathbb{M}_r(n, m)$ of all plane curves with Γ generated by (17) considering the multigerms $\varphi = [\varphi_1, \dots, \varphi_r]$ with $\varphi_i = (t_i^n, \sum_{j \geq m} a_{ij}t_i^j)$ where $a_{ij} \in \mathbb{C}$ are generic, $0 \neq a_{im}^n \neq a_{lm}^n \neq 0$ for every $i, l \in I$ and $i \neq l$. The generic component for the moduli space in this topological class was also considered in [GP16] by other methods.

Similarly to the case presented in the last subsection, the generality hypothesis implies that $L_k = \emptyset$, i.e. $e(k) = 0$ or $L_k = \{1, \dots, e(k)\}$ with $e(k) \leq r$ for every $k > m$ and, by Remark 5.4, $e(k)$ is the maximum integer such that there exists $w \in \mathcal{I}_G$ with $j^k(w) = (b_{1k}t_1^k, \dots, b_{rk}t_r^k)$ for any $b_{ik} \in \mathbb{C}$ and $1 \leq i \leq e(k)$.

In this context we have the following proposition.

PROPOSITION 6.8. *The dimension of the generic component of the moduli space $\mathbb{M}_r(n, m)$ is zero for $r = 1$, $(n, m) \in \{(2, m), (3, 4), (3, 5)\}$, and*

$$\dim_{\mathbb{C}} \mathbb{M}_r(n, m) = r - 2 + \sum_{k>m} (r - e(k))$$

for the other cases.

Proof. Zariski [Zar06] proved that for irreducible plane curves in the topological class determined by $\Gamma \cap \mathbb{N} = \langle n, m \rangle$ with $(n, m) \in \{(2, m), (3, 4), (3, 5)\}$ the moduli space is a single point.

By Theorem 5.10, we can normalize the coefficient a_{1m} in φ_1 and, thus, the number of parameters in the \mathcal{A} -normal form of φ is $r - 1 + \sum_{k>m} \#I \setminus L_k = r - 1 + \sum_{k>m} (r - e(k))$. Moreover, if $\sum_{k>m} (r - e(k)) \geq 1$ there exist $j = \min\{k > m; L_k \neq I\}$ and $l = \min\{i \in I \setminus L_j\}$ such that $a_{lj} \neq 0$. In this way, we can normalize such coefficient taking $c \in \mathbb{C}^*$ with $c^{j-m} = a_{lj}$, $\rho_i(t_i) = ct_i$ for all $i \in I$ and $\sigma(X, Y) = (c^n X, c^m Y)$, that is, $(\rho_1, \dots, \rho_r, \sigma) \cdot \varphi = \psi = [\psi_1, \dots, \psi_r]$ where the coefficient of t_i^j of ψ_l is equal to 1. In this case, we obtain

$$\dim_{\mathbb{C}} \mathbb{M}_r(n, m) = r - 2 + \sum_{k>m} (r - e(k)).$$

We show that, except for the above particular cases considered by Zariski, we have $\sum_{k>m} (r - e(k)) \geq 1$. More specifically, we show that $(r - e(m + 1)) + (r - e(m + 2)) \geq 1$.

Let us compute $e(m + 1)$ that, as mentioned previously, it is equivalent to evaluate the maximum number of equations $j^{m+1}(\varphi_i^*(\omega)/nt_i^n) = b_{i,m+1}t_i^{m+1}$ that admit solution for any $b_{i,m+1} \in \mathbb{C}$ with $0 \leq i \leq e(m + 1)$ and $\omega = p dX + q dY \in \Omega_{\tilde{\mathcal{A}}_1}$, that is, $p \in \langle X, Y \rangle^2$ and $q \in \langle X^2, Y \rangle$.

Note that it is sufficient to consider $\nu_i(\varphi_i^*(\omega)/nt_i^n) = m + 1$. Thus, if $m \neq n + 1$, then $q = 0$, and if $m + 1 \not\equiv 0 \pmod n$, then $p = 0$. In particular, for $m \neq n + 1$ and $m + 1 \not\equiv 0 \pmod n$ we have $e(m + 1) = 0$.

For $m = n + 1$ and $m + 1 \not\equiv 0 \pmod n$ we get $\omega = \alpha Y dY$, $j^{m+1}(\varphi_i^*(\omega)/nt_i^n) = (m/n)\alpha a_{i,m}^2 t_i^{m+1}$, and $e(m + 1) = 1$. Similarly, if $m \neq n + 1$ and $m + 1 \equiv 0 \pmod n$, then $\omega = \sum_{l=2}^{(m+1)/n} \alpha_l X^l dX$, $j^{m+1}(\varphi_i^*(\omega)/nt_i^n) = \sum_{l=2}^{(m+1)/n} \alpha_l t_i^{ln}$ and $e(m + 1) = 1$.

If $m = n + 1$ and $m + 1 \equiv 0 \pmod n$, then $n = 2, m = 3$ and we have $\omega = \alpha_1 X^2 dX + \alpha_2 Y dY$, that is, $j^4(\varphi_i^*(\omega)/2t_i^2) = (\alpha_1 + \frac{3}{2}\alpha_2 a_{i,3}^2)\alpha_2 t_i^4$. Thus, we get $e(m + 1) = \min\{2, r\}$.

Now, we compute $e(m + 2)$ for the cases $e(m + 1) > 0$. As before it is sufficient to consider $\omega = p dX + q dY \in \Omega_{\tilde{\mathcal{A}}_1}$ with $\nu_i(p) \leq m + 2$ and $\nu_i(q) \leq n + 2$.

In the case $m = n + 1$ and $m + 1 \not\equiv 0 \pmod n$, if $n > 3$ we have $\omega = \alpha Y dY$,

$$j^{m+2} \left(\frac{\varphi_i^*(\omega)}{nt_i^n} \right) = \frac{m}{n} \alpha a_{i,m}^2 t_i^{m+1} + \left(\frac{2m + 1}{n} \right) \alpha a_{i,m} a_{i,m+1} t_i^{m+2}$$

and $e(m + 2) = 0$. If $n = 3$ and, thus, $m = 4$, we have $\omega = \alpha_1 X^2 dX + \alpha_2 Y dY$,

$$j^6 \left(\frac{\varphi_i^*(\omega)}{3t_i^3} \right) = \frac{4}{3} \alpha_2 a_{i,4}^2 t_i^5 + (\alpha_1 + 3\alpha_2 a_{i,4} a_{i,5}) t_i^6 \quad \text{and} \quad e(m + 2) = 1.$$

When $m \neq n + 1$ and $m + 1 \equiv 0 \pmod n$ we have the possibilities: if $m \neq n + 2$ and $n \neq 2$ then $\omega = \sum_{l=2}^{(m+1)/n} \alpha_l X^l dX$, $j^{m+2}(\varphi_i^*(\omega)/nt_i^n) = \sum_{l=2}^{(m+1)/n} \alpha_l t_i^{ln}$, thus $e(m + 2) = 0$. If $m \neq n + 2$ and $n = 2$, then $\omega = (\sum_{l=2}^{(m+1)/2} \alpha_l X^l + \alpha_1 XY) dX + \alpha_0 X^2 dY$,

$$j^{m+2} \left(\frac{\varphi_i^*(\omega)}{2t_i^2} \right) = \sum_{l=2}^{(m+1)/2} \alpha_l t_i^{2l} + \left(\alpha_1 a_{i,m} + \alpha_0 \frac{m}{2} a_{i,m} \right) t_i^{m+2}$$

and $e(m + 2) = 1$. If $m = n + 2$, then $n = 3, m = 5, \omega = \alpha_1 X^2 dX + \alpha_2 Y dY$, and

$$j^7 \left(\frac{\varphi_i^*(\omega)}{3t_i^3} \right) = \alpha_1 t_i^6 + \frac{5}{3} \alpha_2 a_{i,5}^2 t_i^7,$$

consequently $e(m + 2) = 1$.

Finally, if $n = 2$ and $m = 3$, then $\omega = (\alpha_1 X^2 + \alpha_2 XY) dX + (\alpha_3 Y + \alpha_4 X^2) dY$,

$$j^5 \left(\frac{\varphi_i^*(\omega)}{2t_i^2} \right) = \left(\alpha_1 + \frac{3}{2} \alpha_3 a_{i,3}^2 \right) t_i^4 + \left(\alpha_2 + \frac{3}{2} \alpha_4 + \frac{7}{2} a_{i,4} \alpha_3 \right) a_{i,3} t_i^5 \quad \text{and} \quad e(m + 2) = 1.$$

These computations give us that, except to the cases $r = 1$ and $(n, m) \in \{(2, m), (3, 4), (3, 5)\}$, we have that $(r - e(m + 1)) + (r - e(m + 2)) \geq 1$. □

To illustrate the above result in a specific case, in what follows $n = 2, m = 3$, and $r \geq 2$.

We compute $e(k)$ for any $k > m = 3$ which allows us to exhibit normal forms to the generic case and an explicit formula for $\dim_{\mathbb{C}} \mathbb{M}_r(2, 3)$.

Our strategy is the same as that of the previous proposition, that is, to evaluate the maximum number of equations $j^k(\varphi_i^*(\omega)/2t_i^2) = b_{i,k} t^k$ that admit solution for any $b_{i,k} \in \mathbb{C}$ with $\omega = p dX + q dY \in \Omega_{\tilde{\mathcal{A}}_1}$, that is, $p \in \langle X, Y \rangle^2$ and $q \in \langle X^2, Y \rangle$. To achieve this goal, for every $h \geq 2$ we consider

$$\begin{aligned} \omega_{2h} &= \left(\sum_{l=0}^{\lfloor h/3 \rfloor} \alpha_{h-3l,2l} X^{h-3l} Y^{2l} \right) dX + \left(\sum_{l=0}^{\lfloor (h-2)/3 \rfloor} \beta_{h-2-3l,2l+1} X^{h-2-3l} Y^{2l+1} \right) dY, \\ \omega_{2h+1} &= \left(\sum_{l=0}^{\lfloor (h-1)/3 \rfloor} \alpha_{h-1-3l,2l+1} X^{h-1-3l} Y^{2l+1} \right) dX + \left(\sum_{l=0}^{\lfloor h/3 \rfloor} \beta_{h-3l,2l} X^{h-3l} Y^{2l} \right) dY. \end{aligned}$$

Any $\omega \in \Omega_{\tilde{\mathcal{A}}_1}$ can be uniquely expressed as $\omega = \sum_{l \geq 4} \omega_l$. As

$$\nu_i \left(\frac{\varphi_i^*(X^{h-3l} Y^{2l} dX)}{2t_i^2} \right) = \nu_i \left(\frac{\varphi_i^*(X^{h-2-3l} Y^{2l+1} dY)}{2t_i^2} \right) = 2h$$

and

$$\nu_i \left(\frac{\varphi_i^*(X^{h-1-3l} Y^{2l+1} dX)}{2t_i^2} \right) = \nu_i \left(\frac{\varphi_i^*(X^{h-3l} Y^{2l} dY)}{2t_i^2} \right) = 2h + 1$$

for any $i \in I$, in order to evaluate $e(k)$ it is sufficient to consider $\omega = \sum_{l=4}^k \omega_l$.

PROPOSITION 6.9. *If $\varphi = [\varphi_1, \dots, \varphi_r]$ is a generic multigerms admitting value semiring Γ with generators as in (17), $n = 2, m = 3$, and $r \geq 2$, then $e(4) = 2, e(5) = 1$, and for $k \geq 6$:*

$$e(k) = \min \left\{ 2 \left\lfloor \frac{k}{6} \right\rfloor + 1, r \right\} \quad \text{if } k \not\equiv 4 \pmod{6}, \quad e(k) = \min \left\{ 2 \left\lfloor \frac{k}{6} \right\rfloor + 3, r \right\} \quad \text{if } k \equiv 4 \pmod{6},$$

and $e(6\lfloor (r - 2)/2 \rfloor + 5) = e(3r - 1) = r$ if r is even.

Proof. We remark that, by the last proposition, $e(4) = 2$ and $e(5) = 1$.

First, we consider the system $j^{c+1}(\varphi_i^*(\omega_c + \omega_{c+1})/2t_i^2) = 0$ for every $i \in I$ and $c \geq 5$.

Using $\varphi_i = (t_i^2, \sum_{j \geq 3} a_{ij} t_i^j)$ we obtain

$$\begin{aligned}
 j^{2h+1} \left(\frac{\varphi_i^*(\omega_{2h})}{2t_i^2} \right) &= \left(\sum_{l=0}^{[h/3]} a_{i3}^{2l} \alpha_{h-3l,2l} + \sum_{l=0}^{[(h-2)/3]} \frac{3}{2} a_{i3}^{2l+2} \beta_{h-2-3l,2l+1} \right) t_i^{2h} \\
 &\quad + \left(\sum_{l=1}^{[h/3]} 2l a_{i3}^{2l-1} a_{i4} \alpha_{h-3l,2l} + \sum_{l=0}^{[(h-2)/3]} \frac{6l+7}{2} a_{i3}^{2l+1} a_{i4} \beta_{h-2-3l,2l+1} \right) t_i^{2h+1}; \\
 j^{2h+2} \left(\frac{\varphi_i^*(\omega_{2h+1})}{2t_i^2} \right) &= \left(\sum_{l=0}^{[(h-1)/3]} a_{i3}^{2l+1} \alpha_{h-1-3l,2l+1} + \sum_{l=0}^{[h/3]} \frac{3}{2} a_{i3}^{2l+1} \beta_{h-3l,2l} \right) t_i^{2h+1} \\
 &\quad + \left(\sum_{l=0}^{[(h-1)/3]} (2l+1) a_{i3}^{2l} a_{i4} \alpha_{h-1-3l,2l+1} + \sum_{l=0}^{[h/3]} (3l+2) a_{i3}^{2l} a_{i4} \beta_{h-3l,2l} \right) t_i^{2h+2}.
 \end{aligned}$$

If $c = 2h$, then to vanish the coefficient of t_i^c in $j^{c+1}(\varphi_i^*(\omega_c + \omega_{c+1})/2t_i^2)$ for every $i \in I$ we obtain the system $N \cdot W_p^t = 0$ with

$$N = \begin{pmatrix} 1 & a_{13}^2 & a_{13}^4 & \cdots & a_{13}^{2[h/3]+p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{r3}^2 & a_{r3}^4 & \cdots & a_{r3}^{2[h/3]+p} \end{pmatrix} \quad \text{with} \quad \begin{cases} p = 0 & \text{if } c \equiv 0, 2 \pmod 6, \\ p = 2 & \text{if } c \equiv 4 \pmod 6, \end{cases}$$

$$W_0 = \left(\alpha_{h,0}, \alpha_{h-3,2} + \frac{3}{2} \beta_{h-2,1}, \dots, \alpha_{h-3[h/3],2[h/3]} + \frac{3}{2} \beta_{h-3[h/3]+1,2[h/3]-1} \right),$$

and $W_2 = (W_0, \frac{3}{2} \beta_{h-2-3[h/3],2[h/3]+1})$, where W_p^t denotes the transpose of W_p .

Considering the solution of $N \cdot W_p^t = 0$, the associated system to vanish the coefficient of t_i^{c+1} in $j^{c+1}(\varphi_i^*(\omega_c + \omega_{c+1})/2t_i^2)$ for every $i \in I$ is $M \cdot Z^t = 0$ where

$$M = \begin{pmatrix} a_{14} a_{13} & a_{14} a_{13}^3 & \cdots & a_{14} a_{13}^{2[h/3]-1} & a_{13} & a_{13}^3 & \cdots & a_{13}^{2[h/3]+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{r4} a_{r3} & a_{r4} a_{r3}^3 & \cdots & a_{r4} a_{r3}^{2[h/3]-1} & a_{r3} & a_{r3}^3 & \cdots & a_{r3}^{2[h/3]+1} \end{pmatrix}, \tag{19}$$

$$Z = \left(\frac{1}{2} \beta_{h-3l+1,2l-1}; 1 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor, \alpha_{h-1-3l,2l+1} + \frac{3}{2} \beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor - 1, \frac{3}{2} \beta_{h-3[h/3],2[h/3]} \right)$$

if $c \equiv 0 \pmod 6$ or

$$Z = \left(\frac{1}{2} \beta_{h-3l+1,2l-1}; 1 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor, \alpha_{h-1-3l,2l+1} + \frac{3}{2} \beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor \right) \quad \text{if } c \equiv 2, 4 \pmod 6.$$

Note that the rank of M is $\min\{2[h/3] + 1, r\} = \min\{2[c/6] + 1, r\}$. If $2[c/6] + 1 \leq r$, then the solution of the above system implies that $\omega_c = 0$.

For $c = 2h + 1$ we vanish the coefficient of t_i^c in $j^{c+1}(\varphi_i^*(\omega_c + \omega_{c+1})/2t_i^2)$ for every $i \in I$ solving the system $N \cdot W^t = 0$ where

$$N = \begin{pmatrix} a_{13} & a_{13}^3 & \cdots & a_{13}^{2[h/3]+1} \\ \vdots & \vdots & & \vdots \\ a_{r3} & a_{r3}^3 & \cdots & a_{r3}^{2[h/3]+1} \end{pmatrix},$$

$$W = \left(\alpha_{h-1-3l,2l+1} + \frac{3}{2}\beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor \right) \quad \text{if } c \equiv 3, 5 \pmod 6$$

or

$$W = \left(\alpha_{h-1-3l,2l+1} + \frac{3}{2}\beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor - 1, \frac{3}{2}\beta_{h-3\lfloor h/3\rfloor,2\lfloor h/3\rfloor} \right) \quad \text{if } c \equiv 1 \pmod 6.$$

Taking the solution of the system $N \cdot W^t = 0$ in order to vanish the coefficient of t_i^{c+1} in $j^{c+1}(\varphi_i^*(\omega_c + \omega_{c+1})/2t_i^2)$ for every $i \in I$ we obtain the system $M \cdot Z^t = 0$ where

$$M = \begin{pmatrix} a_{14} & a_{14}a_{13}^2 & \cdots & a_{14}a_{13}^{2\lfloor h/3\rfloor-2} & 1 & a_{13}^2 & \cdots & a_{13}^{2\lfloor h/3\rfloor} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{r4} & a_{r4}a_{r3}^2 & \cdots & a_{r4}a_{r3}^{2\lfloor h/3\rfloor-2} & 1 & a_{r3}^2 & \cdots & a_{r3}^{2\lfloor h/3\rfloor} \end{pmatrix} \quad \text{if } c \equiv 1 \pmod 6,$$

$$M = \begin{pmatrix} a_{14} & a_{14}a_{13}^2 & \cdots & a_{14}a_{13}^{2\lfloor h/3\rfloor} & 1 & a_{13}^2 & \cdots & a_{13}^{2\lfloor h/3\rfloor+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{r4} & a_{r4}a_{r3}^2 & \cdots & a_{r4}a_{r3}^{2\lfloor h/3\rfloor} & 1 & a_{r3}^2 & \cdots & a_{r3}^{2\lfloor h/3\rfloor+2} \end{pmatrix} \quad \text{if } c \equiv 3, 5 \pmod 6,$$

$$Z = \left(\frac{1}{2}\beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor - 1, \alpha_{h+1,0}, \alpha_{h+1-3l,2l} + \frac{3}{2}\beta_{h+2-3l,2l-1}; 1 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor \right) \quad \text{if } c \equiv 1 \pmod 6,$$

$$Z = \left(\frac{1}{2}\beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor, \alpha_{h+1,0}, \alpha_{h+1-3l,2l} + \frac{3}{2}\beta_{h+2-3l,2l-1}; 1 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor, \frac{3}{2}\beta_{h-1-3\lfloor h/3\rfloor,2\lfloor h/3\rfloor+1} \right) \quad \text{if } c \equiv 3 \pmod 6$$

and

$$Z = \left(\frac{1}{2}\beta_{h-3l,2l}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor, \alpha_{h+1,0}, \alpha_{h-2-3l,2l+2} + \frac{3}{2}\beta_{h-1-3l,2l+1}; 0 \leq l \leq \left\lfloor \frac{h}{3} \right\rfloor \right) \quad \text{if } c \equiv 5 \pmod 6.$$

Remark that the rank of M is $\min\{2\lfloor h/3\rfloor + 1, r\} = \min\{2\lfloor c/6\rfloor + 1, r\}$ if $c \equiv 1 \pmod 6$ or $\min\{2\lfloor h/3\rfloor + 3, r\} = \min\{2\lfloor c/6\rfloor + 3, r\}$ if $c \equiv 3, 5 \pmod 6$. In particular, $\omega_c = 0$ if $2\lfloor c/6\rfloor + 1 \leq r$ for $c \equiv 1 \pmod 6$ or $2\lfloor c/6\rfloor + 3 \leq r$ for $c \equiv 3, 5 \pmod 6$.

In this way, to study the system $j^k(\varphi_i^*(\omega)/2t_i^2) = b_{i,k}t_i^k$ with $b_{i,k} \in \mathbb{C}$, $i \in I$ for $r \leq 2\lfloor (k-1)/6\rfloor + 1$ or $2\lfloor (k-1)/6\rfloor + 3 \leq r$ it is sufficient to consider $\omega = \omega_{k-1} + \omega_k \in \Omega_{\tilde{\mathcal{A}}_1}$. For this situation the cases are the same as those considered above, where we obtained the system $M \cdot Z^t = (b_{1,k}, \dots, b_{r,k})^t$ with M and Z described previously taking $k = c + 1$. Hence, $e(k) = \text{rank}(M)$ for all $k \geq 6$.

As $\lfloor k/6\rfloor = \lfloor c/6\rfloor$ for $c \not\equiv 5 \pmod 6$ and $\lfloor k/6\rfloor = \lfloor c/6\rfloor + 1$ for $c \equiv 5 \pmod 6$, we get

$$e(k) = \min \left\{ 2 \left\lfloor \frac{k}{6} \right\rfloor + 1, r \right\} \quad \text{if } k \not\equiv 4 \pmod 6 \quad \text{and} \quad e(k) = \min \left\{ 2 \left\lfloor \frac{k}{6} \right\rfloor + 3, r \right\} \quad \text{if } k \equiv 4 \pmod 6.$$

We obtain the same conclusion if $r = 2\lfloor (k-1)/6\rfloor + 2$ and $k \not\equiv 5 \pmod 6$.

For $r = 2\lfloor (k-1)/6\rfloor + 2$ and $k \equiv 5 \pmod 6$, that is, $k = 3r - 1 = 6\lfloor (r-2)/2\rfloor + 5$, the condition $j^{k-1}(\varphi_i^*(\omega_{k-2} + \omega_{k-1})/2t_i^2) = 0$ for every $i \in I$ produces a system $M \cdot Z^t = 0$ with $\text{rank}(M) = \min\{2\lfloor (k-2)/6\rfloor + 3, r\} = \min\{r + 1, r\} = r$. In this way, we have an extra

variable for the system $j^k(\varphi_i^*(\omega_{k-2} + \omega_{k-1} + \omega_k)/2t_i^2) = b_{i,k}t_i^k$ that can be expressed as $M_1 \cdot Z_1^t = (b_{1,k}, \dots, b_{r,k})^t$ where M_1 is the matrix M as in (19) with an extra column depending on a_{i3}, a_{i4} and a_{i5} . This allows us to conclude that

$$e\left(6\left[\frac{r-2}{2}\right] + 5\right) = e(3r - 1) = \text{rank}(M_1) = \min\left\{2\left[\frac{k-1}{6}\right] + 1 + 1, r\right\} = r. \quad \square$$

As a consequence of Propositions 6.8 and 6.9 we obtain an explicit formula for the dimension of the generic component of $\mathbb{M}_r(2, 3)$.

COROLLARY 6.10. *For plane curves that admit value semiring generated by (17) with $n = 2$, $m = 3$, and r branches, the dimension of the generic component $\mathbb{M}_r(2, 3)$ of the moduli space is*

$$\dim_{\mathbb{C}} \mathbb{M}_r(2, 3) = \begin{cases} \frac{(r-1)(3r-5)}{2} & \text{if } r \text{ is odd,} \\ \frac{(r-1)(3r-5)+1}{2} & \text{if } r \text{ is even.} \end{cases}$$

Proof. The case $r = 1$ is immediate.

By Proposition 6.8 and the above result we have that

$$\begin{aligned} \dim_{\mathbb{C}} \mathbb{M}_r(2, 3) &= r - 2 + \sum_{k \geq 4} (r - e(k)) = 3r - 5 + \sum_{k \geq 6} (r - e(k)) \\ &= 3r - 5 + 5 \sum_{\substack{k \geq 6 \\ k \not\equiv 4 \pmod{6}}} (r - e(k)) + \sum_{\substack{k \geq 6 \\ k \equiv 4 \pmod{6}}} (r - e(k)). \end{aligned} \tag{20}$$

We remark that for $r = 2$ we have $e(k) = 2$ for every $k \geq 6$, then $\dim_{\mathbb{C}} \mathbb{M}_2(2, 3) = 1$.

If $k \equiv 4 \pmod{6}$, we have $e(k) = \min\{2[k/6] + 3, r\}$ and

$$\sum_{\substack{k \geq 6 \\ k \equiv 4 \pmod{6}}} (r - e(k)) = \sum_{i=1}^{[(r-3)/2]} (r - (2i + 3)) = \left[\frac{r-3}{2}\right] \left(r - 4 - \left[\frac{r-3}{2}\right]\right). \tag{21}$$

If $k \not\equiv 4 \pmod{6}$, then $e(k) = \min\{2[k/6] + 1, r\}$ and $e(6[(r-2)/2] + 5) = e(3r - 1) = r$ if r is even. Thus,

$$\begin{aligned} &\sum_{\substack{k \geq 6 \\ k \not\equiv 4 \pmod{6}}} (r - e(k)) \\ &= \begin{cases} \sum_{i=1}^{[(r-4)/2]} (r - (2i + 1)) = \left[\frac{r-1}{2}\right] \left(r - 2 - \left[\frac{r-1}{2}\right]\right) - 1 & \text{if } k \equiv 5 \pmod{6} \text{ and } r \text{ even,} \\ \sum_{i=1}^{[(r-1)/2]} (r - (2i + 1)) = \left[\frac{r-1}{2}\right] \left(r - 2 - \left[\frac{r-1}{2}\right]\right) & \text{otherwise.} \end{cases} \end{aligned} \tag{22}$$

For $r \geq 3$, considering (21) and (22) in (20) the result follows. □

ACKNOWLEDGEMENTS

We would like to thank Professor Yohann Genzmer for valuable discussions and we express our sincere gratitude to Professor Abramo Hefez and the anonymous referee for their careful reading of the manuscript and their comments and suggestions.

CONFLICTS OF INTEREST

None.

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