

Kolmogorov, Linear and Pseudo-Dimensional Widths of Classes of s -Monotone Functions in \mathbb{L}_p , $0 < p < 1$

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Abstract. Let B_p be the unit ball in \mathbb{L}_p , $0 < p < 1$, and let Δ_+^s , $s \in \mathbb{N}$, be the set of all s -monotone functions on a finite interval I , i.e., Δ_+^s consists of all functions $x: I \mapsto \mathbb{R}$ such that the divided differences $[x; t_0, \dots, t_s]$ of order s are nonnegative for all choices of $(s + 1)$ distinct points $t_0, \dots, t_s \in I$. For the classes $\Delta_+^s B_p := \Delta_+^s \cap B_p$, we obtain exact orders of Kolmogorov, linear and pseudo-dimensional widths in the spaces \mathbb{L}_q , $0 < q < p < 1$:

$$d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{psd}} \asymp d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{kol}} \asymp d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{lin}} \asymp n^{-s}.$$

1 Introduction, Preliminaries, and the Main Result

The general theory of widths deals with approximation of infinite-dimensional function classes by finite-dimensional manifolds which are optimal in a certain sense. It has numerous applications in numerical analysis (what is the optimal rate of convergence attainable by any numerical method for a specific problem?), learning theory (among various hypothesis classes what can be considered optimal?), image compression (what is the best compression ratio?), and many other areas.

Let X be a real linear space of vectors x with (quasi)norm $\|x\|_X$, and W and M be nonempty subsets of X . The *deviation* of W from M is defined by

$$E(W, M)_X := \sup_{x \in W} \inf_{y \in M} \|x - y\|_X.$$

The *Kolmogorov n -width* of W is defined by

$$d_n(W)_X^{\text{kol}} := \inf_{M^n} E(W, M^n)_X, \quad n \geq 0,$$

where the infimum is taken over all affine subsets $M^n \subseteq X$ of dimension $\leq n$.

The *linear n -width* of W is defined by

$$d_n(W)_X^{\text{lin}} := \inf_{M^n} \inf_A \sup_{x \in W} \|x - Ax\|_X, \quad n \geq 0,$$

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where the left-hand infimum is taken over all affine subsets $M^n \subseteq X$ of dimension at most n , and the middle infimum is taken over all continuous affine maps A from affine subsets of X containing W into M^n .

Finally, we will also have estimates for yet another width, the *pseudo-dimensional width* which was introduced by Maiorov and Ratsaby [11, 12, 15] using the concept of pseudo-dimension due to Pollard [13]. Namely, let $M = M(T)$ be a set of real-valued functions $x(\cdot)$ defined on the set T , and denote

$$\text{Sgn}(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a \leq 0. \end{cases}$$

The *pseudo-dimension* $\text{dim}_{ps}(M)$ of the set M is the largest integer n such that there exist points $t_1, \dots, t_n \in T$ and a vector $(y_1, \dots, y_n) \in \mathbb{R}^n$ for which

$$\text{card}\{(\text{Sgn}(x(t_1) + y_1), \dots, \text{Sgn}(x(t_n) + y_n)) \mid x \in M\} = 2^n.$$

If n can be arbitrarily large, then $\text{dim}_{ps}(M) := \infty$.

The *pseudo-dimensional n -width* of W is defined by

$$d_n(W)_X^{\text{psd}} := \inf_{M^n} \sup_{x \in W} \inf_{y \in M^n} \|x - y\|_X,$$

where the left-hand infimum is taken over all subsets $M^n \subseteq X$ such that $\text{dim}_{ps}(M^n) \leq n$. It is known (see [6]) that if M is an arbitrary affine subset in a space X of real-valued functions and $\text{dim } M < \infty$, then $\text{dim}_{ps}(M) = \text{dim}(M)$. Clearly,

$$(1.1) \quad d_n(W)_X^{\text{psd}} \leq d_n(W)_X^{\text{kol}} \leq d_n(W)_X^{\text{lin}}.$$

Let $s \in \mathbb{N}$, and $\Delta_+^s := \Delta_+^s(I)$ be the set of functions $x: I \mapsto \mathbb{R}$ on a finite interval I such that the divided differences $[x; t_0, \dots, t_s]$ of order s of x are nonnegative for all choices of $(s + 1)$ distinct points $t_0, \dots, t_s \in I$. We call functions $x \in \Delta_+^s$ *s -monotone* on I .

It is well known (see [1, 14, 16]) that if x is s -monotone on $[a, b]$, $s \geq 2$, then $x^{(\nu)}$ exists on (a, b) for $\nu \leq s - 2$, and, in fact, $x^{(\nu)} \in \Delta_+^{s-\nu}(a, b)$. In particular, $x^{(s-2)}$ exists, is convex, and therefore is locally absolutely continuous in (a, b) , and has left and right (nondecreasing) derivatives $x_-^{(s-1)}$ and $x_+^{(s-1)}$ there. Moreover, the set E where $x^{(s-1)}$ fails to exist is countable, and $x^{(s-1)}$ is continuous on $(a, b) \setminus E$. Throughout this paper, if a function $x: I \mapsto \mathbb{R}$ possesses both the left-hand and the right-hand derivatives $x_-^{(k)}(t)$ and $x_+^{(k)}(t)$, of order $k \in \mathbb{N}$, at a point $t \in I$, then we denote $x^{(k)}(t) := (x_-^{(k)}(t) + x_+^{(k)}(t))/2$. Evidently, this notation is compatible with that of the derivative $x^{(k)}(t)$ if it exists. We also write $x^{(0)}(t) := x(t)$, $t \in I$.

Given a function space X and $W \subseteq X$, we denote by $\Delta_+^s W$ the subset of s -monotone functions $x \in W$, i.e., $\Delta_+^s W := \Delta_+^s \cap W$.

Let $\mathbb{L}_p := \mathbb{L}_p(I)$, $0 < p \leq \infty$, be the space of all functions x on I with (quasi)norm $\|x\|_{\mathbb{L}_p(I)} := (\int_I |x(t)|^p dt)^{1/p}$. By $B_p := B_p(I)$ we denote the unit ball in \mathbb{L}_p . Evidently

$\Delta_+^s B_p \subset \mathbb{L}_q, 0 < q \leq p \leq \infty$, but $\Delta_+^s B_p \not\subset \mathbb{L}_q, 0 < p < q \leq \infty$. For the integers $r \in \mathbb{N}$, we denote the Sobolev classes

$$\mathbb{W}_p^r := \mathbb{W}_p^r(I) := \{x \mid x^{(r-1)} \in AC_{loc}, \|x^{(r)}\|_{\mathbb{L}_p} \leq 1\}, \quad 0 < p \leq \infty.$$

Recall that for $1 \leq p, q \leq \infty$, the orders of most widths of the classical Sobolev classes \mathbb{W}_p^r in \mathbb{L}_q are well known. They are asymptotically $n^{-r+\alpha(p,q)}$, where $0 \leq \alpha(p, q) \leq 1/2$. In contrast, for $0 < p < 1$ the behavior of the widths differs essentially. Let

$$\mathbb{W}_{p,\infty}^r := \{x \mid x \in \mathbb{W}_p^r, \|x\|_{\mathbb{L}_\infty} \leq 1\}, \quad 0 < p < 1.$$

It was proved in [3] that if $0 < p < 1$, then

$$d_n(\mathbb{W}_{p,\infty}^r)_{\mathbb{L}_q}^{\text{psd}} \asymp d_n(\mathbb{W}_{p,\infty}^r)_{\mathbb{L}_q}^{\text{kol}} \asymp d_n(\mathbb{W}_{p,\infty}^r)_{\mathbb{L}_q}^{\text{lin}} \asymp 1, \quad 0 < q \leq \infty.$$

The aim of this paper is to show that s -monotone functions $x \in \Delta_+^s B_p, 0 < p < 1$, can be approximated well in $\mathbb{L}_q, 0 < q < p < 1$. Our main result is:

Theorem 1 *If $s, n \in \mathbb{N}$ and $0 < q < p < 1$ then*

$$(1.2) \quad d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{psd}} \asymp d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{kol}} \asymp d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{lin}} \asymp n^{-s}, \quad n \geq s.$$

2 Upper Bounds

This section is devoted to proving that a function $x \in \Delta_+^s B_p, 0 < p < 1$, can be well approximated in $\mathbb{L}_q, 0 < q < p$, by piecewise polynomials associated with it, in a linear fashion. Specifically, we will show that for $x \in \Delta_+^s \mathbb{L}_p$ there is a piecewise polynomial $\sigma_{s,n}(\cdot; x; I)$ with $2n - 2$ prescribed knots (see construction below), such that

$$(2.1) \quad \|x(\cdot) - \sigma_{s,n}(\cdot; x; I)\|_{\mathbb{L}_q(I)} \leq c \|x\|_{\mathbb{L}_p(I)} n^{-s}, \quad n \geq 1, \quad 0 < q < p < 1,$$

where $c = c(s, p, q)$.

2.1 Auxiliary Results

The following lemma is due to Bullen [1] (see also [9, Lemma 8.3] for discussion of the cases when some or all interpolation points coincide).

Lemma 1 *Let $s \in \mathbb{N}$, $f \in \Delta_+^s(a, b)$, and let $L_{s-1}(f, t; t_1, \dots, t_k)$ be the Lagrange (Hermite–Taylor) polynomial of degree $\leq s - 1$ interpolating f at the points $t_i, 1 \leq i \leq s$, where $a =: t_0 < t_1 \leq \dots \leq t_s < t_{s+1} := b$. Then*

$$(-1)^{s-i} (f(t) - L_{s-1}(f, t; t_1, \dots, t_s)) \geq 0, \quad t \in (t_i, t_{i+1}), \quad i = 0, \dots, s.$$

In other words, $f - L_{s-1}$ changes sign at t_1, \dots, t_s .

Lemma 2 Let $n \in \mathbb{N}$, $0 < p < 1$, and let $(\alpha_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with nonnegative entries. Then

$$\sum_{j=1}^n \left(\sum_{i=1}^n \alpha_{ij}^p \right)^{1/p} \leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} \right)^p \right)^{1/p}.$$

Proof It is well known (see, e.g., [5, (2.11.5)]) and easy to prove that, in the case $0 < p < 1$,

$$\|\mathbf{x} + \mathbf{y}\|_{l_p^n} \geq \|\mathbf{x}\|_{l_p^n} + \|\mathbf{y}\|_{l_p^n}, \quad \text{for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n.$$

Therefore, for any $\mathbf{x}_j \in \mathbb{R}_+^n$, $1 \leq j \leq n$, we have

$$\left\| \sum_{j=1}^n \mathbf{x}_j \right\|_{l_p^n} \geq \sum_{j=1}^n \|\mathbf{x}_j\|_{l_p^n},$$

and hence choosing $\mathbf{x}_j = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{nj})$, $1 \leq j \leq n$, we have

$$\begin{aligned} \left(\sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} \right)^p \right)^{1/p} &= \left\| \left(\sum_{j=1}^n \alpha_{1j}, \sum_{j=1}^n \alpha_{2j}, \dots, \sum_{j=1}^n \alpha_{nj} \right) \right\|_{l_p^n} = \left\| \sum_{j=1}^n \mathbf{x}_j \right\|_{l_p^n} \\ &\geq \sum_{j=1}^n \|\mathbf{x}_j\|_{l_p^n} = \sum_{j=1}^n \left(\sum_{i=1}^n \alpha_{ij}^p \right)^{1/p}. \end{aligned}$$

Lemma 3 Let $n \in \mathbb{N}$, $0 < p, q < 1$, and suppose that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are n -tuples with nonnegative entries, and $\mathbf{c} := (c_{ij})_{i,j=1}^n$ is a nonnegative $n \times n$ matrix. Given a function

$$f_{q,n}(\mathbf{w}; \mathbf{a}) := \left(\sum_{i=1}^n (a_i \omega_i)^q \right)^{1/q},$$

where $\mathbf{w} := (\omega_1, \dots, \omega_n) \in \mathbb{R}_+^n$, and the set

$$\Omega_p^n(\mathbf{b}; \mathbf{c}) := \mathbb{R}_+^n \cap \left\{ \mathbf{w} : \sum_{i=1}^n \left(b_i \sum_{j=1}^n c_{ij} \omega_j \right)^p \leq 1 \right\},$$

we have

$$\max_{\mathbf{w} \in \Omega_p^n(\mathbf{b}; \mathbf{c})} f_{q,n}(\mathbf{w}; \mathbf{a}) \leq n^{1/q-1} \max_{1 \leq j \leq n} \left\{ a_j \left(\sum_{i=1}^n (b_i c_{ij})^p \right)^{-1/p} \right\}.$$

Proof By Jensen’s inequality, for every $\mathbf{w} \in \mathbb{R}_+^n$, we have

$$f_{q,n}(\mathbf{w}; \mathbf{a}) = \left(\sum_{i=1}^n (a_i \omega_i)^q \right)^{1/q} \leq n^{1/q-1} \sum_{i=1}^n a_i \omega_i =: g_{q,n}(\mathbf{w}; \mathbf{a}).$$

It follows from Lemma 2 that

$$\sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} \right)^p \leq 1 \quad \Rightarrow \quad \sum_{j=1}^n \left(\sum_{i=1}^n \alpha_{ij}^p \right)^{1/p} \leq 1.$$

Therefore, taking $\alpha_{ij} := b_i c_{ij} \omega_j$, we conclude that the set $\Omega_p^n(\mathbf{b}; \mathbf{c})$ is a subset of the simplex

$$S_p^n(\mathbf{b}; \mathbf{c}) := \mathbb{R}_+^n \cap \left\{ \mathbf{w} : \sum_{j=1}^n \left(\sum_{i=1}^n (b_i c_{ij})^p \right)^{1/p} \omega_j \leq 1 \right\}.$$

Since $g_{q,n}(\cdot; \mathbf{a})$ is a linear function it achieves its maximum on $S_p^n(\mathbf{b}; \mathbf{c})$ at the vertices

$$\mathbf{z}_j = \left(\sum_{i=1}^n (b_i c_{ij})^p \right)^{-1/p} \mathbf{e}^j, \quad 1 \leq j \leq n,$$

of $S_p^n(\mathbf{b}; \mathbf{c})$ (here, \mathbf{e}^j , $1 \leq j \leq n$, is the standard basis of \mathbb{R}^n). Therefore,

$$\begin{aligned} \max_{\mathbf{w} \in \Omega_p^n(\mathbf{b}; \mathbf{c})} f_{q,n}(\mathbf{w}; \mathbf{a}) &\leq \max_{\mathbf{w} \in \Omega_p^n(\mathbf{b}; \mathbf{c})} g_{q,n}(\mathbf{w}; \mathbf{a}) \leq \max_{\mathbf{w} \in S_p^n(\mathbf{b}; \mathbf{c})} g_{q,n}(\mathbf{w}; \mathbf{a}) \\ &= n^{1/q-1} \max_{1 \leq j \leq n} \left\{ a_j \left(\sum_{i=1}^n (b_i c_{ij})^p \right)^{-1/p} \right\}, \end{aligned}$$

and the proof is complete. ■

The following lemma is an immediate corollary of a stronger Theorem 1 in [8], taking into account [4] (see also [2, Theorem 4.6.3]).

Lemma 4 ([8]) *Let $s \in \mathbb{N}$, $0 < p \leq \infty$, $I := (-1, 1)$, and $x \in \Delta_+^s \mathbb{L}_p(I)$. Denote by*

$$T_{s-1}(t) := T_{s-1}(t; x; 0) := \sum_{k=0}^{s-1} \frac{x^{(k)}(0)}{k!} t^k, \quad t \in I,$$

the Taylor polynomial of degree $\leq s-1$ at $t = 0$. Then, there exists a constant $c = c(s, p)$ such that

$$\|x - T_{s-1}\|_{\mathbb{L}_p(I)} \leq c \|x\|_{\mathbb{L}_p(I)}.$$

2.2 Proof of the Upper Bounds in Theorem 1

Fix $n \geq 1$, and let $\beta \in \mathbb{N}$, to be prescribed. For every $-n \leq i \leq n$, denote

$$t_i := \text{sign}(i) \left(1 - \left(\frac{n - |i|}{n} \right)^\beta \right).$$

Also,

$$I_i := \begin{cases} [t_{i-1}, t_i) & i = 1, \dots, n, \\ (t_i, t_{i+1}] & i = -1, \dots, -n. \end{cases}$$

Note that $t_{-i} = -t_i$ and $|I_{-i}| = |I_i|$ for all $-n \leq i \leq n$. Straightforward computations show that $\beta(n - i)^{\beta-1}n^{-\beta} \leq |I_i| \leq (2^\beta - 1)(n - i)^{\beta-1}n^{-\beta}$ for $1 \leq i \leq n - 1$, $|I_n| = n^{-\beta}$, and $|I_{i+1}| \leq |I_i| \leq (2^\beta - 1)|I_{i+1}|$ for all $1 \leq i \leq n - 1$.

Let $x \in \Delta_+^s(I)$. Recall that $x^{(s-1)}$ is nondecreasing, and so has left and right derivatives $x_-^{(s-1)}(\tau)$ and $x_+^{(s-1)}(\tau)$, $\tau \in I$. We recall our notation $x^{(s-1)}(\tau) := (x_+^{(s-1)}(\tau) + x_-^{(s-1)}(\tau)) / 2$, and put

$$T_{s-1}(t; x; \tau) := \sum_{k=0}^{s-1} \frac{x^{(k)}(\tau)}{k!} (t - \tau)^k, \quad t \in I.$$

Finally, we denote

$$T_{s-1}(t; x; I_i) := \begin{cases} T_{s-1}(t; x; t_{i-1}) & i = 1, \dots, n, \\ T_{s-1}(t; x; t_{i+1}) & i = -1, \dots, -n, \end{cases}$$

and set

$$(2.2) \quad \sigma_{s,n}(t) := \sigma_{s,n}(t; x; I) := T_{s-1}(t; x; I_i), \quad t \in I_i, \quad i = \pm 1, \dots, \pm n.$$

We will estimate the distance of $\sigma_{s,n}(\cdot; x; I)$ from $x \in \Delta_+^s B_p$. First we assume that x satisfies

$$(2.3) \quad x^{(k)}(0) = 0, \quad k = 0, \dots, s - 1.$$

It follows from Lemma 1 (or may be proved directly) that in this case $x^{(k)}(t) \geq 0$, $t \in I_+ := [0, 1)$, and $(-1)^{s-k}x^{(k)}(t) \geq 0$, $t \in I_- := (-1, 0]$, $k = 0, \dots, s - 1$. We restrict our discussion to I_+ , the estimates for I_- follow by the observation that $y(t) := (-1)^s x(-t)$, $t \in I$, satisfies $y^{(k)}(t) = (-1)^{s-k}x^{(k)}(-t) \geq 0$, $t \in I_+$, $k = 0, \dots, s - 1$, and that $\sigma_{s,n}(t; y; I) = (-1)^s \sigma_{s,n}(-t; x; I)$. Without loss of generality we assume that $\|x\|_{\mathbb{L}_p(I_+)} \neq 0$.

If $n = 1$, then $\sigma_{s,1}(t) \equiv 0$, $t \in I$, and by Hölder's inequality we have

$$\|x(\cdot) - \sigma_{s,1}\|_{\mathbb{L}_q(I)} = \|x\|_{\mathbb{L}_q(I)} \leq 2^{1/q-1/p} \|x\|_{\mathbb{L}_p(I)}.$$

From now on, we assume that $n > 1$, and denote

$$\omega_i := \omega_i(x^{(s-1)}; I_i) := x^{(s-1)}(t_i) - x^{(s-1)}(t_{i-1}), \quad i = 1, \dots, n - 1.$$

First, if $s = 1$ and $t \in I_i, 1 \leq i \leq n - 1$, then it is readily seen that

$$(2.4) \quad \|x - \sigma_{1,n}\|_{L_q(I_i)} \leq |I_i|^{1/q} \omega_i.$$

If $s > 1$ and $t \in I_i, 1 \leq i \leq n - 1$, then Taylor’s formula with integral remainder and integration by parts yield

$$\begin{aligned} x(t) - \sigma_{s,n}(t) &= x(t) - T_{s-1}(t; x; I_i) \\ &= \frac{1}{(s-2)!} \int_{t_{i-1}}^t (x^{(s-1)}(\tau) - x^{(s-1)}(t_{i-1})) (t - \tau)^{s-2} d\tau. \end{aligned}$$

Hence, using monotonicity of $x^{(s-1)}$ we conclude that

$$(2.5) \quad \|x - \sigma_{s,n}\|_{L_q(I_i)} \leq \frac{|I_i|^{s-1+1/q}}{(s-1)!} \omega_i, \quad i = 1, \dots, n - 1.$$

Taking into account (2.4) we see that (2.5) is valid for all $s \geq 1$. It now remains to consider that case $t \in I_n$. For any $s \geq 1$, if $t \in I_n$ then by Lemma 1 and the fact that $x^{(k)}(t_{n-1}) \geq 0, k = 0, \dots, s - 1$, we have $0 \leq x(t) - \sigma_{s,n}(t) = x(t) - T_{s-1}(t; x; I_n) \leq x(t)$. By Hölder’s inequality we get

$$(2.6) \quad \|x - \sigma_{s,n}\|_{L_q(I_n)} \leq \|x\|_{L_p(I_n)} |I_n|^{1/q-1/p}.$$

Combining (2.5) and (2.6) we obtain

$$\begin{aligned} \|x - \sigma_{s,n}\|_{L_q(I_+)}^q &= \sum_{i=1}^n \|x - \sigma_{s,n}\|_{L_q(I_i)}^q \\ &\leq \frac{1}{((s-1)!)^q} \sum_{i=1}^{n-1} (|I_i|^{s-1+1/q} \omega_i)^q + \|x\|_{L_p(I_n)}^q |I_n|^{1-q/p}. \end{aligned}$$

Suppose now that $t \in I_i, 2 \leq i \leq n$, is fixed. Since by (2.3), $T_{s-1}(\cdot; x; I_1) \equiv 0$, we have

$$x(t) = x(t) - T_{s-1}(t; x; I_i) + \sum_{j=2}^i (T_{s-1}(t; x; I_j) - T_{s-1}(t; x; I_{j-1}))$$

and note that

$$\begin{aligned} T_{s-1}(t; x; I_j) - T_{s-1}(t; x; I_{j-1}) &= \sum_{k=0}^{s-1} (x^{(k)}(t_{j-1}) - T_{s-1}^{(k)}(t_{j-1}; x; I_{j-1})) \frac{(t - t_{j-1})^k}{k!} \\ &= \sum_{k=0}^{s-1} (x^{(k)}(t_{j-1}) - T_{s-k-1}(t_{j-1}; x^{(k)}; I_{j-1})) \frac{(t - t_{j-1})^k}{k!}. \end{aligned}$$

Lemma 1 implies (this is also not difficult to show directly) that $x(t) \geq T_{s-1}(t; x; I_i)$ and $x^{(k)}(t_{j-1}) \geq T_{s-k-1}(t_{j-1}; x^{(k)}; I_{j-1})$, $0 \leq k \leq s-2$. Therefore,

$$\begin{aligned} & T_{s-1}(t; x; I_j) - T_{s-1}(t; x; I_{j-1}) \\ & \geq (x^{(s-1)}(t_{j-1}) - T_0(t_{j-1}; x^{(s-1)}; I_{j-1})) \frac{(t - t_{j-1})^{s-1}}{(s-1)!} \\ & = \frac{(t - t_{j-1})^{s-1}}{(s-1)!} \omega_{j-1}, \end{aligned}$$

and, hence, for any $t \in I_i$, $2 \leq i \leq n$,

$$(2.7) \quad x(t) \geq \sum_{j=2}^i \frac{(t - t_{j-1})^{s-1}}{(s-1)!} \omega_{j-1}.$$

Denoting $\bar{t}_i := (t_i + t_{i-1})/2$, for $2 \leq j \leq i \leq n$ and $t \in [\bar{t}_i, t_i)$, we have

$$\begin{aligned} t - t_{j-1} &= (t - \bar{t}_i) + (\bar{t}_i - t_{i-1}) + \sum_{k=j}^{i-1} (t_k - t_{k-1}) \geq \frac{1}{2} \sum_{k=j}^i (t_k - t_{k-1}) \\ &= \frac{1}{2} \sum_{k=j}^i |I_k| \geq \frac{1}{2} (i - j + 1) |I_i|, \end{aligned}$$

since $|I_1| \geq |I_2| \geq \dots \geq |I_n|$.

Combining this estimate with (2.7) we obtain

$$x(t) \geq \frac{|I_i|^{s-1}}{2^{s-1}(s-1)!} \sum_{j=2}^i (i - j + 1)^{s-1} \omega_{j-1}, \quad t \in [\bar{t}_i, t_i), \quad 2 \leq i \leq n,$$

which implies

$$\begin{aligned} (2.8) \quad \|x\|_{\mathbb{L}_p(I_+)}^p &\geq \sum_{i=2}^n \|x\|_{\mathbb{L}_p[\bar{t}_i, t_i]}^p \\ &\geq \frac{1}{2^{(s-1)p+1} ((s-1)!)^p} \sum_{i=2}^n |I_i|^{(s-1)p+1} \left(\sum_{j=2}^i (i - j + 1)^{s-1} \omega_{j-1} \right)^p \\ &= \sum_{i=1}^{n-1} \left(\frac{|I_{i+1}|^{s-1+1/p}}{2^{s-1+1/p}(s-1)!} \sum_{j=1}^i (i - j + 1)^{s-1} \omega_j \right)^p \\ &\geq \sum_{i=1}^{n-1} \left(\bar{c} |I_i|^{s-1+1/p} \sum_{j=1}^i (i - j + 1)^{s-1} \omega_j \right)^p, \end{aligned}$$

where $\bar{c} = 2^{-(\beta+1)(s-1+1/p)} ((s-1)!)^{-1}$.

We can rewrite (2.8) in the following equivalent form

$$(2.9) \quad \sum_{i=1}^{n-1} \left(\frac{\bar{c}|I_i|^{s-1+1/p}}{\|x\|_{L_p(I_+)}} \sum_{j=1}^i (i-j+1)^{s-1} \omega_j \right)^p \leq 1.$$

We are interested in estimating the sum in (2.2) from above subject to (2.9). In other words, we want to find (estimate) the maximum value of the function

$$(f_{q,n-1}(\mathbf{w}; \mathbf{a}))^q := \sum_{i=1}^{n-1} (a_i \omega_i)^q,$$

on the set

$$\Omega_p^{n-1}(\mathbf{b}; \mathbf{c}) := \mathbb{R}_+^{n-1} \cap \left\{ \mathbf{w} : \sum_{i=1}^{n-1} \left(b_i \sum_{j=1}^{n-1} c_{ij} \omega_j \right)^p \leq 1 \right\},$$

where $a_i := |I_i|^{s-1+1/q}$, $b_i := \bar{c}|I_i|^{s-1+1/p} \|x\|_{L_p(I_+)}^{-1}$ and $c_{ij} := (i-j+1)_+^{s-1} := (\max\{i-j+1, 0\})^{s-1}$, $i, j = 1, \dots, n-1$. We now estimate $\sum_{i=1}^{n-1} (b_i c_{ij})^p$, and then apply Lemma 3 in order to estimate this maximum value.

$$\begin{aligned} \sum_{i=1}^{n-1} (b_i c_{ij})^p &\leq c \|x\|_{L_p(I_+)}^{-p} \sum_{i=1}^{n-1} |I_i|^{(s-1)p+1} (i-j+1)_+^{(s-1)p} \\ &\leq c \|x\|_{L_p(I_+)}^{-p} \sum_{i=j}^{n-1} |I_i|^{(s-1)p+1} (i-j+1)^{(s-1)p} \\ &\leq c \|x\|_{L_p(I_+)}^{-p} \sum_{i=j}^{n-1} (n-i)^{(\beta-1)(sp-p+1)} n^{-\beta(sp-p+1)} (i-j+1)^{(s-1)p} \\ &\leq c \|x\|_{L_p(I_+)}^{-p} \left(\frac{n-j}{n} \right)^{\beta(sp-p+1)}. \end{aligned}$$

We now take $\beta \in \mathbb{N}$ to be such that

$$(2.10) \quad \beta \geq p(sq - q + 1)/(p - q).$$

Then, Lemma 3 (with $n - 1$ instead of n) implies

$$\begin{aligned} & \max_{\mathbf{w} \in \Omega_p^{n-1}(\mathbf{b}; \mathbf{c})} (f_{q,n-1}(\mathbf{w}; \mathbf{a}))^q \\ & \leq n^{1-q} \max_{1 \leq j \leq n-1} \left\{ a_j^q \left(\sum_{i=1}^{n-1} (b_i c_{ij})^p \right)^{-q/p} \right\} \\ & \leq c \|x\|_{\mathbb{L}_p(I_+)}^q n^{1-q} \max_{1 \leq j \leq n-1} \left\{ |I_j|^{sq-q+1} \left(\frac{n-j}{n} \right)^{-\beta q(sp-p+1)/p} \right\} \\ & \leq c \|x\|_{\mathbb{L}_p(I_+)}^q n^{1-q-\beta+\beta q/p} \max_{1 \leq j \leq n-1} \left\{ (n-j)^{\beta(1-q/p)-(sq-q+1)} \right\} \\ & \leq c \|x\|_{\mathbb{L}_p(I_+)}^q n^{-sq}. \end{aligned}$$

Therefore, using the above as well as the observation that, if β satisfies (2.10), then $|I_n|^{1-q/p} = n^{-\beta(1-q/p)} < n^{-sq}$, we immediately get from (2.2)

$$\|x - \sigma_{s,n}\|_{\mathbb{L}_q(I_+)}^q \leq c \|x\|_{\mathbb{L}_p(I_+)}^q n^{-sq} + \|x\|_{\mathbb{L}_p(I_n)}^q |I_n|^{1-q/p} \leq c \|x\|_{\mathbb{L}_p(I_+)}^q n^{-sq},$$

and so (2.1) is proved for all $x \in \Delta_+^s \mathbb{L}_p$ satisfying (2.3).

In general, if $x \in \Delta_+^s \mathbb{L}_p$, then the function

$$\tilde{x}(t) := x(t) - T_{s-1}(t; x; 0), \quad t \in I,$$

satisfies (2.3), and by Lemma 4 $\|\tilde{x}\|_{\mathbb{L}_p(I)} \leq c \|x\|_{\mathbb{L}_p(I)}$, where c depends only on s and p . Therefore, it is enough to set

$$\sigma_{s,n}(t; x; I) := \sigma_{s,n}(t; \tilde{x}; I) + T_{s-1}(t; x; 0), \quad t \in I,$$

in order to complete the proof of (2.1) for all $x \in \Delta_+^s \mathbb{L}_p$.

Denote by $\Sigma_{\beta,s,n} := \Sigma_{\beta,s,n}(I)$, where β satisfies (2.10), the space of piecewise polynomials $\sigma: I \mapsto \mathbb{R}$, of order s (of degree $\leq s - 1$), with knots at t_i , $i = \pm 1, \dots, \pm(n - 1)$. Then, for $x \in \Delta_+^s B_p$, clearly $\sigma_{s,n}(\cdot; x; I) \in \Sigma_{\beta,s,n}$. Also, by our construction, the mapping $A: \text{span}(\Delta_+^s B_p) \mapsto \Sigma_{\beta,s,n}$ defined by (2.2) is linear. Since $\dim(\Sigma_{\beta,s,n}) = s(2n - 1)$, it follows by (1.1) that

$$d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{psd}} \leq d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{kol}} \leq d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{lin}} \leq cn^{-s}, \quad n \geq s, \quad 0 < q < p < 1,$$

where $c = c(s, p, q)$. This proves the upper bound in (1.2).

3 Lower Bounds

3.1 Auxiliary Results

The following lemma can be proved in exactly the same way as [10, Lemma 2.2, p. 489] (also see [12, Claim 1]).

Lemma 1 Let $m \in \mathbb{N}$ and $V^m := \{(v_1, \dots, v_m) : v_i = \pm 1, i = 1, \dots, m\}$. Then there exists a subset $V^{(m)} \subset V^m$ of cardinality $\geq 2^{m/16}$ such that for any $\mathbf{v}, \mathbf{u} \in V^{(m)}$, $\mathbf{v} \neq \mathbf{u}$, the distance $\|\mathbf{v} - \mathbf{u}\|_{l_1^m} \geq m/2$.

The following property of the pseudo-dimension is well known (see [17], [6]) and immediately follows from the definition.

Lemma 2 Let $T \neq \emptyset$ and $M := M(T)$ be a family of functions $x: T \mapsto \mathbb{R}$. Fix any function $y: T \mapsto \mathbb{R}$,

$$\dim_{ps}\{z : z = y + x, x \in M\} = \dim_{ps}(M).$$

Lemma 3 ([3, Lemma 1]) Let $I := (0, 1)$, and let $a > 0, \varepsilon > 0$, and $m \in \mathbb{N}$, such that $m \geq 16(8 + \log_2(a/\varepsilon))$, be given. Suppose that a set $\Phi^{(m)}$ of functions $\varphi \in \mathbb{L}_\infty(I)$ is such that

$$\begin{aligned} \text{card}(\Phi^{(m)}) &\geq 2^{m/16}, \\ \|\varphi\|_{\mathbb{L}_\infty(I)} &\leq a, \quad \varphi \in \Phi^{(m)}, \end{aligned}$$

and for some $0 < q < 1$,

$$\|\phi_1 - \phi_2\|_{\mathbb{L}_q(I)} \geq \varepsilon, \quad \phi_1 \neq \phi_2, \quad \phi_1, \phi_2 \in \Phi^{(m)}.$$

Then for any $n \in \mathbb{N}$ such that $n \leq \left(16(8 + \log_2(a/\varepsilon))\right)^{-1} m$ we have

$$d_n(\Phi^{(m)})_{\mathbb{L}_q(I)}^{\text{psd}} \geq 2^{-2-1/q}(2^q - 1)^{1/q}\varepsilon.$$

3.2 Proof of the Lower Bounds in Theorem 1

Let $\varphi \in C^\infty(\mathbb{R})$ be nonnegative with $\text{supp } \varphi = I_+ = [0, 1]$, $\|\varphi\|_{\mathbb{L}_\infty(I)} = 1$, and $\varphi(t) = 1$ if $t \in [1/4, 3/4]$. Denote $\vartheta_s := \|\varphi^{(s)}\|_{\mathbb{L}_\infty(I)}^{-1}$.

For $s \in \mathbb{N}$, let

$$\phi_s(t) := \vartheta_s \varphi(t), \quad t \in \mathbb{R},$$

and for $m \in \mathbb{N}$ to be prescribed, take $t_i^* := t_{m,i}^* := i/m, 0 \leq i \leq m$, and $I_i^* := I_{m,i}^* := [t_{i-1}^*, t_i^*], 1 \leq i \leq m$. Denote

$$\kappa := (sp + 1)^{1/p} 2^{-1/p} s!,$$

and, for each $1 \leq i \leq m$, set

$$\phi_{s,m,i}(t) := \kappa m^{-s} \phi_s(m(t - t_{i-1}^*)), \quad t \in \mathbb{R}.$$

Then, $\text{supp } \phi_{s,m,i} = I_i^*$, $\|\phi_{s,m,i}^{(s)}\|_{\mathbb{L}_\infty(I)} = \kappa, 0 \leq \phi_{s,m,i}(t) \leq \kappa \vartheta_s m^{-s}, t \in I$, and $\phi_{s,m,i}(t) = \kappa \vartheta_s m^{-s}, t \in [t_{i-1}^* + 1/(4m), t_i^* - 1/(4m)]$.

Write

$$\Phi_s^{(m)} := \left\{ \phi : \phi = \sum_{i=1}^m v_i \phi_{s,m,i}, (v_1, \dots, v_m) \in V^{(m)} \right\},$$

where $V^{(m)}$ is the class of sign-vectors defined in Lemma 1. Then, for all $\phi \in \Phi_s^{(m)}$,

$$\|\phi\|_{\mathbb{L}_\infty(I)} \leq \kappa \vartheta_s m^{-s} \quad \text{and} \quad \|\phi^{(s)}\|_{\mathbb{L}_\infty(I)} \leq \kappa.$$

Set

$$\psi_s(t) := \frac{\kappa}{s!} t_+^s, \quad t \in I.$$

It is easy to check that $\|\psi_s\|_{\mathbb{L}_p(I)} = 2^{-1/p}$, and $\psi_s^{(s)}(t) = \kappa, t \in I_+$.

Clearly, for all $\phi \in \Phi_s^{(m)}$, $\psi_s^{(s)}(t) + \phi^{(s)}(t) \geq 0$ a.e. on I , and

$$\|\psi_s + \phi\|_{\mathbb{L}_p(I)}^p \leq \|\psi_s\|_{\mathbb{L}_p(I)}^p + \|\phi\|_{\mathbb{L}_p(I)}^p \leq 1/2 + \kappa \vartheta_s m^{-s}.$$

Thus, for $m \geq (2\kappa)^{1/s} \vartheta_s^{1/s}$ we have $\psi_s + \Phi_s^{(m)} \subset \Delta_+^s B_p$, and so applying Lemma 2 we have

$$(3.1) \quad d_n(\Delta_+^s B_p)_{\mathbb{L}_q(I)}^{\text{psd}} \geq d_n(\psi_s + \Phi_s^{(m)})_{\mathbb{L}_q(I)}^{\text{psd}} = d_n(\Phi_s^{(m)})_{\mathbb{L}_q(I)}^{\text{psd}}, \quad n \geq 1.$$

For any two different vectors $\mathbf{v} := (v_1, \dots, v_m)$ and $\mathbf{u} := (u_1, \dots, u_m)$ in $V^{(m)}$, let

$$\phi_1 := \sum_{i=1}^m v_i \phi_{s,m,i} \quad \text{and} \quad \phi_2 := \sum_{i=1}^m u_i \phi_{s,m,i}$$

be the two corresponding functions in $\Phi_s^{(m)}$. Since $\|\mathbf{v} - \mathbf{u}\|_1^m \geq m/2$, then there exist $\lceil m/4 \rceil$ indices $i_1, \dots, i_{\lceil m/4 \rceil}$ such that $v_{i_k} = -u_{i_k}, k = 1, \dots, \lceil m/4 \rceil$. Hence,

$$\begin{aligned} \|\phi_1 - \phi_2\|_{\mathbb{L}_q(I)}^q &= \int_I \left| \sum_{i=1}^m (v_i - u_i) \phi_{s,m,i}(t) \right|^q dt = \sum_{i=1}^m \int_{I_i^*} |v_i - u_i|^q (\phi_{s,m,i}(t))^q dt \\ &\geq \kappa^q \vartheta_s^q m^{-sq} \sum_{k=1}^{\lceil m/4 \rceil} |v_{i_k} - u_{i_k}|^q \int_{t_{i_k}^* - 1 + \frac{1}{4m}}^{t_{i_k}^* - \frac{1}{4m}} dt \\ &= \kappa^q \vartheta_s^q m^{-sq} (2m)^{-1} \sum_{k=1}^{\lceil m/4 \rceil} 2^q \\ &\geq \kappa^q 2^{q-3} \vartheta_s^q m^{-sq} =: \varepsilon^q. \end{aligned}$$

If we set $a := \kappa \vartheta_s m^{-s}$, and given $n \in \mathbb{N}$, we take $m = \lceil 80(\kappa 2^{3/q-1} + 1) \rceil n$, then applying Lemma 3, we conclude that

$$d_n(\Phi_s^{(m)})_{\mathbb{L}_q(I)} \geq cn^{-s}, \quad n \geq 1,$$

where $c = c(s, p, q)$. By virtue of (1.1) and (3.1) this implies

$$d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{lin}} \geq d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{kol}} \geq d_n(\Delta_+^s B_p)_{\mathbb{L}_q}^{\text{psd}} \geq cn^{-s}, \quad n \geq s,$$

where $c = c(s, p, q)$. This completes the proof of the lower bound and so of Theorem 1. ■

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