THE HAMILTON-JACOBI EQUATION FOR A CHARGED PARTICLE IN A COMBINED GRAVITATIONAL AND ELECTROMAGNETIC FIELD

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The equation of motion of a charged particle in a combined gravitational and electromagnetic field is cast in the classical Hamilton-Jacobi form and then applied to the special case of a Schwarzschild metric, leading to the well established equation of planetary motion.

1. Consider a particle of charge e and rest-mass m_o in a gravitational field characterised by the metric tensor $g_{\mu\lambda}$ and an electromagnetic field characterised by the potential ϕ_μ. Its equation of motion is then given by the following variation principle [1]

$$\delta \int L^* ds = 0 ,$$

$$L^*\left(q^\alpha, \frac{dq^\alpha}{ds}\right) = m_o c g_{\mu\lambda}\left[\frac{dq^\mu}{ds}\frac{dq^\lambda}{ds}\right]^{1/2} + e\phi_\mu \frac{dq^\mu}{ds} . \tag{1}$$

Here s is an arbitrary parameter, c the velocity of light, and the coordinates are $q^\mu \equiv (q^k, t)$, the latin indices denoting the 3 space coordinates. We shall use the summation convention throughout.

We put $L^* ds = L\, dt$, $\dot{q}^k = \dfrac{dq^k}{dt}$, so that $L(q^k, t, \dot{q}^k) = L' + L''$, where

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$$\left.\begin{aligned} L' &= m_o c\,[g_{k\ell}\,\dot{q}^k \dot{q}^\ell + 2g_{4k}\,\dot{q}^k + g_{44}]^{1/2}\,, \\ L'' &= e(\phi_k\,\dot{q}^k + \phi_4)\;. \end{aligned}\right\} \qquad (2)$$

The momentum p_k canonically conjugate to q^k is then

$$p_k = \frac{\partial L}{\partial \dot{q}^k} = \frac{1}{2L'}\,\frac{\partial L'^2}{\partial \dot{q}^k} + \frac{\partial L''}{\partial \dot{q}^k}\,,$$ assuming $L' \neq 0$, or

$$p_k = \frac{m_o^2 c^2}{L'}\,(g_{k\ell}\,\dot{q}^\ell + g_{4k}) + e\phi_k\;. \qquad (3)$$

Introduce g^{km} by the relation $g^{km}\,g_{im} = \delta^k_i$. Then, contracting p_k by g^{km}, we obtain

$$p_k g^{km} = \frac{m_o^2 c^2}{L'}\,(\dot{q}^m + g_{4k}\,g^{km}) + eg^{km}\,\phi_k$$

or $$\dot{q}^m = \frac{L'}{m_o^2 c^2}\,(p_k - e\phi_k)g^{km} - g_{4k}\,g^{km}$$

or $$\dot{q}^m = \frac{L'}{m_o^2 c^2}\,p'_k\,g^{km} - f^m\,, \qquad (4)$$

where we have put

$$p'_k = p_k - e\phi_k\,,$$

$$f^m = g_{4k}g^{km}\;.$$

In order to express L' as a function of p'_k and q^k we consider

$$L'^2 = m_o^2 c^2 (g_{k\ell}\,\dot{q}^k \dot{q}^\ell + 2g_{4k}\dot{q}^k + g_{44})\;.$$

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Substituting $\dot{q}^m$ from (4) in the expression for L'^2 we get

$$L'^2 = m_o^2 c^2 g_{k\ell}\left(\frac{L'}{m_o^2c^2}p'_s g^{sk} - f^k\right)\left(\frac{L'}{m_o^2c^2}p'_m g^{m\ell} - f^\ell\right)$$

$$+ 2m_o^2c^2 g_{4k}\left(\frac{L'}{m_o^2c^2}p'_s g^{sk} - f^k\right) + m_o^2c^2 g_{44}$$

$$= \frac{L'^2}{m_o^2c^2}p'_m p'_\ell g^{m\ell} - m_o^2c^2 g_{4k}f^k + m_o^2c^2 g_{44},$$

by virtue of $g_{k\ell}f^\ell = g_{4k}$. Therefore, finally,

$$\frac{L'}{m_o^2c^2} = \left(\frac{g_{4k}f^k - g_{44}}{p'_m p'_\ell g^{m\ell} - m_o^2c^2}\right)^{1/2}. \tag{5}$$

The Hamiltonian is therefore given by

$$H = p_m \dot{q}^m - L$$

$$= (p'_m + e\phi_m)\left(\frac{L'}{m_o^2c^2}p'_k g^{km} - f^m\right) - L' - L''$$

$$= \frac{L'}{m_o^2c^2}\left(p'_m p'_k g^{km} - m_o^2c^2\right) - p'_m f^m$$

$$+ \frac{eL'}{m_o^2c^2}\phi_m p'_k g^{km} - e\phi_m f^m - L''. \tag{6}$$

From (2) and (4)

$$L'' = \frac{e}{m_o^2c^2}L'\phi_m p'_k g^{km} - ef^m\phi_m + e\phi_4 \tag{7}$$

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so that, substituting (7) and (5) in (6) ,

$$H = \frac{L'}{m_o^2 c^2}(p'_m p'_k g^{km} - m_o^2 c^2) - p'_m f^m - e\phi_4$$

or

$$H(p,q,t) = [(p'_m p'_k g^{km} - m_o^2 c^2)(g_{4k} f^k - g_{44})]^{1/2} - p'_m f^m - e\phi_4 . \qquad (8)$$

Consider now the special case of the Minkowski metric

$$ds^2 = - \delta_{km} dq^k dq^m + c^2 dt^2 ,$$

so that

$$g^{km} = - \delta_{km} , \quad g_{44} = c^2 , \quad f^m = 0 \qquad \text{and}$$

$$H = [c^2(p_1'^2 + p_2'^2 + p_3'^2 + m_o^2 c^2)]^{1/2} - e\phi_4 .$$

When there is no electromagnetic field the Hamiltonian reduces to the familiar expression

$$H = [c^2 (p_1^2 + p_2^2 + p_3^2 + m_o^2 c^2)]^{1/2}$$

for a relativistic free particle.

The Hamilton-Jacobi equation for a charged particle in a combined gravitational and electromagnetic field is then, from (8),

$$H\left(\frac{\partial S}{\partial q^k}, q^k, t\right) + \frac{\partial S}{\partial t} = 0$$

or

$$\left[\left\{\left(\frac{\partial S}{\partial q^m} - e\phi_m\right)\left(\frac{\partial S}{\partial q^k} - e\phi_k\right) g^{km} - m_o^2 c^2\right\}\left\{g_{4k} f^k - g_{44}\right\}\right]^{1/2} - \left(\frac{\partial S}{\partial q^m} - e\phi_m\right) f^m - e\phi_4 + \frac{\partial S}{\partial t} = 0 . \qquad (9)$$

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When there is no electromagnetic field (9) reduces to

$$\left[\left(\frac{\partial S}{\partial q^m}\frac{\partial S}{\partial q^k} g^{km} - m_o^2 c^2\right)\left(g_{4k} f^k - g_{44}\right)\right]^{1/2}$$

$$- \frac{\partial S}{\partial q^m} f^m + \frac{\partial S}{\partial t} = 0 \ . \qquad (10)$$

A coordinate-symmetric formulation of the same problem based on the work of H. Rund has been given by Vanstone (c. f. preceding paper.)

2. As an elementary application of (10) we consider the familiar problem of the advance of the perihelion of a planet in a Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

$$+ c^2\left(1 - \frac{2m}{r}\right) dt^2 , \qquad (11)$$

where

$$m = \frac{GM}{c^2} ,$$

G = const. of gravitation

M = mass of the sun .

$$\text{Then } g^{ik} \equiv \begin{bmatrix} -\left(1 - \frac{2m}{r}\right) & 0 & 0 \\ 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & -\frac{1}{r^2}\sin^2\theta \end{bmatrix} ,$$

$$g_{44} = c^2(1 - \frac{2m}{r}) \ , \quad g_{4k} = 0 \ , \quad f^m = 0 \ , \text{ and}$$

H does not contain t .

The Hamilton-Jacobi equation for the motion is then

$$H\left(q^k, \frac{\partial S}{\partial q^k}\right) = E$$

or

$$\left[c^2(1 - \frac{2m}{r})\left\{(1 - \frac{2m}{r})(\frac{\partial S}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial S}{\partial \theta})^2 + \frac{1}{r^2 \sin^2 \theta}(\frac{\partial S}{\partial \phi})^2 + m_o^2 c^2\right\}\right]^{1/2} = E \ . \qquad (12)$$

We note first of all that ϕ is a cyclic variable, and separate the variables by putting

$$S(r, \theta, \phi) = S_r(r) + S_\theta(\theta) + S_\phi(\phi) \ .$$

Then $p_\phi = \frac{dS_\phi}{d\phi} = \alpha_1 = \text{constant}$, and $\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\alpha_1^2}{\sin^2 \theta} = \alpha_2$

$= \text{constant}$, or $p_\theta = \frac{dS_\theta}{d\theta} = \left(\alpha_2^2 - \frac{\alpha_1^2}{\sin^2 \theta}\right)^{1/2}$.

For a planet moving in the plane $\theta = \frac{\pi}{2}$ we may take $\alpha_2 = \alpha_1$, $p_\theta = 0$, so that

$$S(r, \phi) = S_r + \alpha_1 \phi \ ,$$

where S_r satisfies the equation

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$$\frac{E^2}{c^2} = \left(1 - \frac{2m}{r}\right)^2 \left(\frac{dS_r}{dr}\right)^2 + \frac{\alpha_1^2}{r^2}\left(1 - \frac{2m}{r}\right) + m_o^2 c^2 \left(1 - \frac{2m}{r}\right),$$

or
$$S(r,\phi) = \int_{r_o}^{r} \left[\frac{\frac{E^2}{c^2} - \frac{\alpha_1^2}{r^2}(1 - \frac{2m}{r}) - m_o^2 c^2 (1 - \frac{2m}{r})}{(1 - \frac{2m}{r})^2} \right]^{1/2} dr$$

$$+ \alpha_1 \phi + \text{const.} \qquad (13)$$

We now set $\frac{\partial S}{\partial \alpha_1} = \beta_1 = \text{const.}$ We have then

$$\beta_1 = - \int_{r_o}^{r} \left[\frac{\frac{E^2}{c^2} - \frac{\alpha_1^2}{r^2}(1 - \frac{2m}{r}) - m_o^2 c^2 (1 - \frac{2m}{r})}{(1 - \frac{2m}{r})^2} \right]^{-1/2} \frac{\alpha_1}{r^2}(1 - \frac{2m}{r})^{-1} dr + \phi$$

Differentiating with respect to r and squaring we get

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \left[\frac{1}{\alpha_1^2} \quad \frac{E^2}{c^2} - \frac{\alpha_1^2}{r^2}(1 - \frac{2m}{r}) - m_o^2 c^2 (1 - \frac{2m}{r}) \right] .$$

Introducing $u = \frac{1}{r}$ and differentiating with respect to ϕ we obtain the familiar equation for the orbit

$$\frac{d^2 u}{d\phi^2} = \frac{m}{h^2} - u + 3mu^2 ,$$

where

$$\frac{1}{h^2} = \frac{m_o^2 c^2}{\alpha_1^2} .$$

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In the limit as $m_o \to 0$ one obtains the familiar equation of motion of a photon in a gravitational field.

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