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SHAPIRO'S UNCERTAINTY PRINCIPLE IN THE DUNKL SETTING

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Abstract

The Dunkl transform $\mathcal{F}_k$ is a generalisation of the usual Fourier transform to an integral transform invariant under a finite reflection group. The goal of this paper is to prove a strong uncertainty principle for orthonormal bases in the Dunkl setting which states that the product of generalised dispersions cannot be bounded for an orthonormal basis. Moreover, we obtain a quantitative version of Shapiro's uncertainty principle on the time–frequency concentration of orthonormal sequences and show, in particular, that if the elements of an orthonormal sequence and their Dunkl transforms have uniformly bounded dispersions then the sequence is finite.

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1. Introduction

In an unpublished manuscript [20], Shapiro proved a number of uncertainty inequalities for orthonormal sequences that are stronger than the corresponding inequalities for a single function. In particular, he proved that for any orthonormal sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $L^2(\mathbb{R})$,

$$\sup_n(\|x\varphi_n\|^2_{L^2(\mathbb{R})} + \|\xi\mathcal{F}(\varphi_n)\|^2_{L^2(\mathbb{R})}) = \infty,$$

where $\mathcal{F}$ is the Fourier transform defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx$$

and extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way.

A quantitative version of Shapiro's result has been proved by Jaming and Powell [12]: if $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(\mathbb{R})$ then for all $N \geq 1$,

$$\sum_{n=1}^{N}(\|x\varphi_n\|^2_{L^2(\mathbb{R})} + \|\xi\mathcal{F}(\varphi_n)\|^2_{L^2(\mathbb{R})}) \geq N^2. \tag{1.1}$$

The latter inequality is sharp. The equality cases have been entirely described (see [12]) and are given by the sequence of Hermite functions. The higher-dimensional version of (1.1) involving generalised dispersions was obtained by Malinnikova [14]. That is, for $s > 0$ and $\{\varphi_n\}_{n=1}^{\infty}$ an orthonormal sequence in $L^2(\mathbb{R}^d)$,

$$\sum_{n=1}^{N} (\| |x|^s \varphi_n\|_{L^2(\mathbb{R}^d)}^2 + \| |\xi|^s \mathcal{F}(\varphi_n)\|_{L^2(\mathbb{R}^d)}^2) \geq C\, N^{1+s/d}. \tag{1.2}$$

We refer the reader to [1, 2, 8, 11, 16] for numerous results and discussions on time–frequency localisation of orthonormal sequences and bases.

The goal of this paper is to provide an analogue of inequality (1.2) for the Dunkl transform, which is a generalisation of the usual Fourier transform to an integral transform invariant under a finite reflection group. We show also that the product of generalised dispersions cannot be bounded for an orthonormal basis.

In order to describe our results, we first need to introduce some notation (further details can be found in Section 2.2). In this paper we consider the Dunkl operators (see [5]) T_j, $j = 1, \dots, d$, associated to an arbitrary finite reflection group G and a nonnegative multiplicity function k. These are differential-difference operators, generalising the usual partial derivatives, and they play a useful role in the algebraic description of exactly solvable quantum many-body systems of Calogero–Moser–Sutherland type. Among the extensive literature, we refer to [13, 15].

The Dunkl kernel $\mathcal{K}_k$ on $\mathbb{R}^d \times \mathbb{R}^d$ associated with G and k was introduced by Dunkl in [5, 6]. It generalises the usual exponential function (to which it reduces in the case $k = 0$) and can be characterised as the solution of a joint eigenvalue problem for the associated Dunkl operators. This kernel is of special interest as it gives rise to a corresponding integral transform on $\mathbb{R}^d$. The Dunkl transform $\mathcal{F}_k$ associated with G and k involves a weight function w_k and is defined for an integrable function f on $\mathbb{R}^d$ with respect to the measure $d\mu_k(x) = w_k(x)\, dx$ by

$$\mathcal{F}_k(f)(\xi) := c_k \int_{\mathbb{R}^d} \mathcal{K}_k(-i\xi, x) f(x)\, d\mu_k(x), \quad \xi \in \mathbb{R}^d,$$

and extended to $L^2(\mathbb{R}^d, \mu_k)$ by a Parseval-type relation when c_k is a suitable constant.

For $p = 1$ or 2, we will denote by $L_k^p(\mathbb{R}^d) = L^p(\mathbb{R}^d, \mu_k)$ the spaces of complex-valued measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{L_k^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p \; d\mu_k(x) \right)^{1/p} < \infty.$$

Our first result will be the following strong uncertainty principle for orthonormal bases of $L_k^2(\mathbb{R}^d)$.

THEOREM A. *Let $s > 0$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L_k^2(\mathbb{R}^d)$. Then*

$$\sup_n (\| |x|^s \varphi_n\|_{L_k^2} \| |\xi|^s \mathcal{F}_k(\varphi_n)\|_{L_k^2}) = \infty.$$

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This theorem shows that there does not exist an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ for $L_k^2(\mathbb{R}^d)$ such that the sequence $\{\|\,|x|^s\varphi_n\|_{L_k^2}\,\|\,|\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}\}_{n=1}^{\infty}$ is bounded. It is not difficult to construct an infinite orthonormal sequence in $L_k^2(\mathbb{R}^d)$ with bounded product of dispersions (see Remark 3.9).

Our next result will be the following quantitative dispersion inequality.

THEOREM B. *Let $s > 0$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. Then for all $N \geq 1$,*

$$\sum_{n=1}^{N}(\|\,|x|^s\varphi_n\|_{L_k^2}^2 + \|\,|\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2) \geq c(k,s)N^{1+s/(2\gamma+d)}. \tag{1.3}$$

This theorem implies in particular that, if the elements of an orthonormal sequence and their Dunkl transforms have uniformly bounded dispersions, then the sequence is finite. Moreover, it implies that

$$\sup_n(\|\,|x|^s\varphi_n\|_{L_k^2}^2 + \|\,|\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2) = \infty.$$

When the multiplicity function k is identically 0 (therefore $\gamma = 0$), the Dunkl transform coincides with the usual Fourier transform $\mathcal{F}$, and then inequality (1.3) coincides with the inequality (1.2).

The remainder of the paper is organised as follows. The next section is devoted to some preliminaries on the Dunkl transform. In Section 3 we prove Theorems A and B.

2. Preliminaries

2.1. Notation. Throughout this paper, we denote by $|x|$ and $\langle x, y\rangle$ the usual norm and scalar product on $\mathbb{R}^d$. The unit sphere of $\mathbb{R}^d$ is denoted by $\mathbb{S}^{d-1}$ and we endow it with the (nonnormalised) Lebesgue measure $d\sigma$, that is, $r^{d-1}\,dr\,d\sigma(\zeta)$ is the polar decomposition of the Lebesgue measure.

If A is a subset of $\mathbb{R}^d$, then we denote by $A^c = \mathbb{R}^d\backslash A$ the complement of A in $\mathbb{R}^d$, and by χ_A the characteristic function of A. Given a multi-index $n \in \mathbb{N}^d$, we write $|n| = n_1 + \cdots + n_d$ and, for $r > 0$, $\mathcal{B}(0,r) = \{x \in \mathbb{R}^d : |x| \leq r\}$ is the closed ball in $\mathbb{R}^d$ centred at 0 and of radius r.

2.2. The Dunkl transform. Let us fix some notation and present some necessary material on the Dunkl transform. Let G be a finite reflection group on $\mathbb{R}^d$ associated with a root system R, and R_+ the positive subsystem of R (see [4, 6, 19]). We denote by k a nonnegative multiplicity function defined on R with the property that k is G-invariant. We associate with k the index

$$\gamma := \gamma(k) = \sum_{\xi\in R_+} k(\xi) \geq 0$$

and the weight function w_k defined by

$$w_k(x) = \prod_{\xi\in R_+} |\langle\xi, x\rangle|^{2k(\xi)}.$$

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Further, we introduce the Mehta-type constant c_k given by

$$c_k = \Big(\int_{\mathbb{R}^d} e^{-1/2|x|^2}\, d\mu_k(x)\Big)^{-1},$$

where $d\mu_k(x) = w_k(x)\,dx$. Moreover,

$$\int_{\mathbb{S}^{d-1}} w_k(x)\,d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+d/2-1}\Gamma(\gamma+d/2)} := d_k.$$

By using the homogeneity of w_k, it is shown in [19] that for a radial function $f \in L_k^1(\mathbb{R}^d)$ the function $\widetilde{f}$, defined on $[0,\infty)$ by $f(x) = \widetilde{f}(|x|)$ for $x \in \mathbb{R}^d$, is integrable with respect to the measure $r^{2\gamma+d-1}\,dr$. More precisely,

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)w_k(x)\,dx &= \int_0^\infty\Big(\int_{\mathbb{S}^{d-1}} w_k(ry)\,d\sigma(y)\Big)\widetilde{f}(r)r^{d-1}\,dr\\ &= d_k\int_0^\infty \widetilde{f}(r)r^{2\gamma+d-1}\,dr.\end{aligned}$$

As introduced by Dunkl in [5], the Dunkl operators T_j, $1 \le j \le d$, on $\mathbb{R}^d$ associated with the reflection group G and the multiplicity function k are the first-order differential-difference operators given by

$$T_jf(x) = \frac{\partial f}{\partial x_j} + \sum_{\xi\in R_+} k(\xi)\xi_j\frac{f(x) - f(\sigma_\xi(x))}{\langle \xi, x\rangle}, \quad x \in \mathbb{R}^d,$$

where f is an infinitely differentiable function on $\mathbb{R}^d$, $\xi_j = \langle \xi, e_j\rangle$, $(e_1,\ldots,e_d)$ being the canonical basis of $\mathbb{R}^d$, and σ_ξ denotes the reflection with respect to the hyperplane orthogonal to ξ.

The Dunkl kernel $\mathcal{K}_k$ on $\mathbb{R}^d \times \mathbb{R}^d$ was introduced by Dunkl in [6]. For $\xi \in \mathbb{R}^d$ the function $x \mapsto \mathcal{K}_k(x,\xi)$ can be viewed as the solution on $\mathbb{R}^d$ of the initial value problem

$$T_ju(x,\xi) = \xi_ju(x,\xi), \quad 1 \le j \le d; \quad u(0,\xi) = 1.$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ and for all $\lambda \in \mathbb{C}$, $(z,z') \in \mathbb{C}^{2d}$, $(x,\xi) \in \mathbb{R}^{2d}$ (see [17]),

$$\mathcal{K}_k(z,z') = \mathcal{K}_k(z',z), \quad \mathcal{K}_k(\lambda z,z') = \mathcal{K}_k(z,\lambda z'), \quad \overline{\mathcal{K}_k(-i\xi,x)} = \mathcal{K}_k(i\xi,x),$$
$$|\mathcal{K}_k(-i\xi,x)| \le 1.$$

The Dunkl transform $\mathcal{F}_k$ of a function $f \in L_k^1(\mathbb{R}^d) \cap L_k^2(\mathbb{R}^d)$, introduced by Dunkl (see [4]), is given by

$$\mathcal{F}_k(f)(\xi) := c_k\int_{\mathbb{R}^d} \mathcal{K}_k(-i\xi,x)f(x)\,d\mu_k(x), \quad \xi \in \mathbb{R}^d,$$

and extends uniquely to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$ with

$$\mathcal{F}_k^{-1}(f)(\xi) = \mathcal{F}_k(f)(-\xi), \quad \xi \in \mathbb{R}^d,$$

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and

$$\|\mathcal{F}_k(f)\|_{L^2_k} = \|f\|_{L^2_k}. \tag{2.1}$$

Finally, according to [4, 19], we have, for all $f \in L^1_k(\mathbb{R}^d)$,

$$\|\mathcal{F}_k(f)\|_\infty \le c_k \|f\|_{L^1_k},$$

where $\|\cdot\|_\infty$ is the usual essential supremum norm.

3. Time–frequency concentration of orthonormal sequences in $L^2_k(\mathbb{R}^d)$

3.1. Heisenberg-type uncertainty inequality for the Dunkl transform. The Heisenberg–Pauli–Weyl inequality leads to the following classical formulation of the uncertainty principle in the form of a lower bound for the product of the dispersions of a function in $L^2(\mathbb{R}^d)$ and its Fourier transform:

$$\|\,|x|f\|_{L^2(\mathbb{R}^d)}\,\|\,|\xi|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)} \ge \frac{d}{2}\,\|f\|^2_{L^2(\mathbb{R}^d)}, \tag{3.1}$$

with equality if and only if f is a multiple of a Gaussian. Heisenberg's inequality (3.1) may also be written in the form

$$\|\,|x|f\|^2_{L^2(\mathbb{R}^d)} + \|\,|\xi|\mathcal{F}(f)\|^2_{L^2(\mathbb{R}^d)} \ge d\,\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

In this section we will give a slightly simpler proof of the sharp Heisenberg uncertainty inequality for the Dunkl transform which was first proved by Rösler [17] and then by Shimeno [21]. Rösler in [17] used expansions in terms of Dunkl Hermite polynomials and the recurrence relations among them as given in [18]. This generalises a well-known method for the (one-dimensional) classical situation (see, for example, [3]). Shimeno in [21] used expansions in terms of the basis given by Dunkl in [7] and recurrence relations for the classical Laguerre polynomial. Our proof is quite similar to that of Rösler but without using any recurrence relations.

The Dunkl Hermite functions $\{h^k_n\}_{n\in\mathbb{N}^d}$ associated with G and k, introduced by Rösler in [18], are defined by

$$h^k_n(x) = (c_k 2^{-|n|}\, e^{-|x|^2})^{1/2} H^k_n(x), \quad x \in \mathbb{R}^d,$$

where H^k_n represents the Dunkl Hermite polynomials of degree $|n|$, with real coefficients.

It is well known (see [18]) that the sequence $\{h^k_n\}_{n\in\mathbb{N}^d}$ is an orthonormal basis for $L^2_k(\mathbb{R}^d)$ and h^k_n is an eigenfunction for the Dunkl transform associated to the eigenvalue $(-1)^{|n|}$, that is,

$$\mathcal{F}_k(h^k_n) = (-1)^{|n|} h^k_n, \quad n \in \mathbb{N}^d.$$

Now if we denote by $\Delta_k = -\sum_{j=1}^d T_j^2$ the Dunkl Laplacian, then the h^k_n form the family of eigenfunctions of the Dunkl Hermite operator (or Dunkl harmonic oscillator) $\mathcal{L}_k = \Delta_k + |x|^2$ with corresponding eigenvalues $2|n| + 2\gamma + d$, that is,

$$\mathcal{L}_k h^k_n = (2|n| + 2\gamma + d) h^k_n, \quad n \in \mathbb{N}^d.$$

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Moreover, for sufficiently regular functions f,

$$\mathcal{F}_k(\Delta_k f)(\xi) = |\xi|^2 \mathcal{F}_k(f)(\xi), \quad \xi \in \mathbb{R}^d,$$

so that we can define the nonnegative self-adjoint extension of Δ_k (still denoted by the same symbol) defined by

$$\Delta_k f = \mathcal{F}_k^{-1}[|\xi|^2 \mathcal{F}_k(f)], \quad f \in \mathrm{Dom}(\Delta_k), \tag{3.3}$$

where $\mathrm{Dom}(\Delta_k) = \{f \in L_k^2(\mathbb{R}^d) : |\xi|^2 \mathcal{F}_k(f) \in L_k^2(\mathbb{R}^d)\}$.

The Dunkl Hermite operator $\mathcal{L}_k$ is symmetric and positive in $L_k^2(\mathbb{R}^d)$ and it has a natural self-adjoint extension on $L_k^2(\mathbb{R}^d)$, still denoted by the same symbol $\mathcal{L}_k$, whose spectral decomposition is discrete and is given by

$$\mathcal{L}_k f = \sum_{n \in \mathbb{N}^d} (2|n| + 2\gamma + d) \langle f, h_n^k \rangle_k h_n^k = \sum_{m=0}^{\infty} (2m + 2\gamma + d) \mathcal{P}_m^k f \tag{3.4}$$

on the domain $\mathrm{Dom}\mathcal{L}_k$ consisting of all functions $f \in L_k^2(\mathbb{R}^d)$ for which the defining series converges in $L_k^2(\mathbb{R}^d)$. Here $\mathcal{P}_m^k$ are the spectral projections

$$\mathcal{P}_m^k f = \sum_{|n|=m} \langle f, h_n^k \rangle_k h_n^k,$$

and $\langle \cdot, \cdot \rangle_k$ is the usual inner product in the Hilbert space $L_k^2(\mathbb{R}^d)$.

From this it immediately follows that, for each f in the domain of $\mathcal{L}_k$,

$$\langle \mathcal{L}_k f, f \rangle_k = \sum_{n \in \mathbb{N}^d} (2|n| + 2\gamma + d) |\langle f, h_n^k \rangle_k|^2.$$

Theorem 3.1. *For every $f \in L_k^2(\mathbb{R}^d)$,*

$$\| \, |x| f \|_{L_k^2}^2 + \| \, |\xi| \mathcal{F}_k(f) \|_{L_k^2}^2 \ge (2\gamma + d) \|f\|_{L_k^2}^2,$$

with equality if and only if $f(x) = c e^{-|x|^2/2}$ for some $c \in \mathbb{C}$.

Proof. Let $f \in L_k^2(\mathbb{R}^d)$ be a nonzero function such that

$$\| \, |x| \, f \|_{L_k^2}, \quad \| \, |\xi| \, \mathcal{F}_k(f) \|_{L_k^2} < \infty.$$

Then from (3.3) and Parseval's equality for the Dunkl transform,

$$\| \, |\xi| \mathcal{F}_k(f) \|_{L_k^2}^2 = \langle |\xi|^2 \mathcal{F}_k(f), \mathcal{F}_k(f) \rangle_k = \langle \mathcal{F}_k(\Delta_k f), \mathcal{F}_k(f) \rangle_k = \langle \Delta_k f, f \rangle_k.$$

Thus

$$\| \, |x| f \|_{L_k^2}^2 + \| \, |\xi| \mathcal{F}_k(f) \|_{L_k^2}^2 = \langle |x|^2 f, f \rangle_k + \langle \Delta_k f, f \rangle_k = \langle \mathcal{L}_k f, f \rangle_k.$$

It follows by (3.4) that the self-adjoint operator $\mathcal{L}_k$ has only discrete spectra, of which the minimum is $(2\gamma + d)$. Therefore

$$\| \, |x| f \|_{L_\alpha^2}^2 + \| \, |\xi| \mathcal{F}_k(f) \|_{L_k^2}^2 \ge (2\gamma + d) \|f\|_{L_k^2}^2.$$

Further, the equality holds if and only if f is an eigenfunction of $\mathcal{L}_k$ corresponding to the minimum eigenvalue $(2\gamma + d)$, namely f is a scalar multiple of h_0^k, which is a constant multiple of $e^{-|x|^2/2}$. □

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A simple well-known dilation argument allows us to obtain the following corollary (see [17, Proof of Theorem 1.1]).

COROLLARY 3.2. *For every* $f \in L_k^2(\mathbb{R}^d)$,

$$\| |x| f\|_{L_k^2} \| |\xi| \mathcal{F}_k(f)\|_{L_k^2} \geq (\gamma + d/2)\|f\|_{L_k^2}^2,$$

with equality if and only if $f(x) = ce^{-\mu|x|^2/2}$ *for some* $c \in \mathbb{C}$ *and* $\mu > 0$.

If the multiplicity function k is identically 0, then the above inequality coincides with the Heisenberg inequality (3.1).

3.2. Strong uncertainty principle for orthonormal bases. In this section we will prove a strong uncertainty principle for orthonormal bases for $L_k^2(\mathbb{R}^d)$ which shows that the Heisenberg inequality (3.2) for the Dunkl transform can be refined for an orthonormal basis. Our proof is inspired by Malinnikova [14] who proved a similar result in the classical setting. In order to do this, we will need to introduce the time-limiting and the frequency-limiting operators on $L_k^2(\mathbb{R}^d)$ defined by

$$E_S f = \chi_S f, \quad F_\Sigma f = \mathcal{F}_k^{-1}[\chi_\Sigma \mathcal{F}_k(f)],$$

where S and Σ are measurable subsets of $\mathbb{R}^d$ of finite measure, $0 < \mu_k(S), \mu_k(\Sigma) < \infty$.

A straightforward computation shows that $E_S F_\Sigma$ is an integral operator with kernel

$$\mathcal{N}(x, \xi) = c_k \chi_S(x) \mathcal{F}_k(\chi_\Sigma \mathcal{K}_k(ix, \cdot))(\xi).$$

Thus $E_S F_\Sigma$ is a Hilbert–Schmidt operator with (see, for example, [9, Lemma 3.2]),

$$\|E_S F_\Sigma\|_{HS}^2 \leq c_k^2 \mu_k(S)\mu_k(\Sigma). \tag{3.5}$$

The phase space restriction operator is defined by

$$L_{S,\Sigma} = (E_S F_\Sigma)^* E_S F_\Sigma = F_\Sigma E_S F_\Sigma,$$

where $(E_S F_\Sigma)^* = F_\Sigma E_S$.

An elementary calculation of the trace of the self-adjoint operator $L_{S,\Sigma}$ allows as to obtain the following localisation inequality.

THEOREM 3.3. *Let* $\{\varphi_n\}_{n=1}^N$ *be an orthonormal system in* $L_k^2(\mathbb{R}^d)$. *If*

$$\|E_{S^c}\varphi_n\|_{L_k^2}^2 \leq a_n^2 \quad \text{and} \quad \|F_{\Sigma^c}\varphi_n\|_{L_k^2}^2 \leq b_n^2,$$

then

$$\sum_{n=1}^N \Big(1 - \frac{3}{2}a_n - \frac{3}{2}b_n\Big) \leq c_k^2 \mu_k(S)\mu_k(\Sigma).$$

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Proof. We will apply a standard estimate of the trace (see, for example, [10, Theorem 5.6, page 63]) of the time–frequency restriction operator $L_{S,\Sigma}$ to conclude that

$$\operatorname{tr}(L_{S,\Sigma}) = \|E_S F_\Sigma\|_{HS}^2.$$

Then by means of relation (3.5),

$$\sum_{n=1}^{N} \langle L_{S,\Sigma}\varphi_n, \varphi_n\rangle_k \le \operatorname{tr}(L_{S,\Sigma}) \le c_k^2\, \mu_k(S)\mu_k(\Sigma).$$

On the other hand, as the identity operator $I = E_S + E_{S^c} = F_\Sigma + F_{\Sigma^c}$, so

$$\begin{aligned}\langle L_{S,\Sigma}\varphi_n, \varphi_n\rangle_k &= \langle E_S F_\Sigma\varphi_n, F_\Sigma\varphi_n\rangle_k \\ &= \langle \varphi_n, \varphi_n\rangle_k - \langle F_{\Sigma^c}\varphi_n, \varphi_n\rangle_k - \langle F_\Sigma\varphi_n, E_{S^c}\varphi_n\rangle_k - \langle E_S F_\Sigma\varphi_n, F_{\Sigma^c}\varphi_n\rangle_k.\end{aligned}$$

Therefore, $\langle L_{S,\Sigma}\varphi_n, \varphi_n\rangle_k \ge 1 - a_n - 2b_n$ and

$$\sum_{n=1}^{N} (1 - a_n - 2b_n) \le c_k^2\, \mu_k(S)\mu_k(\Sigma). \tag{3.6}$$

If we consider the operator $\tilde{L}_{S,\Sigma} = (F_\Sigma E_S)^* F_\Sigma E_S = E_S F_\Sigma E_S$, we similarly obtain

$$\sum_{n=1}^{N} (1 - 2a_n - b_n) \le c_k^2\, \mu_k(S)\mu_k(\Sigma). \tag{3.7}$$

Combining (3.6) and (3.7), we deduce the desired result. □

Definition 3.4. Let $0 < \varepsilon < 1$ and $f \in L_k^2(\mathbb{R}^d)$. Then:

(1) f is ε-concentrated on S if $\|E_{S^c} f\|_{L_k^2} \le \varepsilon \|f\|_{L_k^2}$,
(2) f is ε-bandlimited on Σ if $\|F_{\Sigma^c} f\|_{L_k^2} \le \varepsilon \|f\|_{L_k^2}$.

It is clear that if f is ε-bandlimited on Σ then, by the Plancherel theorem (2.1), $\mathcal{F}_k(f)$ is ε-concentrated on Σ.

From Theorem 3.3, we can immediately obtain the following corollary.

Corollary 3.5. *Let $a, b > 0$ and $0 < \varepsilon_1, \varepsilon_2 < 1$ such that $\varepsilon_1 + \varepsilon_2 < \frac{2}{3}$. Let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system in $L_k^2(\mathbb{R}^d)$. If φ_n is ε_1-concentrated on $\mathcal{B}(0, a)$ and ε_2-bandlimited on $\mathcal{B}(0, b)$, then*

$$N \le \left(\frac{c_k d_k}{2\gamma + d}\right)^2 \frac{(ab)^{2\gamma+d}}{1 - \frac{3}{2}(\varepsilon_1 + \varepsilon_2)}.$$

Therefore if the generalised dispersions of the elements of an orthonormal sequence are uniformly bounded then this sequence is finite and we can give a bound on the number of elements in that sequence. More precisely, we have the following result.

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Corollary 3.6. *Fix $A, B > 0$. Let $s > 0$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$ that satisfies $\| |x|^s \varphi_n\|_{L_k^2}^{1/s} \le A$ and $\| |\xi|^s \mathcal{F}_k(\varphi_n)\|_{L_k^2}^{1/s} \le B$. Then the sequence is finite, that is,*

$$N \le \Big(\frac{2^{((2\gamma+d)/s)+1}}{2\gamma+d} c_k\, d_k\Big)^2 (AB)^{2\gamma+d}.$$

Proof. Since, for any $r > 0$,

$$\|E_{\mathcal{B}(0,r)^c}\varphi_n\|_{L_k^2} \le r^{-s}\| |x|^s\varphi_n\|_{L_k^2},$$

it follows that

$$\|E_{\mathcal{B}(0,4^{1/s}A)^c}\varphi_n\|_{L_k^2} \le \frac{1}{4A^s}\| |x|^s\varphi_n\|_{L_k^2} \le \frac{1}{4}.$$

In the same way we get

$$\|E_{\mathcal{B}(0,4^{1/s}B)^c}\mathcal{F}_k(\varphi_n)\|_{L_k^2} \le \tfrac{1}{4}.$$

Thus φ_n is $\frac{1}{4}$-concentrated on $\mathcal{B}(0, 4^{1/s}A)$ and $\frac{1}{4}$-bandlimited on $\mathcal{B}(0, 4^{1/s}B)$. The desired result follows from Corollary 3.5. □

Lemma 3.7. *Let S and Σ be measurable subsets of finite measure $\mu_k(S), \mu_k(\Sigma) < \infty$. Then there exists a nonzero function $f \in L_k^2(\mathbb{R}^d)$ such that* $\operatorname{supp} f \subset S^c$ *and* $\operatorname{supp} \mathcal{F}_k(f) \subset \Sigma^c$.

Proof. Let $PW(\Sigma)$ be the space of functions $f \in L_k^2(\mathbb{R}^d)$ such that $\mathcal{F}_k(f)$ is supported on Σ^c. Then from [9, Theorem 4.4(2)], there exists a positive constant $C_k(S,\Sigma)$ such that for all functions $f \in PW(\Sigma)$,

$$\|f\|_{L_k^2} \le C_k(S,\Sigma)\|f\|_{L_k^2(S^c)}.$$

Therefore the trace space $\Lambda = \{f|_{S^c} : f \in PW(\Sigma)\}$ forms a closed subspace in $L_k^2(S^c)$ which is obviously not the whole space. Let g be a nonzero function in $\Lambda^c = L_k^2(S^c)\backslash\Lambda$. Since $g = F_\Sigma g + F_{\Sigma^c} g$, we have that $f = F_{\Sigma^c} g$ is a nonzero function in $L_k^2(\mathbb{R}^d)$ such that f is supported on S^c and $\mathcal{F}_k(f)$ is supported on Σ^c. We extend f by zero on S in order to get the required function. □

Theorem 3.8. *Let $s > 0$ and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for $L_k^2(\mathbb{R}^d)$. Then*

$$\sup_n(\| |x|^s\varphi_n\|_{L_k^2}\, \| |\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}) = \infty.$$

Proof. Assume that there exists an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ such that

$$\| |x|^s\varphi_n\|_{L_k^2}^{1/s}\, \| |\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^{1/s} \le C^2.$$

Let $j \in \mathbb{Z}$ and let

$$A_k = \{\varphi_n : \| |x|^s\varphi_n\|_{L_k^2}^{1/s} \in (2^{-j}C, 2^{-j+1}C]\}.$$

Clearly, $\{\varphi_n\}_{n=1}^\infty = \bigcup_j A_j$, and for $\varphi_n \in A_j$, we have

$$\| |x|^s\varphi_n\|_{L_k^2}^{1/s} \le 2^{-j+1}C \quad \text{and} \quad \| |\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^{1/s} \le C2^j.$$

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By Corollary 3.6, A_j is finite and, if N_j is the number of elements in A_j, then N_j is bounded by a constant $c_{k,s}$ that does not depend on j.

Let $r > 0$. By Lemma 3.7, there is a nonzero function $f \in L^2_k(\mathbb{R}^d)$ with $\|f\|_{L^2_k} = 1$, that vanishes on $\mathcal{B}(0, r)$ with its Dunkl transform. Then, for $j \geq 0$ and $\varphi_n \in A_j$, the Cauchy–Schwartz inequality gives

$$|\langle f, \varphi_n\rangle_k|^2 \leq r^{-2s}\|f\|^2_{L^2_k}\|\,|x|^s\varphi_n\|^2_{L^2_k} \leq (2Cr^{-1})^{2s}4^{-sj}. \tag{3.8}$$

Similarly, for $j < 0$ and $\varphi_n \in A_j$, Parseval's theorem for the Dunkl transform gives

$$|\langle f, \varphi_n\rangle_k|^2 = |\langle \mathcal{F}_k(f), \mathcal{F}_k(\varphi_n)\rangle_k|^2 \leq r^{-2s}\|f\|^2_{L^2_k}\|\,|\xi|^s\mathcal{F}_k(\varphi_n)\|^2_{L^2_k} \leq (Cr^{-1})^{2s}4^{sj}. \tag{3.9}$$

Since $\{\varphi_n\}_{n=1}^\infty$ is a basis for $L^2_k(\mathbb{R}^d)$,

$$1 = \|f\|^2_{L^2_k} = \sum_j \sum_{\varphi_n \in A_j} |\langle f, \varphi_n\rangle_k|^2,$$

and, by combining inequalities (3.8) and (3.9), we obtain

$$\begin{aligned}
1 &\leq (2Cr^{-1})^{2s}\sum_{j=0}^\infty 4^{-sj}N_j + (Cr^{-1})^{2s}\sum_{j=1}^\infty 4^{-sj}N_{-j}\\
&\leq c_{k,s}(2Cr^{-1})^{2s}\sum_{j=0}^\infty 4^{-sj} + c_{k,s}(Cr^{-1})^{2s}\sum_{j=1}^\infty 4^{-sj}\\
&\leq \frac{4c_{k,s}(2C)^{2s}}{3r^{2s}}.
\end{aligned}$$

Choosing r large enough, we get a contradiction. The theorem is proved. □

REMARK 3.9. There is an infinite orthonormal sequence $\{\varphi_n\}_{n=1}^\infty$ in $L^2_k(\mathbb{R}^d)$ with bounded product of dispersions. Indeed, fix $\phi : \mathbb{R}^d \to \mathbb{R}$ a radial, real-valued Schwartz function supported in $\mathcal{B}(0, 2)\backslash\mathcal{B}(0, 1)$ with $\|\phi\|_{L^2_k} = 1$. Consider $\varphi_n(x) = 2^{n(\gamma+d/2)}\phi(2^n x)$. Then

$$\|\varphi_n\|_{L^2_k} = \|\phi\|_{L^2_k}, \quad \operatorname{supp}\varphi_n \subset \mathcal{B}(0, 2^{-n+1})\backslash\mathcal{B}(0, 2^{-n}) \quad \text{and}$$
$$\mathcal{F}_k(\varphi_n)(\xi) = 2^{-n(\gamma+d/2)}\mathcal{F}_k(\phi)(2^{-n}\xi).$$

Therefore, $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal sequence in $L^2_k(\mathbb{R}^d)$ and, for every $s > 0$,

$$\|\,|x|^s\varphi_n\|_{L^2_k} = 2^{-ns}\|\,|x|^s\phi\|_{L^2_k}, \quad \|\,|\xi|^s\mathcal{F}_k(\varphi_n)\|_{L^2_k} = 2^{ns}\|\,|\xi|^s\mathcal{F}_k(\phi)\|_{L^2_k}.$$

Hence, for all n,

$$\|\,|x|^s\varphi_n\|_{L^2_k}\,\|\,|\xi|^s\mathcal{F}_k(\varphi_n)\|_{L^2_k} = \|\,|x|^s\phi\|_{L^2_k}\|\,|\xi|^s\mathcal{F}_k(\phi)\|_{L^2_k} < \infty.$$

3.3. Quantitative dispersion inequality for orthonormal sequences. In this section we will prove Theorem B. To do so let us recall the general form of the Heisenberg-type uncertainty inequality for the Dunkl transform (see [9, Theorem 4.4, (3)] or [22]).

THEOREM 3.10. *Let $s > 0$. Then there exists a constant $C_{s,k}$ such that for all $f \in L^2_k(\mathbb{R}^d)$,*

$$\| |x|^s f\|_{L^2_k} \| |\xi|^s \mathcal{F}_k(f)\|_{L^2_k} \geq C_{s,k} \|f\|^2_{L^2_k}. \tag{3.10}$$

Inequality (3.10) is equivalent to

$$\| |x|^s f\|^2_{L^2_k} + \| |\xi|^s \mathcal{F}_k(f)\|^2_{L^2_k} \geq 2C_{s,k} \|f\|^2_{L^2_k}.$$

Consequently, we immediately obtain the following result.

COROLLARY 3.11. *Let $s > 0$ and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. Then there exists $j_0 \in \mathbb{Z}$ such that,*

$$\forall n \geq 1, \quad \max(\| |x|^s \varphi_n\|_{L^2_k}, \| |\xi|^s \mathcal{F}_k(\varphi_n)\|_{L^2_k}) \geq 2^{s(j_0-1)}. \tag{3.11}$$

THEOREM 3.12. *Let $s > 0$ and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. Then for every $N \geq 1$,*

$$\sum_{n=1}^N (\| |x|^s \varphi_n\|^2_{L^2_k} + \| |\xi|^s \mathcal{F}_k(\varphi_n)\|^2_{L^2_k}) \geq \left(\frac{(2\gamma + d)^2 (4^{2\gamma+d} - 1)}{2^{(2\gamma+d)(4+3/s)+3} c_k^2 d_k^2} \right)^{s/(2\gamma+d)} N^{1+s/(2\gamma+d)}.$$

PROOF. For each $j \in \mathbb{Z}$, we define

$$P_j = \{n : \max(\| |x|^s \varphi_n\|^{1/s}_{L^2_k}, \| |\xi|^s \mathcal{F}_k(\varphi_n)\|^{1/s}_{L^2_k}) \in [2^{j-1}, 2^j)\}.$$

First, by inequality (3.11), we see that P_j is empty for all $j < j_0$. Moreover, since for each $n \in P_j$ ($j \geq j_0$),

$$\| |x|^s \varphi_n\|^{1/s}_{L^2_k} \leq 2^j \quad \text{and} \quad \| |\xi|^s \mathcal{F}_k(\varphi_n)\|^{1/s}_{L^2_k} \leq 2^j,$$

P_j is finite for all $j \geq j_0$, by Corollary 3.6. If we denote by N_j the number of elements in P_j then

$$N_j \leq \left(\frac{2^{(2\gamma+d)/s+1}}{2\gamma + d} c_k\, d_k \right)^2 4^{j(2\gamma+d)}.$$

Therefore, for every $m \geq j_0$, the number of elements in $\bigcup_{j=j_0}^m P_j$ is less than $c_{k,s}\, 4^{m(2\gamma+d)}$, where

$$c_{k,s} = \left(\frac{2^{(2\gamma+d)(1+1/s)+1}}{(2\gamma + d) \sqrt{4^{2\gamma+d} - 1}} c_k\, d_k \right)^2$$

is a constant that does not depend on m.

Now if $N > 2c_{k,s} 4^{j_0(2\gamma+d)}$, then we can choose an integer $m > j_0$ such that

$$2c_{k,s} 4^{(m-1)(2\gamma+d)} < N \leq 2c_{k,s}\, 4^{m(2\gamma+d)}.$$

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Therefore, at least half of $1,\dots,N$ do not belong to $\bigcup_{j=j_0}^{m-1} P_j$ and we obtain

$$\begin{aligned}\sum_{n=1}^{N}(\||x|^s\varphi_n\|_{L_k^2}^2+\||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2) &\geq \sum_{n=1}^{N}\max(\||x|^s\varphi_n\|_{L_k^2}^2,\||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2)\\ &\geq \frac{N}{2}4^{s(m-1)}\\ &\geq \frac{1}{2}\frac{N}{4^s}\Big(\frac{N}{2c_{k,s}}\Big)^{s/(2\gamma+d)}\\ &= \frac{1}{2}\Big(\frac{(2\gamma+d)^2(4^{2\gamma+d}-1)}{2^{(2\gamma+d)(4+2/s)+3}c_k^2d_k^2}\Big)^{s/(2\gamma+d)}N^{1+s/(2\gamma+d)}.\end{aligned}$$

Finally, if $N\leq 2c_{k,s}4^{j_0(2\gamma+d)}$, then from Corollary 3.11,

$$\begin{aligned}\sum_{n=1}^{N}(\||x|^s\varphi_n\|_{L_k^2}^2+\||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2) &\geq \sum_{n=1}^{N}\max(\||x|^s\varphi_n\|_{L_k^2}^2,\||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2)\\ &\geq N4^{s(j_0-1)}\\ &\geq \frac{N}{4^s}\Big(\frac{N}{2c_{k,s}}\Big)^{s/(2\gamma+d)}\\ &= \Big(\frac{(2\gamma+d)^2(4^{2\gamma+d}-1)}{2^{(2\gamma+d)(4+2/s)+3}c_k^2d_k^2}\Big)^{s/(2\gamma+d)}N^{1+s/(2\gamma+d)}.\end{aligned}$$

This completes the proof. □

The last dispersion inequality implies in particular that there does not exist an infinite sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $L_k^2(\mathbb{R}^d)$ such that the two sequences $\{\||x|^s\varphi_n\|_{L_k^2}\}_{n=1}^{\infty}$ and $\{\||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}\}_{n=1}^{\infty}$ are bounded. More precisely, we have the following corollary.

Corollary 3.13. *Let $s>0$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. Then for every $N\geq 1$,*

$$\sup_{1\leq n\leq N}\{\||x|^s\varphi_n\|_{L_k^2}^2,\ \||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2\}\geq \Big(\frac{(2\gamma+d)^2(4^{2\gamma+d}-1)}{2^{(2\gamma+d)(4+4/s)+3}d_k^2}\Big)^{s/(2\gamma+d)}N^{s/(2\gamma+d)}.$$

In particular,

$$\sup_n(\||x|^s\varphi_n\|_{L_k^2}^2+\||\xi|^s\mathcal{F}_k(\varphi_n)\|_{L_k^2}^2)=\infty.$$

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