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On the spectral theory of groups of automorphisms of *S*-adic nilmanifolds

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Abstract. Let $S = \{p_1, \ldots, p_r, \infty\}$ for prime integers $p_1, \ldots, p_r$. Let X be an S-adic compact nilmanifold, equipped with the unique translation-invariant probability measure μ. We characterize the countable groups Γ of automorphisms of X for which the Koopman representation κ on $L^2(X, \mu)$ has a spectral gap. More specifically, let Y be the maximal quotient solenoid of X (thus, Y is a finite-dimensional, connected, compact abelian group). We show that κ does not have a spectral gap if and only if there exists a Γ-invariant proper subsolenoid of Y on which Γ acts as a virtually abelian group,

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1. *Introduction*

Let Γ be a countable group acting measurably on a probability space (X, μ) by measure-preserving transformations. Let $\kappa = \kappa_X$ denote the corresponding Koopman representation of Γ, that is, the unitary representation of Γ on $L^2(X, \mu)$ given by

$$\kappa(\gamma)\xi(x) = \xi(\gamma^{-1}x) \quad \text{for all } \xi \in L^2(X, \mu), x \in X, \gamma \in \Gamma.$$

We say that the action $\Gamma \curvearrowright (X, \mu)$ of Γ on (X, μ) has a *spectral gap* if the restriction κ_0 of κ to the Γ-invariant subspace

$$L^2_0(X, \mu) = \left\{\xi \in L^2(X, \mu) : \int_X \xi(x)\, d\mu(x) = 0\right\}$$

does not weakly contain the trivial representation 1_Γ; equivalently, if κ_0 does not have almost invariant vectors, that is, there is *no* sequence $(\xi_n)_n$ of unit vectors in $L^2_0(X, \mu)$ such that

$$\lim_n \|\kappa_0(\gamma)\xi_n - \xi_n\| = 0 \quad \text{for all } \gamma \in \Gamma.$$

The existence of a spectral gap admits the following useful quantitative version. Let ν be a probability measure on Γ and $\kappa_0(\nu)$ the convolution operator defined on $L^2_0(X, \mu)$ by

$$\kappa_0(\nu)\xi = \sum_{\gamma\in\Gamma} \nu(\gamma)\kappa_0(\gamma)\xi \quad \text{for all } \xi \in L^2_0(X, \mu).$$

Observe that we have $\|\kappa_0(\nu)\| \leq 1$ and hence $r(\kappa_0(\nu)) \leq 1$ for the spectral radius $r(\kappa_0(\nu))$ of $\kappa_0(\mu)$. Assume that ν is aperiodic, that is, the support of ν is not contained in the coset of a proper subgroup of Γ. Then the action of Γ on X has a spectral gap if and only if $r(\kappa_0(\nu)) < 1$ and this is equivalent to $\|\kappa_0(\nu)\| < 1$; for more details, see the survey [**Bekk16**].

In this paper we will be concerned with the case where X is an S-adic nilmanifold, to be introduced below, and Γ is a subgroup of automorphisms of X.

Fix a finite set $\{p_1, \ldots, p_r\}$ of integer primes and set $S = \{p_1, \ldots, p_r, \infty\}$. The product

$$\mathbf{Q}_S := \prod_{p\in S} \mathbf{Q}_p = \mathbf{Q}_\infty \times \mathbf{Q}_{p_1} \times \cdots \times \mathbf{Q}_{p_r}$$

is a locally compact ring, where $\mathbf{Q}_\infty = \mathbf{R}$ and $\mathbf{Q}_p$ is the field of p-adic numbers for a prime p. Let $\mathbf{Z}[1/S] = \mathbf{Z}[1/p_1, \cdots, 1/p_r]$ denote the subring of $\mathbf{Q}$ generated by 1 and $\{1/p_1, \ldots, 1/p_r\}$. Through the diagonal embedding

$$\mathbf{Z}[1/S] \to \mathbf{Q}_S, \quad b \mapsto (b, \cdots, b),$$

we may identify $\mathbf{Z}[1/S]$ with a discrete and cocompact subring of $\mathbf{Q}_S$.

If $\mathbf{G}$ is a linear algebraic group defined over $\mathbf{Q}$, we denote by $\mathbf{G}(R)$ the group of elements of $\mathbf{G}$ with coefficients in R and determinant invertible in R, for every subring R of an overfield of $\mathbf{Q}$.

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$, that is, $\mathbf{U}$ is an algebraic subgroup of the group of $n \times n$ upper triangular unipotent matrices for some $n \geq 1$. The group $\mathbf{U}(\mathbf{Q}_S)$ is a locally compact group and $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$ is a cocompact lattice in $\mathbf{U}(\mathbf{Q}_S)$. The corresponding *S-adic compact nilmanifold*

$$\mathbf{Nil}_S = \mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S])$$

will be equipped with the unique translation-invariant probability measure μ on its Borel subsets.

For $p \in S$, let $\mathrm{Aut}(\mathbf{U}(\mathbf{Q}_p))$ be the group of continuous automorphisms of $\mathbf{U}(\mathbf{Q}_p)$. Set

$$\mathrm{Aut}(\mathbf{U}(\mathbf{Q}_S)) := \prod_{p\in S} \mathrm{Aut}(\mathbf{U}(\mathbf{Q}_p))$$

and denote by $\mathrm{Aut}(\mathbf{Nil}_S)$ the subgroup

$$\{g \in \mathrm{Aut}(\mathbf{U}(\mathbf{Q}_S)) \mid g(\Lambda) = \Lambda\}.$$

Every $g \in \mathrm{Aut}(\mathbf{Nil}_S)$ acts on $\mathbf{Nil}_S$ preserving the probability measure μ.

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The abelian quotient group

$$\overline{\mathbf{U}(\mathbf{Q}_S)} := \mathbf{U}(\mathbf{Q}_S)/[\mathbf{U}(\mathbf{Q}_S), \mathbf{U}(\mathbf{Q}_S)]$$

can be identified with $\mathbf{Q}_S^d$ for some $d \geq 1$ and the image Δ of $\mathbf{U}(\mathbf{Z}[1/S])$ in $\overline{\mathbf{U}(\mathbf{Q}_S)}$ is a cocompact and discrete subgroup of $\overline{\mathbf{U}(\mathbf{Q}_S)}$; so,

$$\mathbf{Sol}_S := \overline{\mathbf{U}(\mathbf{Q}_S)}/\Delta$$

is a solenoid (that is, is a finite-dimensional, connected, compact abelian group; see [**HeRo63**, §25]). We refer to $\mathbf{Sol}_S$ as the *S-adic solenoid* attached to the *S*- adic nilmanifold $\mathbf{Nil}_S$. We equip $\mathbf{Sol}_S$ with the probability measure ν which is the image of μ under the canonical projection $\mathbf{Nil}_S \to \mathbf{Sol}_S$.

Observe that $\mathrm{Aut}(\mathbf{Q}_S^d)$ is canonically isomorphic to $\prod_{s\in S} GL_d(\mathbf{Q}_s)$ and that $\mathrm{Aut}(\mathbf{Sol}_S)$ can be identified with the subgroup $GL_d(\mathbf{Z}[1/S])$. The group $\mathrm{Aut}(\mathbf{Nil}_S)$ acts naturally by automorphisms of $\mathbf{Sol}_S$; we denote by

$$p_S : \mathrm{Aut}(\mathbf{Nil}_S) \to GL_d(\mathbf{Z}[1/S]) \subset GL_d(\mathbf{Q})$$

the corresponding representation.

THEOREM 1. *Let* $\mathbf{U}$ *be an algebraic unipotent group defined over* $\mathbf{Q}$ *and* $S = \{p_1, \ldots, p_r, \infty\}$, *where* $p_1, \ldots, p_r$ *are integer primes. Let* $\mathbf{Nil}_S = \mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S])$ *be the associated S-adic nilmanifold and let* $\mathbf{Sol}_S$ *be the corresponding S-adic solenoid, respectively equipped with the probability measures* μ *and* ν *as above. Let* Γ *be a countable subgroup of* $\mathrm{Aut}(\mathbf{Nil}_S)$. *The following properties are equivalent.*

(i) *The action* $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ *has a spectral gap.*

(ii) *The action* $p_S(\Gamma) \curvearrowright (\mathbf{Sol}_S, \nu)$ *has a spectral gap, where* $p_S : \mathrm{Aut}(\mathbf{Nil}_S) \to GL_d(\mathbf{Z}[1/S])$ *is the canonical homomorphism.*

Actions with spectral gap of groups of automorphisms (or more generally groups of affine transformations) of the *S*-adic solenoid $\mathbf{Sol}_S$ have been completely characterized in [**BeFr20**, Theorem 5]. The following result is an immediate consequence of this characterization and of Theorem 1. For a subset T of $GL_d(\mathbf{K})$ for a field $\mathbf{K}$, we denote by $T^t = \{g^t \mid g \in T\}$ the set of transposed matrices from T.

COROLLARY 2. *With the notation as in Theorem 1, the following properties are equivalent.*

(i) *The action of* Γ *on the S-adic nilmanifold* $\mathbf{Nil}_S$ *does not have a spectral gap.*

(ii) *There exists a non-zero linear subspace* W *of* $\mathbf{Q}^d$ *which is invariant under* $p_S(\Gamma)^t$ *and such that the image of* $p_S(\Gamma)^t$ *in* $GL(W)$ *is a virtually abelian group.*

Here is an immediate consequence of Corollary 2.

COROLLARY 3. *With the notation as in Theorem 1, assume that the linear representation of* $p_S(\Gamma)^t$ *in* $\mathbf{Q}^d$ *is irreducible and that* $p_S(\Gamma)^t$ *is not virtually abelian. Then the action* $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ *has a spectral gap.*

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Recall that the action of a countable group Γ by measure-preserving transformations on a probability space (X, μ) is *strongly ergodic* (see [**Schm81**]) if every sequence $(B_n)_n$ of measurable subsets of X which is asymptotically invariant (that is, which is such that $\lim_n \mu(\gamma B_n \triangle B_n) = 0$ for all $\gamma \in \Gamma$) is trivial (that is, $\lim_n \mu(B_n)(1 - \mu(B_n)) = 0$). It is straightforward to check that the spectral gap property implies strong ergodicity and it is known that the converse does not hold in general.

The following corollary is a direct consequence of Theorem 1 (compare with [**BeGu15**, Corollary 2]).

COROLLARY 4. *With the notation as in Theorem 1, the following properties are equivalent.*

(i) *The action* $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ *has the spectral gap property.*

(ii) *The action* $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ *is strongly ergodic.*

Theorem 1 generalizes our previous work [**BeGu15**], where we treated the real case (that is, the case $S = \infty$). This requires an extension of our methods to the S-adic setting, which is a non-straightforward task; more specifically, we had to establish the following four main tools for our proof:

- a canonical decomposition of the Koopman representation of Γ in $L^2(\mathbf{Nil}_S)$ as a direct sum of certain representations of Γ induced from stabilizers of representations of $\mathbf{U}(\mathbf{Q}_S)$—this fact is valid more generally for compact homogeneous spaces (see Proposition 9);
- a result of Howe and Moore [**HoMo79**] about the decay of matrix coefficients of algebraic groups (see Proposition 11);
- the fact that the irreducible representations of $\mathbf{U}(\mathbf{Q}_S)$ appearing in the decomposition of $L^2(\mathbf{Nil}_S)$ are rational, in the sense that the Kirillov data associated to each one of them are defined over $\mathbf{Q}$ (see Proposition 13);
- a characterization (see Lemma 12) of the projective kernel of the extension of an irreducible representation of $\mathbf{U}(\mathbf{Q}_p)$ to its stabilizer in $\mathrm{Aut}(\mathbf{U}(\mathbf{Q}_p)$.

Another tool we constantly use is a generalized version of Herz's majoration principle (see Lemma 7).

Given a probability measure ν on Γ, our approach does not seem to provide quantitative estimates for the operator norm of the convolution operator $\kappa_0(\nu)$ acting on $L^2_0(\mathbf{Nil}_S, \mu)$ for a general unipotent group $\mathbf{U}$. However, using known bounds for the so-called metaplectic representation of the symplectic group $Sp_{2n}(\mathbf{Q}_p)$, we give such estimates in the case of S-adic Heisenberg nilmanifolds (see §11).

COROLLARY 5. *For an integer* $n \geq 1$*, let* $\mathbf{U} = \mathbf{H}_{2n+1}$ *be the* $(2n+1)$*-dimensional Heisenberg group and* $\mathbf{Nil}_S = \mathbf{H}_{2n+1}(\mathbf{Q}_S)/\mathbf{H}_{2n+1}(\mathbf{Z}[1/S])$*. Let* ν *be a probability measure on the subgroup* $Sp_{2n}(\mathbf{Z}[1/S])$ *of* $\mathrm{Aut}(\mathbf{Nil}_S)$*. Then*

$$\|\kappa_0(\nu)\| \leq \max\{\|\lambda_\Gamma(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},$$

where κ_1 *is the restriction of* κ_0 *to* $L^2_0(\mathbf{Sol}_S)$ *and* λ_Γ *is the regular representation of the group* Γ *generated by the support of* ν*. In particular, in the case where* $n = 1$ *and* ν *is aperiodic, the action of* Γ *on* $\mathbf{Nil}_S$ *has a spectral gap if and only if* Γ *is non-amenable.*

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2. *Extension of representations*

Let G be a locally compact group which we assume to be second countable. We will need the notion of a projective representation. Recall that a mapping $\pi : G \to U(\mathcal{H})$ from G to the unitary group of the Hilbert space $\mathcal{H}$ is a *projective representation* of G if the following assertions hold.

- $\pi(e) = I$.
- For all $g_1, g_2 \in G$, there exists $c(g_1, g_2) \in \mathbf{C}$ such that

$$\pi(g_1 g_2) = c(g_1, g_2)\pi(g_1)\pi(g_2).$$

- The function $g \mapsto \langle \pi(g)\xi, \eta\rangle$ is measurable for all $\xi, \eta \in \mathcal{H}$.

The mapping $c : G \times G \to \mathbf{S}^1$ is a 2-cocycle with values in the unit circle $\mathbf{S}^1$. Every projective unitary representation of G can be lifted to an ordinary unitary representation of a central extension of G (for all this, see [**Mack76**] or [**Mack58**]).

Let N be a closed normal subgroup of G. Let π be an irreducible unitary representation of N on a Hilbert space $\mathcal{H}$. Consider the stabilizer

$$G_\pi = \{g \in G \mid \pi^g \text{ is equivalent to } \pi\}$$

of π in G for the natural action of G on the unitary dual $\widehat{N}$ given by $\pi^g(n) = \pi(g^{-1}ng)$. Then G_π is a closed subgroup of G containing N. The following lemma is a well-known part of Mackey's theory of unitary representations of group extensions.

LEMMA 6. *Let π be an irreducible unitary representation of N on the Hilbert space $\mathcal{H}$. There exists a projective unitary representation $\widetilde{\pi}$ of G_π on $\mathcal{H}$ which extends π. Moreover, $\widetilde{\pi}$ is unique, up to scalars: any other projective unitary representation $\widetilde{\pi}'$ of G_π extending π is of the form $\widetilde{\pi}' = \lambda\widetilde{\pi}$ for a measurable function $\lambda : G_\pi \to \mathbf{S}^1$.*

Proof. For every $g \in G_\pi$, there exists a unitary operator $\widetilde{\pi}(g)$ on $\mathcal{H}$ such that

$$\pi(g(n)) = \widetilde{\pi}(g)\pi(n)\widetilde{\pi}(g)^{-1} \quad \text{for all } n \in N.$$

One can choose $\widetilde{\pi}(g)$ such that $g \mapsto \widetilde{\pi}(g)$ is a projective unitary representation of G_π which extends π (see [**Mack58**, Theorem 8.2]). The uniqueness of π follows from the irreducibility of π and Schur's lemma. □

3. *A weak containment result for induced representations*

Let G be a locally compact group with Haar measure μ_G. Recall that a unitary representation $(\rho, \mathcal{K})$ of G is weakly contained in another unitary representation $(\pi, \mathcal{H})$ of G, if every matrix coefficient

$$g \mapsto \langle \rho(g)\eta \mid \eta\rangle \quad \text{for } \eta \in \mathcal{K}$$

of ρ is the limit, uniformly over compact subsets of G, of a finite sum of matrix coefficients of π; equivalently, if $\|\rho(f)\| \leq \|\pi(f)\|$ for every $f \in C_c(G)$, where $C_c(G)$ is the space of continuous functions with compact support on G and where the operator $\pi(f) \in \mathcal{B}(\mathcal{H})$ is defined by the integral

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$$\pi(f)\xi = \int_G f(g)\pi(g)\xi \, d\mu_G(g) \quad \text{for all } \xi \in \mathcal{H}.$$

The trivial representation 1_G is weakly contained in π if and only if there exists, for every compact subset Q of G and every $\varepsilon > 0$, a unit vector $\xi \in \mathcal{H}$ which is (Q, ε)-invariant, that is, such that

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| \le \varepsilon.$$

Let H be a closed subgroup of G. We will always assume that the coset space $H\backslash G$ admits a non-zero G-invariant (possibly infinite) measure on its Borel subsets. Let $(\sigma, \mathcal{K})$ be a unitary representation of H. We will use the following model for the induced representation $\pi := \mathrm{Ind}_H^G \sigma$. Choose a Borel fundamental domain $X \subset G$ for the action of G on $H\backslash G$. For $x \in X$ and $g \in G$, let $x \cdot g \in X$ and $c(x, g) \in H$ be defined by

$$xg = c(x, g)(x \cdot g).$$

There exists a non-zero G-invariant measure on X for the action $(x, g) \mapsto x \cdot g$ of G on X. The Hilbert space of π is the space $L^2(X, \mathcal{K}, \mu)$ of all square-integrable measurable mappings $\xi : X \to \mathcal{K}$ and the action of G on $L^2(X, \mathcal{K}, \mu)$ is given by

$$(\pi(g)\xi)(x) = \sigma(c(x, g))(\xi(x \cdot g)), \quad g \in G, \ \xi \in L^2(X, \mathcal{K}, \mu), \ x \in X.$$

Observe that, in the case where σ is the trivial representation 1_H, the induced representation $\mathrm{Ind}_H^G 1_H$ is equivalent to *quasi-regular representation* $\lambda_{H\backslash G}$, that is, the natural representation of G on $L^2(H\backslash G, \mu)$ given by right translations.

We will use several times the following elementary but crucial lemma, which can be viewed as a generalization of Herz's majoration principle (see [**BeGu15**, Proposition 17]).

LEMMA 7. *Let $(H_i)_{i \in I}$ be a family of closed subgroups of G such that $H_i\backslash G$ admits a non-zero G-invariant measure. Let $(\sigma_i, \mathcal{K}_i)$ be a unitary representation of H_i. Assume that 1_G is weakly contained in the direct sum $\bigoplus_{i \in I} \mathrm{Ind}_{H_i}^G \sigma_i$. Then 1_G is weakly contained in $\bigoplus_{i \in I} \lambda_{H_i\backslash G}$.*

Proof. Let Q be a compact subset of G and $\varepsilon > 0$. For every $i \in I$, let $X_i \subset G$ be a Borel fundamental domain for the action of G on $H_i\backslash G$ and μ_i a non-zero G-invariant measure on X_i. There exists a family of vectors $\xi_i \in L^2(X_i, \mathcal{K}_i, \mu_i)$ such that $\sum_i \|\xi_i\|^2 = 1$ and

$$\sup_{g \in Q} \sum_i \|\mathrm{Ind}_{H_i}^G \sigma_i(g)\xi_i - \xi_i\|^2 \le \varepsilon.$$

Define φ_i in $L^2(X_i, \mu_i)$ by $\varphi_i(x) = \|\xi_i(x)\|$. Then $\sum_i \|\varphi_i\|^2 = 1$ and, denoting by $(x, g) \mapsto x \cdot_i g$ the action of G on X_i and by $c_i : X_i \times G \to H_i$ the associated map as above, we have

$$\begin{aligned}\|(\mathrm{Ind}_{H_i}^G \sigma_i(g))\xi_i - \xi_i\|^2 &= \int_{X_i} \|\sigma_i(c_i(x, g))(\xi_i(x \cdot_i g)) - \xi_i(x)\|^2 \, d\mu_i(x) \\ &\ge \int_{X_i} |\|\sigma_i(c_i(x, g))(\xi_i(x \cdot_i g))\| - \|\xi_i(x)\||^2 \, d\mu_i(x)\end{aligned}$$

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$$= \int_{X_i} |\|\xi_i(x \cdot_i g)\| - \|\xi_i(x)\||^2 \, d\mu_i(x)$$
$$= \int_{X_i} |\varphi_i(x \cdot_i g) - \varphi(x)|^2 \, d\mu_i(x)$$
$$= \|\lambda_{H_i\backslash G}(g)\varphi_i - \varphi_i\|^2,$$

for every $g \in G$, and the claim follows. □

4. *Decay of matrix coefficients of unitary representations*

We recall a few general facts about the decay of matrix coefficients of unitary representations, Recall that the projective kernel of a (genuine or projective) representation π of the locally compact group G is the closed normal subgroup P_π of G consisting of the elements $g \in G$ such that $\pi(g)$ is a scalar multiple of the identity operator, that is, such that $\pi(g) = \lambda_\pi(g)I$ for some $\lambda_\pi(g) \in \mathbf{S}^1$.

Observe also that, for $\xi, \eta \in \mathcal{H}$, the absolute value of the matrix coefficient

$$C^\pi_{\xi,\eta} : g \mapsto \langle \pi(g)\xi, \eta\rangle$$

is constant on cosets modulo P_π. For a real number p with $1 \le p < +\infty$, the representation π is said to be *strongly L^p modulo P_π*, if there is a dense subspace $D \subset \mathcal{H}$ such that $|C^\pi_{\xi,\eta}| \in L^p(G/P_\pi)$ for all $\xi, \eta \in D$.

PROPOSITION 8. *Assume that the unitary representation π of the locally compact group G is strongly L^p modulo P_π for $1 \le p < +\infty$. Let k be an integer $k \ge p/2$. Then the tensor power $\pi^{\otimes k}$ is contained in an infinite multiple of* $\mathrm{Ind}_{P_\pi}^G \lambda_\pi^k$, *where λ_π is the unitary character of P_π associated to π.*

Proof. Observe that $\sigma := \pi^{\otimes k}$ is square-integrable modulo P_π for every integer $k \ge p/2$. It follows (see [**HoMo79**, Proposition 4.2] or [**HoTa92**, Ch. V, Proposition 1.2.3]) that σ is contained in an infinite multiple of $\mathrm{Ind}_{P_\sigma}^G \lambda_\sigma = \mathrm{Ind}_{P_\pi}^G \lambda_\pi^k$. □

5. *The Koopman representation of the automorphism group of a homogeneous space*

We establish a decomposition result for the Koopman representation of a group of automorphisms of an S-adic compact nilmanifold. We will state the result in the general context of a compact homogeneous space.

Let G be a locally compact group and Λ a lattice in G. We assume that Λ is cocompact in G. The homogeneous space $X := G/\Lambda$ carries a probability measure μ on the Borel subsets of X which is invariant by translations with elements from G. Every element from

$$\mathrm{Aut}(X) := \{\gamma \in \mathrm{Aut}(G) \mid \gamma(\Lambda) \subset \Lambda\}$$

induces a Borel isomorphism of X, which leaves μ invariant, as follows from the uniqueness of μ.

Given a subgroup Γ of $\mathrm{Aut}(X)$, the following crucial proposition gives a decomposition of the associated Koopman Γ on $L^2(X, \mu)$ as direct sum of certain induced representations of Γ.

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PROPOSITION 9. *Let G be a locally compact group and Λ a cocompact lattice in G, and let Γ be a countable subgroup of* $\mathrm{Aut}(X)$ *for* $X := G/\Lambda$. *Let κ denote the Koopman representation of Γ associated to the action $\Gamma \curvearrowright X$. There exists a family $(\pi_i)_{i\in I}$ of irreducible unitary representations of G such that κ is equivalent to a direct sum*

$$\bigoplus_{i\in I} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\widetilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where $\widetilde{\pi}_i$ is an irreducible projective representation of the stabilizer G_i of π_i in $\mathrm{Aut}(G) \ltimes G$ *extending π_i, and where W_i is a finite-dimensional projective unitary representation of $\Gamma_i := \Gamma \cap G_i$.*

Proof. We extend κ to a unitary representation, again denoted by κ, of $\Gamma \ltimes G$ on $L^2(X, \mu)$ given by

$$\kappa(\gamma, g)\xi(x) = \xi(\gamma^{-1}(gx)) \quad \text{for all } \gamma \in \Gamma, g \in G, \xi \in L^2(X, \mu),\ x \in X.$$

Identifying Γ and G with subgroups of $\Gamma \ltimes G$, we have

$$\kappa(\gamma^{-1})\kappa(g)\kappa(\gamma) = \kappa(\gamma^{-1}(g)) \quad \text{for all } \gamma \in \Gamma,\ g \in G. \qquad (*)$$

Since Λ is cocompact in G, we can consider the decomposition of $L^2(X, \mu)$ into G-isotypical components: we have (see [GGPS69, Ch. I, §3, Theorem])

$$L^2(X, \mu) = \bigoplus_{\pi\in\Sigma} \mathcal{H}_\pi,$$

where Σ is a certain set of pairwise non-equivalent irreducible unitary representations of G; for every $\pi \in \Sigma$, the space $\mathcal{H}_\pi$ is the union of the closed $\kappa(G)$-invariant subspaces $\mathcal{K}$ of $\mathcal{H} := L^2(X, \mu)$ for which the corresponding representation of G in $\mathcal{K}$ is equivalent to π; moreover, the multiplicity of every π is finite, that is, every $\mathcal{H}_\pi$ is a direct sum of finitely many irreducible unitary representations of G.

Let γ be a fixed automorphism in Γ. Let κ^γ be the conjugate representation of κ by γ, that is, $\kappa^\gamma(g) = \kappa(\gamma g \gamma^{-1})$ for all $g \in \Gamma \ltimes G$. On the one hand, for every $\pi \in \Sigma$, the isotypical component of $\kappa^\gamma|_G$ corresponding to π is $\mathcal{H}_{\pi^{\gamma^{-1}}}$. On the other hand, relation $(*)$ shows that $\kappa(\gamma)$ is a unitary equivalence between $\kappa|_G$ and $\kappa^\gamma|_G$. It follows that

$$\kappa(\gamma)(\mathcal{H}_\pi) = \mathcal{H}_{\pi^\gamma} \quad \text{for all } \gamma \in \Gamma;$$

so, Γ permutes the $\mathcal{H}_\pi$s among themselves according to its action on $\widehat{G}$.

Write $\Sigma = \bigcup_{i\in I} \Sigma_i$, where the Σ_i are the Γ-orbits in Σ, and set

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{\pi\in\Sigma_i} \mathcal{H}_\pi.$$

Every $\mathcal{H}_{\Sigma_i}$ is invariant under $\Gamma \ltimes G$ and we have an orthogonal decomposition

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\Sigma_i}.$$

Fix $i \in I$. Choose a representation π_i in Σ_i and set $\mathcal{H}_i = \mathcal{H}_{\pi_i}$. Let Γ_i denote the stabilizer of π_i in Γ. The space $\mathcal{H}_i$ is invariant under Γ_i. Let V_i be the corresponding representation of $\Gamma_i \ltimes G$ on $\mathcal{H}_i$.

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Choose a set S_i of representatives for the cosets in

$$\Gamma/\Gamma_i = (\Gamma \ltimes G)/(\Gamma_i \ltimes G)$$

with $e \in S_i$. Then $\Sigma_i = \{\pi_i^s : s \in S_i\}$ and the Hilbert space $\mathcal{H}_{\Sigma_i}$ is the sum of mutually orthogonal spaces:

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{s \in S_i} \mathcal{H}_i^s.$$

Moreover, $\mathcal{H}_i^s$ is the image under $\kappa(s)$ of $\mathcal{H}_i$ for every $s \in S_i$. This means that the restriction κ_i of κ to $\mathcal{H}_{\Sigma_i}$ of the Koopman representation κ of Γ is equivalent to the induced representation $\mathrm{Ind}_{\Gamma_i}^{\Gamma} V_i$.

Since every $\mathcal{H}_i$ is a direct sum of finitely many irreducible unitary representations of G, we can assume that $\mathcal{H}_i$ is the tensor product

$$\mathcal{H}_i = \mathcal{K}_i \otimes \mathcal{L}_i$$

of the Hilbert space $\mathcal{K}_i$ of π_i with a finite-dimensional Hilbert space $\mathcal{L}_i$, in such a way that

$$V_i(g) = \pi_i(g) \otimes I_{\mathcal{L}_i} \quad \text{for all } g \in G. \tag{$**$}$$

Let $\gamma \in \Gamma_i$. By $(*)$ and $(**)$ above, we have

$$V_i(\gamma)(\pi_i(g) \otimes I_{\mathcal{L}_i})V_i(\gamma)^{-1} = \pi_i(\gamma g \gamma^{-1}) \otimes I_{\mathcal{L}_i} \tag{$***$}$$

for all $g \in G$. On the other hand, let G_i be the stabilizer of π_i in $\mathrm{Aut}(G) \ltimes G$; then π_i extends to an irreducible projective representation $\widetilde{\pi}_i$ of G_i (see §2). Since

$$\widetilde{\pi}_i(\gamma)\pi_i(g)\,\widetilde{\pi}_i(\gamma^{-1}) = \pi_i(\gamma g \gamma^{-1}) \quad \text{for all } g \in G,$$

it follows from $(***)$ that $(\widetilde{\pi}_i(\gamma^{-1}) \otimes I_{\mathcal{L}_i})V_i(\gamma)$ commutes with $\pi_i(g) \otimes I_{\mathcal{L}_i}$ for all $g \in G$. As π_i is irreducible, there exists a unitary operator $W_i(\gamma)$ on $\mathcal{L}_i$ such that

$$V_i(\gamma) = \widetilde{\pi}_i(\gamma) \otimes W_i(\gamma).$$

It is clear that W_i is a projective unitary representation of $\Gamma_i \ltimes G$, since V_i is a unitary representation of $\Gamma_i \ltimes G$. □

6. *Unitary dual of solenoids*

Let p be either a prime integer or $p = \infty$. Define an element e_p in the unitary dual group $\widehat{\mathbf{Q}_p}$ of the additive group of $\mathbf{Q}_p$ (recall that $\mathbf{Q}_\infty = \mathbf{R}$) by $e_p(x) = e^{2\pi i x}$ if $p = \infty$ and $e_p(x) = \exp(2\pi i\{x\})$ otherwise, where $\{x\} = \sum_{j=m}^{-1} a_j p^j$ denotes the 'fractional part' of a p-adic number $x = \sum_{j=m}^{\infty} a_j p^j$ for integers $m \in \mathbf{Z}$ and $a_j \in \{0, \ldots, p-1\}$. Observe that $\mathrm{Ker}(e_p) = \mathbf{Z}$ if $p = \infty$ and that $\mathrm{Ker}(e_p) = \mathbf{Z}_p$ if p is a prime integer, where $\mathbf{Z}_p$ is the ring of p-adic integers. The map

$$\mathbf{Q}_p \to \widehat{\mathbf{Q}_p}, \quad y \mapsto (x \mapsto e_p(xy))$$

is an isomorphism of topological groups (see [**BeHV08**, §D.4]).

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Fix an integer $d \geq 1$. Then $\widehat{\mathbf{Q}_p^d}$ will be identified with $\mathbf{Q}_p^d$ by means of the map

$$\mathbf{Q}_p^d \to \widehat{\mathbf{Q}_p^d}, \quad y \mapsto x \mapsto e_p(x \cdot y),$$

where $x \cdot y = \sum_{i=1}^d x_i y_i$ for $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbf{Q}_p^d$.

Let $S = \{p_1, \ldots, p_r, \infty\}$, where $p_1, \ldots, p_r$ are integer primes. For an integer $d \geq 1$, consider the *S-adic solenoid*

$$\mathbf{Sol}_S = \mathbf{Q}_S^d / \mathbf{Z}[1/S]^d,$$

where $\mathbf{Z}[1/S]^d = \mathbf{Z}[1/p_1, \ldots, 1/p_r]^d$ is embedded diagonally in $\mathbf{Q}_S = \prod_{p \in S} \mathbf{Q}_p$. Then $\widehat{\mathbf{Sol}_S}$ is identified with the annihilator of $\mathbf{Z}[1/S]^d$ in $\mathbf{Q}_S^d$, that is, with $\mathbf{Z}[1/S]^d$ embedded in $\mathbf{Q}_S^d$ via the map

$$\mathbf{Z}[1/S]^d \to \mathbf{Q}_S^d, \quad b \mapsto (b, -b, \cdots, -b).$$

Under this identification, the dual action of the automorphism group

$$\mathrm{Aut}(\mathbf{Q}_S^d) \cong GL_d(\mathbf{R}) \times GL_d(\mathbf{Q}_{p_1}) \times \cdots \times GL_d(\mathbf{Q}_{p_r}).$$

on $\widehat{\mathbf{Q}_S^d}$ corresponds to the right action on $\mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \cdots \times \mathbf{Q}_{p_r}^d$ given by

$$((g_\infty, g_1, \ldots, g_r), (a_\infty, a_1, \ldots, a_r)) \mapsto (g_\infty^t a_\infty, g_1^t a_1, \ldots, g_r^t a_r),$$

where $(g, a) \mapsto ga$ is the usual (left) linear action of $GL_d(\mathbf{k})$ on $\mathbf{k}^d$ for a field $\mathbf{k}$.

7. *Unitary representations of unipotent groups*

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$. The Lie algebra $\mathfrak{u}$ is defined over $\mathbf{Q}$ and the exponential map $\exp : \mathfrak{u} \to U$ is a bijective morphism of algebraic varieties.

Let p be either a prime integer or $p = \infty$. The irreducible unitary representations of $U_p := \mathbf{U}(\mathbf{Q}_p)$ are parametrized by Kirillov's theory as follows.

The Lie algebra of U_p is $\mathfrak{u}_p = \mathfrak{u}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$, where $\mathfrak{u}(\mathbf{Q})$ is the Lie algebra over $\mathbf{Q}$ consisting of the $\mathbf{Q}$-points in $\mathfrak{u}$.

Fix an element f in the dual space $\mathfrak{u}_p^* = \mathcal{H}om_{\mathbf{Q}_p}(\mathfrak{u}_p, \mathbf{Q}_p)$ of $\mathfrak{u}_p$. There exists a polarization $\mathfrak{m}$ for f, that is, a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{u}_p$ such that $f([\mathfrak{m}, \mathfrak{m}]) = 0$ and which is of maximal dimension. The induced representation $\mathrm{Ind}_M^{U_p} \chi_f$ is irreducible, where $M = \exp(\mathfrak{m})$ and χ_f is the unitary character of M defined by

$$\chi_f(\exp X) = e_p(f(X)) \quad \text{for all } X \in \mathfrak{m},$$

where $e_p \in \widehat{\mathbf{Q}_p}$ is as in §6. The unitary equivalence class of $\mathrm{Ind}_M^{U_p} \chi_f$ only depends on the co-adjoint orbit $\mathrm{Ad}^*(U_p) f$ of f. The resulting map

$$\mathfrak{u}_p^* / \mathrm{Ad}^*(U_p) \to \widehat{U_p}, \quad \mathcal{O} \mapsto \pi_{\mathcal{O}},$$

called the Kirillov map, from the orbit space $\mathfrak{u}_p^* / \mathrm{Ad}^*(U_p)$ of the co-adjoint representation to the unitary dual $\widehat{U}_p$ of U_p, is a bijection. In particular, U_p is a so-called type I locally compact group. For all of this, see [**Kiri62**] or [**CoGr89**] in the case of $p = \infty$ and [**Moor65**] in the case of a prime integer p.

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The group $\mathrm{Aut}(U_p)$ of continuous automorphisms of U_p can be identified with the group of $\mathbf{Q}_p$-points of the algebraic group $\mathrm{Aut}(\mathfrak{u})$ of automorphisms of the Lie algebra $\mathfrak{u}$ of $\mathbf{U}$. Notice also that the natural action of $\mathrm{Aut}(U_p)$ on $\mathfrak{u}_p$ as well as its dual action on $\mathfrak{u}_p^*$ are algebraic.

Let $\pi \in \widehat{U_p}$ with corresponding Kirillov orbit $\mathcal{O}_\pi$ and $g \in \mathrm{Aut}(U_p)$. Then $g(\mathcal{O}_\pi)$ is the Kirillov orbit associated to the conjugate representation π^g.

LEMMA 10. *Let π be an irreducible unitary representation of U_p. The stabilizer G_π of π in $\mathrm{Aut}(U_p)$ is an algebraic subgroup of $\mathrm{Aut}(U_p)$.*

Proof. Let $\mathcal{O}_\pi \subset \mathfrak{u}_p^*$ be the Kirillov orbit corresponding to π. Then G_π is the set of $g \in \mathrm{Aut}(U_p)$ such that $g(\mathcal{O}_\pi) = \mathcal{O}_\pi$. As $\mathcal{O}_\pi$ is an algebraic subvariety of $\mathfrak{u}_p^*$, the claim follows. □

8. *Decay of matrix coefficients of unitary representations of S-adic groups*

Let p be an integer prime or $p = \infty$ and let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}_p$. Set $U_p := \mathbf{U}(\mathbf{Q}_p)$.

Let π be an irreducible unitary representation of U_p. Recall (see Lemma 10) that the stabilizer G_π of π in $\mathrm{Aut}(U_p)$ is an algebraic subgroup of $\mathrm{Aut}(U_p)$. Recall also (see Lemma 6) that π extends to a projective representation of G_π. The following result was proved in [**BeGu15**, Proposition 22] in the case where $p = \infty$, using arguments from [**HoMo79**]. The proof in the case where p is a prime integer is along similar lines and will be omitted.

PROPOSITION 11. *Let π be an irreducible unitary representation of U_p and let $\widetilde{\pi}$ be a projective unitary representation of G_π which extends π. There exists a real number $r \geq 1$, only depending on the dimension of G_π, such that $\widetilde{\pi}$ is strongly L^r modulo its projective kernel.*

We will need later a precise description of the projective kernel of a representation $\widetilde{\pi}$ as above.

LEMMA 12. *Let π be an irreducible unitary representation of U_p and $\widetilde{\pi}$ a projective unitary representation of G_π which extends π. Let $\mathcal{O}_\pi \subset \mathfrak{u}_p^*$ be the corresponding Kirillov orbit of π. For $g \in \mathrm{Aut}(U_p)$, the following properties are equivalent.*

(i) *g belongs to the projective kernel $P_{\widetilde{\pi}}$ of $\widetilde{\pi}$.*

(ii) *For every $u \in U_p$, we have*

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \exp(\mathrm{Ker}(f)).$$

Proof. We can assume that $\pi = \mathrm{Ind}_M^{U_p} \chi_{f_0}$, for $f_0 \in \mathcal{O}_\pi$, and $M = \exp \mathfrak{m}$ for a polarization $\mathfrak{m}$ of f_0.

Let $g \in \mathrm{Aut}(U_p)$. If g is in the stabilizer G_π of π in $\mathrm{Aut}(U_p)$, recall (see Proof of Lemma 6) that

$$\pi(g(u)) = \widetilde{\pi}(g)\pi(u)\widetilde{\pi}(g^{-1}) \quad \text{for all } u \in U_p.$$

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Since π is irreducible, it follows from Schur's lemma that $g \in P_{\widetilde{\pi}}$ if and only if

$$\pi(g(u)) = \pi(u) \quad \text{for all } u \in U_p$$

that is,

$$g(u)u^{-1} \in \text{Ker}(\pi) \quad \text{for all } u \in U_p.$$

Now we have (see [**BeGu15**, Lemma 18])

$$\text{Ker}(\pi) = \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f),$$

and so $g \in P_{\widetilde{\pi}}$ if and only if

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p.$$

Let $g \in P_{\widetilde{\pi}}$. Denote by $X \mapsto g(X)$ the automorphism of $\mathfrak{u}_p$ corresponding to g. Let $u = \exp(X)$ for $X \in \mathfrak{u}_p$ and $f \in \mathcal{O}_\pi$. Set $u_t = \exp(tX)$. By the Campbell Hausdorff formula, there exist $Y_1, \ldots Y_r \in \mathfrak{u}_p$ such that

$$g(u_t)(u_t)^{-1} = \exp(tY_1 + t^2 Y_2 + \cdots + t^r Y_r),$$

for every $t \in \mathbf{Q}_p$. Since

$$1 = \chi_f(g(u_t)(u_t)^{-1}) = e_p(f(tY_1 + t^2 Y_2 + \cdots + t^r Y_2)), \tag{$*$}$$

it follows that the polynomial

$$t \mapsto Q(t) = tf(Y_1) + t^2 f(Y_2) + \cdots + t^r f(Y_r)$$

takes its values in $\mathbf{Z}$ if $p = \infty$, and in $\mathbf{Z}_p$ (and so Q has bounded image) otherwise. This clearly implies that $Q(t) = 0$ for all $t \in \mathbf{Q}_p$; in particular, we have

$$\log(g(u)u^{-1}) = Y_1 + Y_2 + \cdots + Y_r \in \text{Ker}(f).$$

This shows that (i) implies (ii).

Conversely, assume that (ii) holds. Then clearly

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p$$

and so $g \in P_{\widetilde{\pi}}$. □

9. *Decomposition of the Koopman representation for a nilmanifold*

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$. Let $S = \{p_1, \ldots, p_r, \infty\}$, where $p_1, \ldots, p_r$ are integer primes. Set

$$U := \mathbf{U}(\mathbf{Q}_S) = \prod_{p \in S} U_p.$$

Since U is a type I group, the unitary dual $\widehat{U}$ of U can be identified with the cartesian product $\prod_{p \in S} \widehat{U_p}$ via the map

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$$\prod_{p\in S} \widehat{U_p} \to \widehat{U}, \quad (\pi_p)_{p\in S} \mapsto \bigotimes_{p\in S} \pi_p,$$

where $\bigotimes_{p\in S} \pi_p = \pi_\infty \otimes \pi_{p_1} \otimes \cdots \otimes \pi_{p_r}$ is the tensor product of the π_p.

Let $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$ and consider the corresponding *S-adic compact nilmanifold*

$$\mathbf{Nil}_S := U/\Lambda,$$

equipped with the unique U-invariant probability measure μ on its Borel subsets.

The associated *S-adic solenoid* is

$$\mathbf{Sol}_S = \overline{U}/\overline{\Lambda},$$

where $\overline{U} := U/[U, U]$ is the quotient of U by its closed commutator subgroup $[U, U]$ and where $\overline{\Lambda}$ is the image of $\mathbf{U}(\mathbf{Z}[1/S])$ in $\overline{U}$.

Set

$$\mathrm{Aut}(U) := \prod_{p\in S} \mathrm{Aut}(\mathbf{U}(\mathbf{Q}_p))$$

and denote by $\mathrm{Aut}(\mathbf{Nil}_S)$ the subgroup of all $g \in \mathrm{Aut}(U)$ with $g(\Lambda) = \Lambda$. Observe that $\mathrm{Aut}(\mathbf{Nil}_S)$ is a discrete subgroup of $\mathrm{Aut}(U)$, where every $\mathrm{Aut}(U_p)$ is endowed with its natural (locally compact) topology and $\mathrm{Aut}(U)$ with the product topology.

Let Γ be a subgroup of $\mathrm{Aut}(\mathbf{Nil}_S)$. Let κ be the Koopman representation of $\Gamma \ltimes U$ on $L^2(\mathbf{Nil}_S)$ associated to the action $\Gamma \ltimes U \curvearrowright \mathbf{Nil}_S$. By Proposition 9, there exists a family $(\pi_i)_{i\in I}$ of irreducible representations of U, such that κ is equivalent to

$$\bigoplus_{i\in I} \mathrm{Ind}_{\Gamma_i \ltimes U}^{\Gamma \ltimes U}(\widetilde{\pi_i} \otimes W_i),$$

where $\widetilde{\pi_i}$ is an irreducible projective representation $\widetilde{\pi_i}$ of the stabilizer G_i of π_i in $\mathrm{Aut}(U) \ltimes U$ extending π_i, and where W_i is a projective unitary representation of $G_i \cap (\Gamma \ltimes U)$.

Fix $i \in I$. We have $\pi_i = \bigotimes_{p\in S} \pi_{i,p}$ for irreducible representations $\pi_{i,p}$ of U_p.

We will need the following more precise description of π_i. Recall that $\mathfrak{u}$ is the Lie algebra of $\mathbf{U}$ and that $\mathfrak{u}(\mathbf{Q})$ denotes the Lie algebra over $\mathbf{Q}$ consisting of the $\mathbf{Q}$-points in $\mathfrak{u}$.

Let $\mathfrak{u}^*(\mathbf{Q})$ be the set of $\mathbf{Q}$-rational points in the dual space $\mathfrak{u}^*$; so, $\mathfrak{u}^*(\mathbf{Q})$ is the subspace of $f \in \mathfrak{u}^*$ with $f(X) \in \mathbf{Q}$ for all $X \in \mathfrak{u}(\mathbf{Q})$. Observe that, for $f \in \mathfrak{u}^*(\mathbf{Q})$, we have $f(X) \in \mathbf{Q}_p$ for all $X \in \mathfrak{u}_p = \mathfrak{u}(\mathbf{Q}_p)$.

A polarization for $f \in \mathfrak{u}^*(\mathbf{Q})$ is a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{u}(\mathbf{Q})$ such that $f([\mathfrak{m}, \mathfrak{m}]) = 0$ and which is of maximal dimension with this property.

PROPOSITION 13. *Let $\pi_i = \bigotimes_{p\in S} \pi_{i,p}$ be one of the irreducible representations of $U = \mathbf{U}(\mathbf{Q}_S)$ appearing in the decomposition $L^2(\mathbf{Nil}_S)$ as above. There exist $f_i \in \mathfrak{u}^*(\mathbf{Q})$ and a polarization $\mathfrak{m}_i \subset \mathfrak{u}(\mathbf{Q})$ for f_i with the following property: for every $p \in S$, the representation $\pi_{i,p}$ is equivalent to $\mathrm{Ind}_{M_{i,p}}^{U} \chi_{f_i}$, where:*

- *$M_{i,p} = \exp(\mathfrak{m}_{i,p})$ for $\mathfrak{m}_{i,p} = \mathfrak{m}_i \otimes_{\mathbf{Q}} Q_p$;*
- *χ_{f_i} is the unitary character of $M_{i,p}$ given by $\chi_{f_i}(\exp X) = e_p(f_i(X))$, for all $X \in \mathfrak{m}_{i,p}$, with $e_p \in \widehat{\mathbf{Q}_p}$ as in §6.*

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Proof. The same result is proved in [**Moor65**, Theorem 11] (see also [**Fox89**, Theorem 1.2]) for the Koopman representation of $\mathbf{U}(\mathbf{A})$ in $L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$, where $\mathbf{A}$ is the ring of adeles of $\mathbf{Q}$. We could check that the proof, which proceeds by induction of the dimension of $\mathbf{U}$, carries over to the Koopman representation on $L^2(\mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S]))$, with the appropriate changes. We prefer to deduce our claim from the result for $\mathbf{U}(\mathbf{A})$, as follows.

It is well known (see [**Weil74**]) that

$$\mathbf{A} = \left(\mathbf{Q}_S \times \prod_{p \notin S} \mathbf{Z}_p\right) + \mathbf{Q}$$

and that

$$\left(\mathbf{Q}_S \times \prod_{p \notin S} \mathbf{Z}_p\right) \cap \mathbf{Q} = \mathbf{Z}[1/S].$$

This gives rise to a well-defined projection $\varphi : \mathbf{A}/\mathbf{Q} \to \mathbf{Q}_S/\mathbf{Z}[1/S]$ given by

$$\varphi((a_S, (a_p)_{p\notin S}) + \mathbf{Q}) = a_S + \mathbf{Z}[1/S] \quad \text{for all } a_S \in \mathbf{Q}_S,\ (a_p)_{p\notin S} \in \prod_{p\notin S} \mathbf{Z}_p;$$

so the fiber over a point $a_S + \mathbf{Z}[1/S] \in \mathbf{Q}_S/\mathbf{Z}[1/S]$ is

$$\varphi^{-1}(a_S + \mathbf{Z}[1/S]) = \{(a_S, (a_p)_{p\notin S}) + \mathbf{Q} \mid a_p \in \mathbf{Z}_p \text{ for all } p\}.$$

This induces an identification of $\mathbf{U}(\mathbf{Q_S})/\mathbf{U}(\mathbf{Z}[1/S]) = \mathbf{Nil}_S$ with the double coset space $K_S\backslash\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})$, where K_S is the compact subgroup

$$K_S = \prod_{p\notin S} \mathbf{U}(\mathbf{Z}_p)$$

of $\mathbf{U}(\mathbf{A})$. Observe that this identification is equivariant under translation by elements from $\mathbf{U}(\mathbf{Q}_S)$. In this way, we can view $L^2(\mathbf{Nil}_S)$ as the $\mathbf{U}(\mathbf{Q}_S)$-invariant subspace $L^2(K_S\backslash\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$ of $L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$.

Choose a system T of representatives for the $\mathrm{Ad}^*(\mathbf{U}(\mathbf{Q}))$-orbits in $\mathfrak{u}^*(\mathbf{Q})$. By [**Moor65**, Theorem 11], for every $f \in T$, we can find a polarization $\mathfrak{m}_f \subset \mathfrak{u}(\mathbf{Q})$ for f with the following property: setting

$$\mathfrak{m}_f(\mathbf{A}) = \mathfrak{m}_f \otimes_{\mathbf{Q}} \mathbf{A},$$

we have a decomposition

$$L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})) = \bigoplus_{f\in T} \mathcal{H}_f$$

into irreducible $\mathbf{U}(\mathbf{A})$-invariant subspaces $\mathcal{H}_f$ such that the representation π_f of $\mathbf{U}(\mathbf{A})$ in $\mathcal{H}_f$ is equivalent to $\mathrm{Ind}_{M_f(\mathbf{A})}^{\mathbf{U}(\mathbf{A})}\chi_f$, where

$$M_f(\mathbf{A}) = \exp(\mathfrak{m}_f(\mathbf{A}))$$

and $\chi_{f,\mathbf{A}}$ is the unitary character of $M_f(\mathbf{A})$ given by

$$\chi_{f,\mathbf{A}}(\exp X) = e(f(X)) \quad \text{for all } X \in \mathfrak{m}_f(\mathbf{A});$$

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here, e is the unitary character of $\mathbf{A}$ defined by

$$e((a_p)_p) = \prod_{p \in \mathcal{P} \cup \{\infty\}} e_p(a_p) \quad \text{for all } (a_p)_p \in \mathbf{A},$$

where $\mathcal{P}$ is the set of integer primes.

We have

$$L^2(K_S \backslash \mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})) = \bigoplus_{f \in T} \mathcal{H}_f^{K_S},$$

where $\mathcal{H}_f^{K_S}$ is the space of K_S-fixed vectors in $\mathcal{H}_f$. It is clear that the representation of $\mathbf{U}(\mathbf{Q}_S)$ in $\mathcal{H}_f^{K_S}$ is equivalent to

$$\mathrm{Ind}_{M_f(\mathbf{Q}_S)}^{\mathbf{U}(\mathbf{Q}_S)} \bigg(\bigotimes_{p \in S} \chi_{f,p} \bigg) = \bigotimes_{p \in S} (\mathrm{Ind}_{M_f(\mathbf{Q}_p)}^{\mathbf{U}(\mathbf{Q}_p)} \chi_{f,p}),$$

where $\chi_{f,p}$ is the unitary character of $M_f(\mathbf{Q}_p)$ given by

$$\chi_{f,p}(\exp X) = e_p(f(X)) \quad \text{for all } X \in \mathfrak{m}_f(\mathbf{Q}_p).$$

Since $M_f(\mathbf{Q}_p)$ is a polarization for f, each of the $\mathbf{U}(\mathbf{Q}_p)$-representations $\mathrm{Ind}_{M_f(\mathbf{Q_p})}^{\mathbf{U(Q_p)}} \chi_{f,p}$ and, hence, each of the $\mathbf{U}(\mathbf{Q}_S)$-representations

$$\mathrm{Ind}_{M_f(\mathbf{Q}_S)}^{\mathbf{U}(\mathbf{Q}_S)} \bigg(\bigotimes_{p \in S} \chi_{f,p} \bigg)$$

is irreducible. This proves the claim. □

We establish another crucial fact about the representations π_i in the following proposition.

PROPOSITION 14. *With the notation of Proposition 13, let $\mathcal{O}_{\mathbf{Q}}(f_i)$ be the co-adjoint orbit of f_i under $\mathbf{U}(\mathbf{Q})$ and set*

$$\mathfrak{k}_{i,p} = \bigcap_{f \in \mathcal{O}_{\mathbf{Q}}(f_i)} \mathfrak{k}_p(f),$$

where $\mathfrak{k}_p(f)$ is the kernel of f in $\mathfrak{u}_p$. Let $K_{i,p} = \exp(\mathfrak{k}_{i,p})$ and $K_i = \prod_{p \in S} K_{i,p}$.

(i) *K_i is a closed normal subgroup of U and $K_i \cap \Lambda = K_i \cap \mathbf{U}(\mathbf{Z}[1/S])$ is a lattice in K_i.*

(ii) *Let $P_{\widetilde{\pi}_i}$ be the projective kernel of the extension $\widetilde{\pi}_i$ of π_i to the stabilizer G_i of π_i in $\mathrm{Aut}(U) \ltimes U$. For $g \in G_i$, we have $g \in P_{\widetilde{\pi}_i}$ if and only if $g(u) \in uK_i$ for every $u \in U$.*

Proof. (i) Let

$$\mathfrak{k}_{i,\mathbf{Q}} = \bigcap_{f \in \mathcal{O}_{\mathbf{Q}}(f_i)} \mathfrak{k}_{\mathbf{Q}(f)},$$

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where $\mathfrak{k}_{\mathbf{Q}}(f)$ is the kernel of f in $\mathfrak{u}(\mathbf{Q})$. Observe that $\mathfrak{k}_{i,\mathbf{Q}}$ is an ideal in $\mathfrak{u}(\mathbf{Q})$, since it is $\mathrm{Ad}(\mathbf{U}(\mathbf{Q}))$-invariant. So, we have

$$\mathfrak{k}_{i,\mathbf{Q}} = \mathfrak{k}_i(\mathbf{Q})$$

for an ideal $\mathfrak{k}_i$ in $\mathfrak{u}$. Since $f \in \mathfrak{u}^*(\mathbf{Q})$ for $f \in \mathcal{O}_{\mathbf{Q}}(f_i)$, we have

$$\mathfrak{k}_{i,p}(f) = \mathfrak{k}_{i,\mathbf{Q}}(f) \otimes_{\mathbf{Q}} \mathbf{Q}_p$$

and hence

$$\mathfrak{k}_{i,p} = \mathfrak{k}_i(\mathbf{Q}_p).$$

Let $\mathbf{K}_i = \exp(\mathfrak{k}_i)$. Then $\mathbf{K}_i$ is a normal algebraic $\mathbf{Q}$-subgroup of $\mathbf{U}$ and we have $K_{i,p} = \mathbf{K}_i(\mathbf{Q}_p)$ for every p; so,

$$K_i = \prod_{s\in S} \mathbf{K}_i(\mathbf{Q}_p) = \mathbf{K}_i(\mathbf{Q}_S)$$

and $K_i \cap \Lambda = \mathbf{K}_i(\mathbf{Z}[1/S])$ is a lattice in K_i. This proves (i).

To prove (ii), observe that

$$P_{\widetilde{\pi}_i} = \prod_{p\in S} P_{i,p},$$

where $P_{i,p}$ is the projective kernel of $\widetilde{\pi_{i,p}}$.

Fix $p \in S$ and let $g \in G_i$. By Lemma 12, $g \in P_{i,p}$ if and only if $g(u) \in uK_{i,p}$ for every $u \in U_p = \mathbf{U}(\mathbf{Q}_p)$. This finishes the proof. □

10. *Proof of Theorem 1*

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$ and $S = \{p_1, \ldots, p_r, \infty\}$, where $p_1, \ldots, p_r$ are integer primes. Set $U := \mathbf{U}(\mathbf{Q}_S)$ and $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$. Let $\mathbf{Nil}_S = U/\Lambda$ and $\mathbf{Sol}_S$ be the S-adic nilmanifold and the associated S-adic solenoid as in §9. Denote by μ the translation-invariant probability measure on $\mathbf{Nil}_S$ and let ν be the image of μ under the canonical projection $\varphi : \mathbf{Nil}_S \to \mathbf{Sol}_S$. We identify $L^2(\mathbf{Sol}_S) = L^2(\mathbf{Sol}_S, \nu)$ with the closed $\mathrm{Aut}(\mathbf{Nil}_S)$-invariant subspace

$$\{f \circ \varphi \mid f \in L^2(\mathbf{Sol}_S)\}$$

of $L^2(\mathbf{Nil}_S) = L^2(\mathbf{Nil}_S, \mu)$. We have an orthogonal decomposition into $\mathrm{Aut}(\mathbf{Nil}_S)$-invariant subspaces

$$L^2(\mathbf{Nil}_S) = \mathbf{C}\mathbf{1}_{\mathbf{Nil}_S} \oplus L^2_0(\mathbf{Sol}_S) \oplus \mathcal{H},$$

where

$$L^2_0(\mathbf{Sol}_S) = \left\{ f \in L^2(\mathbf{Sol}_S) \,\middle|\, \int_{\mathbf{Nil}_S} f\, d\mu = 0 \right\}$$

and where $\mathcal{H}$ is the orthogonal complement of $L^2(\mathbf{Sol}_S)$ in $L^2(\mathbf{Nil}_S)$.

Let Γ be a subgroup of $\mathrm{Aut}(\mathbf{Nil}_S)$. Let κ be the Koopman representation of Γ on $L^2(\mathbf{Nil}_S)$ and denote by κ_1 and κ_2 the restrictions of κ to $L^2_0(\mathbf{Sol}_S)$ and $\mathcal{H}$, respectively.

Let Σ_1 be a set of representatives for the Γ-orbits in $\widehat{\mathbf{Sol}_S} \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$. We have

$$\kappa_1 \cong \bigoplus_{\chi \in \Sigma_1} \lambda_{\Gamma/\Gamma_\chi},$$

where Γ_χ is the stabilizer of χ in Γ and $\lambda_{\Gamma/\Gamma_\chi}$ is the quasi-regular representation of Γ on $\ell^2(\Gamma/\Gamma_\chi)$.

By Proposition 9, there exists a family $(\pi_i)_{i\in I}$ of irreducible representations of U, such that κ_2 is equivalent to a direct sum

$$\bigoplus_{i\in I} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\widetilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where $\widetilde{\pi}_i$ is an irreducible projective representation of the stabilizer G_i of π_i in $\mathrm{Aut}(U)$ and where W_i is a projective unitary representation of $\Gamma_i := \Gamma \cap G_i$.

PROPOSITION 15. *For $i \in I$, let $\widetilde{\pi}_i$ be the (projective) representation of G_i and let Γ_i be as above. There exists a real number $r \geq 1$ such that $\widetilde{\pi}_i|_{\Gamma_i}$ is strongly L^r modulo $P_{\widetilde{\pi}_i} \cap \Gamma_i$, where $P_{\widetilde{\pi}_i}$ is the projective kernel of $\widetilde{\pi}_i$.*

Proof. By Proposition 11, there exists a real number $r \geq 1$ such that the representation $\widetilde{\pi}_i$ of the algebraic group G_i is strongly L^r modulo $P_{\widetilde{\pi}_i}$. In order to show that $\widetilde{\pi}_i|_{\Gamma_i}$ is strongly L^r modulo $P_{\widetilde{\pi}_i} \cap \Gamma_i$, it suffices to show that $\Gamma_i P_{\widetilde{\pi}_i}$ is closed in G_i (compare with the proof of [**HoMo79**, Proposition 6.2]).

Let K_i be the closed G_i-invariant normal subgroup K_i of U as described in Proposition 14. Then $\overline{\Lambda} = K_i\Lambda/K_i$ is a lattice in the unipotent group $\overline{U} = U/K_i$. By Proposition 14(ii), $P_{\widetilde{\pi}_i}$ coincides with the kernel of the natural homomorphism $\varphi : \mathrm{Aut}(U) \to \mathrm{Aut}(\overline{U})$. Hence, we have

$$\Gamma_i P_{\widetilde{\pi}_i} = \varphi^{-1}(\varphi(\Gamma_i)).$$

Now, $\varphi(\Gamma_i)$ is a discrete (and hence closed) subgroup of $\mathrm{Aut}(\overline{U})$, since $\varphi(\Gamma_i)$ preserves $\overline{\Lambda}$ (and so $\varphi(\Gamma_i) \subset \mathrm{Aut}(\overline{U}/\overline{\Lambda})$). It follows from the continuity of φ that $\varphi^{-1}(\varphi(\Gamma_i))$ is closed in $\mathrm{Aut}(U)$. □

Proof of Theorem 1. We have to show that, if 1_Γ is weakly contained in κ_2, then 1_Γ is weakly contained in κ_1. It suffices to show that, if 1_Γ is weakly contained in κ_2, then there exists a finite-index subgroup H of Γ such that 1_H is weakly contained in $\kappa_1|_H$ (see [**BeFr20**, Theorem 2]).

We proceed by induction on the integer

$$n(\Gamma) := \sum_{p\in S} \dim \mathrm{Zc}_p(\Gamma),$$

where $\mathrm{Zc}_p(\Gamma)$ is the Zariski closure of the projection of Γ in $GL_n(\mathbf{Q}_p)$.

If $n(\Gamma) = 0$, then Γ is finite and there is nothing to prove.

Assume that $n(\Gamma) \geq 1$ and that the claim above is proved for every countable subgroup H of $\mathrm{Aut}(\mathbf{Nil}_S)$ with $n(H) < n(\Gamma)$.

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Let $I_{\text{fin}} \subset I$ be the set of all $i \in I$ such that $\Gamma_i = G_i \cap \Gamma$ has finite index in Γ and set $I_\infty = I \setminus I_{\text{fin}}$. With $V_i = \widetilde{\pi_i}|_{\Gamma_i} \otimes W_i$, set

$$\kappa_2^{\text{fin}} = \bigoplus_{i \in I_{\text{fin}}} \operatorname{Ind}_{\Gamma_i}^{\Gamma} V_i \quad \text{and} \quad \kappa_2^{\infty} = \bigoplus_{i \in I_\infty} \operatorname{Ind}_{\Gamma_i}^{\Gamma} V_i.$$

Two cases can occur.

First case: 1_Γ is weakly contained in κ_2^∞. Observe that $n(\Gamma_i) < n(\Gamma)$ for $i \in I_\infty$. Indeed, otherwise $\mathrm{Zc}_p(\Gamma_i)$ and $\mathrm{Zc}_p(\Gamma)$ would have the same connected component C_p^0 for every $p \in S$, since $\Gamma_i \subset \Gamma$. Then

$$C^0 := \prod_{p \in S} C_p^0$$

would stabilize π_i and $\Gamma \cap C^0$ would therefore be contained in Γ_i. Since $\Gamma \cap C^0$ has finite index in Γ, this would contradict the fact that Γ_i has infinite index in Γ.

By restriction, 1_{Γ_i} is weakly contained in $\kappa_2|_{\Gamma_i}$ for every $i \in I$. Hence, by the induction hypothesis, 1_{Γ_i} is weakly contained in $\kappa_1|_{\Gamma_i}$ for every $i \in I_\infty$. Now, on the one hand, we have

$$\kappa_1|_{\Gamma_i} \cong \bigoplus_{\chi \in T_i} \lambda_{\Gamma_i/\Gamma_\chi \cap \Gamma_i},$$

for a subset T_i of $\widehat{\mathbf{Sol}_S} \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$. It follows that $\operatorname{Ind}_{\Gamma_i}^{\Gamma} 1_{\Gamma_i} = \lambda_{\Gamma/\Gamma_i}$ is weakly contained in

$$\bigoplus_{\chi \in T_i} \operatorname{Ind}_{\Gamma_i}^{\Gamma}(\lambda_{\Gamma_i/\Gamma_\chi \cap \Gamma_i}) = \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi \cap \Gamma_i},$$

for every $i \in I_\infty$. On the other hand, since 1_Γ is weakly contained in

$$\kappa_2^\infty \cong \bigoplus_{i \in I_\infty} \operatorname{Ind}_{\Gamma_i}^{\Gamma}(\widetilde{\pi_i}|_{\Gamma_i} \otimes W_i),$$

Lemma 7 shows that 1_Γ is weakly contained in $\bigoplus_{i \in I_\infty} \lambda_{\Gamma/\Gamma_i}$. It follows that 1_Γ is weakly contained in

$$\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi \cap \Gamma_i}.$$

Hence, by Lemma 7 again, 1_Γ is weakly contained in

$$\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi}.$$

This shows that 1_Γ is weakly contained in κ_1.

Second case: 1_Γ is weakly contained in κ_2^{fin}. By the Noetherian property of the Zariski topology, we can find finitely many indices $i_1, \ldots, i_r$ in I_{fin} such that, for every $p \in S$, we have

$$\mathrm{Zc}_p(\Gamma_{i_1}) \cap \cdots \cap \mathrm{Zc}_p(\Gamma_{i_r}) = \bigcap_{i \in I_{\text{fin}}} \mathrm{Zc}_p(\Gamma_i),$$

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Set $H := \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_r}$. Observe that H has finite index in Γ. Moreover, it follows from Lemma 10 that $\mathrm{Zc}_p(\Gamma_{i_1}) \cap \cdots \cap \mathrm{Zc}_p(\Gamma_{i_r})$ stabilizes $\pi_{i,p}$ for every $i \in I_{\mathrm{fin}}$ and $p \in S$. Hence, H is contained in Γ_i for every $i \in I_{\mathrm{fin}}$.

By Proposition 9, we have a decomposition of $\kappa_2^{\mathrm{fin}}|_H$ into the direct sum

$$\bigoplus_{i \in I_{\mathrm{fin}}} (\widetilde{\pi}_i \otimes W_i)|_H.$$

By Propositions 11 and 15, there exists a real number $r \geq 1$, which is independent of i, such that $(\widetilde{\pi}_i \otimes W_i)|_H$ is a strongly L^r representation of H modulo its projective kernel P_i. Observe that P_i is contained in the projective kernel $P_{\widetilde{\pi}_i}$ of $\widetilde{\pi}_i$, since $P_i = P_{\widetilde{\pi}_i} \cap H$. Hence (see Proposition 8), there exists an integer $k \geq 1$ such that $(\kappa_2^{\mathrm{fin}}|_H)^{\otimes k}$ is contained in a multiple of the direct sum

$$\bigoplus_{i \in I_{\mathrm{fin}}} \mathrm{Ind}_{P_i}^H \rho_i,$$

for representations ρ_i of P_i. Since 1_H is weakly contained in $\kappa_2^{\mathrm{fin}}|H$ and hence in $(\kappa_2^{\mathrm{fin}}|_H)^{\otimes k}$, using Lemma 7, it follows that 1_H is weakly contained in

$$\bigoplus_{i \in I_{\mathrm{fin}}} \lambda_{H/P_i}.$$

Let $i \in I$. We claim that P_i is contained in Γ_χ for some character χ from $\widehat{\mathbf{Sol}}_S \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$. Once proved, this will imply, again by Lemma 7, 1_H is weakly contained in $\kappa_1|_H$. Since H has finite index in Γ, this will show that 1_Γ is weakly contained in κ_1 and conclude the proof.

To prove the claim, recall from Proposition 14 that there exists a closed normal subgroup K_i of U with the following properties: $K_i\Lambda/K_i$ is a lattice in the unipotent algebraic group U/K_i, K_i is invariant under $P_{\widetilde{\pi}_i}$ and $P_{\widetilde{\pi}_i}$ acts as the identity on U/K_i. Observe that $K_i \neq U$, since π_i is not trivial on U. We can find a non-trivial unitary character χ of U/K_i which is trivial on $K_i\Lambda/K_i$. Then χ lifts to a non-trivial unitary character of U which is fixed by $P_{\widetilde{\pi}_i}$ and hence by P_i. Observe that $\chi \in \widehat{\mathbf{Sol}}_S$, since χ is trivial on Λ. □

11. *An example: the S-adic Heisenberg nilmanifold*

As an example, we study the spectral gap property for groups of automorphisms of the S-adic Heisenberg nilmanifold, proving Corollary 5. We will give a quantitative estimate for the norm of associated convolution operators, as we did in [**BeHe11**] in the case of real Heisenberg nilmanifolds (that is, in the case $S = \{\infty\}$).

Let $\mathbf{K}$ be an algebraically closed field containing $\mathbf{Q}_p$ for $p = \infty$ and for all prime integers p. For an integer $n \geq 1$, consider the symplectic form β on $\mathbf{K}^{2n}$ given by

$$\beta((x, y), (x', y')) = (x, y)^t J(x', y') \quad \text{for all } (x, y), (x', y') \in \mathbf{K}^{2n},$$

where J is the $(2n \times 2n)$-matrix

$$J = \begin{pmatrix} 0 & I \\ -I_n & 0 \end{pmatrix}.$$

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The symplectic group

$$Sp_{2n} = \{g \in GL_{2n}(\mathbf{K}) \mid {}^t g J g = J\}$$

is an algebraic group defined over $\mathbf{Q}$.

The $(2n+1)$-dimensional Heisenberg group is the unipotent algebraic group $\mathbf{H}$ defined over $\mathbf{Q}$, with underlying set $\mathbf{K}^{2n} \times \mathbf{K}$ and product

$$((x, y), s)((x', y'), t) = ((x + x', y + y'), s + t + \beta((x, y), (x', y'))),$$

for $(x, y), (x', y') \in \mathbf{K}^{2n}$, $s, t \in \mathbf{K}$.

The group Sp_{2n} acts by rational automorphisms of $\mathbf{H}$, given by

$$g((x, y), t) = (g(x, y), t) \quad \text{for all } g \in Sp_{2n}, (x, y) \in \mathbf{K}^{2n}, t \in \mathbf{K}.$$

Let p be either an integer prime or $p = \infty$. Set $H_p = \mathbf{H}(\mathbf{Q}_p)$. The center Z of H_p is $\{(0, 0, t) \mid t \in \mathbf{Q}_p\}$. The unitary dual $\widehat{H_p}$ of H_p consists of the equivalence classes of the following representations:

- the unitary characters of the abelianized group H_p/Z;
- for every $t \in \mathbf{Q}_p \setminus \{0\}$, the infinite-dimensional representation π_t defined on $L^2(\mathbf{Q}_p^n)$ by the formula

$$\pi_t((a, b), s)\xi(x) = e_p(ts)e_p(\langle a, x - b\rangle)\xi(x - b)$$

  for $((a, b), s) \in H_p$, $\xi \in L^2(\mathbf{Q}_p^n)$, and $x \in \mathbf{Q}_p^n$, where $e_p \in \widehat{\mathbf{Q}_p}$ is as in §6.

For $t \neq 0$, the representation π_t is, up to unitary equivalence, the unique irreducible unitary representation of H_p whose restriction to the center Z is a multiple of the unitary character $s \mapsto e_p(ts)$.

For $g \in Sp_{2n}(\mathbf{Q}_p)$ and $t \in \mathbf{Q}_p \setminus \{0\}$, the representation π_t^g is unitary equivalent to π_t, since both representations have the same restriction to Z. This shows that $Sp_{2n}(\mathbf{Q}_p)$ stabilizes π_t. We denote the corresponding projective representation of $Sp_{2n}(\mathbf{Q}_p)$ by $\omega_t^{(p)}$. The representation $\omega_t^{(p)}$ has different names: it is called the *metaplectic representation*, *Weil's representation* or the *oscillator representation*. The projective kernel of $\omega_t^{(p)}$ coincides with the (finite) center of $Sp_{2n}(\mathbf{Q}_p)$ and $\omega_t^{(p)}$ is strongly $L^{4n+2+\varepsilon}$ on $Sp_{2n}(\mathbf{Q}_p)$ for every $\varepsilon > 0$ (see [**HoMo79**, Proposition 6.4] or [**Howe82**, Proposition 8.1]).

Let $S = \{p_1, \ldots, p_r, \infty\}$, where $p_1, \ldots, p_r$ are integer primes. Set $H := \mathbf{H}(\mathbf{Q}_S)$ and

$$\Lambda := \mathbf{H}(\mathbf{Z}[1/S]) = \{((x, y), s) : x, y \in \mathbf{Z}^n[1/S], s \in \mathbf{Z}[1/S]\}.$$

Let $\mathbf{Nil}_S = H/\Lambda$; the associated S-adic solenoid is $\mathbf{Sol}_S = \mathbf{Q}_S^{2n}/\mathbf{Z}[1/S]^{2n}$. The group $Sp_{2n}(\mathbf{Z}[1/S])$ is a subgroup of $\mathrm{Aut}(\mathbf{Nil}_S)$. The action of $Sp_{2n}(\mathbf{Z}[1/S])$ on $\mathbf{Sol}_S$ is induced by its representation by linear bijections on $\mathbf{Q}_S^{2n}$.

Let Γ be a subgroup of $Sp_{2n}(\mathbf{Z}[1/S])$. The Koopman representation κ of Γ on $L^2(\mathbf{Nil}_S)$ decomposes as

$$\kappa = \mathbf{1}_{\mathbf{Nil}_S} \oplus \kappa_1 \oplus \kappa_2,$$

where κ_1 is the restriction of κ to $L^2_0(\mathbf{Sol}_S)$ and κ_2 the restriction of κ to the orthogonal complement of $L^2(\mathbf{Sol}_S)$ in $L^2(\mathbf{Nil}_S)$. Since $Sp_{2n}(\mathbf{Q}_p)$ stabilizes every infinite-dimensional representation of H_p, it follows from Proposition 13 that there exists a subset $I \subset \mathbf{Q}$ such that κ_2 is equivalent to a direct sum

$$\bigoplus_{t\in I}\Big(\bigotimes_{p\in S}(\omega_t^{(p)}|_\Gamma \otimes W_i)\Big),$$

where W_i is an projective representation of Γ.

Let ν be a probability measure on Γ. We can give an estimate of the norm of $\kappa_2(\nu)$ as in [**BeHe11**] in the case of $S = \{\infty\}$. Indeed, by a general inequality (see [**BeGu15**, Proposition 30]), we have

$$\|\kappa_2(\nu)\| \le \|(\kappa_2 \otimes \overline{\kappa_2})^{\otimes k}(\nu)\|^{1/2k},$$

for every integer $k \ge 1$, where $\overline{\kappa_2}$ denotes the representation conjugate to κ_2. Since $\omega_t^{(p)}$ is strongly $L^{4n+2+\varepsilon}$ on $Sp_{2n}(\mathbf{Q}_p)$ for any $t \in I$ and $p \in S$, Proposition 8 implies that $(\kappa_2 \otimes \overline{\kappa_2})^{\otimes(n+1)}$ is contained in an infinite multiple of the regular representation λ_Γ of Γ. Hence,

$$\|\kappa_2(\nu)\| \le \|\lambda_\Gamma(\nu)\|^{1/2n+2}$$

and so,

$$\|\kappa_0(\nu)\| \le \max\{\|\lambda_\Gamma(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},$$

where κ_0 is the restriction of κ to $L^2_0(\mathbf{Nil}_S)$.

Assume that ν is aperiodic. If Γ is not amenable then $\|\lambda_\Gamma(\nu)\| < 1$ by Kesten's theorem (see [**BeHV08**, Appendix G]); so, in this case, the action of Γ on $\mathbf{Nil}_S$ has a spectral gap if and only if $\|\kappa_1(\nu)\| < 1$, as stated in Theorem 1.

Observe that, if Γ is amenable, then the action of Γ on $\mathbf{Nil}_S$ or $\mathbf{Sol}_S$ does not have a spectral gap; indeed, by a general result (see [**JuRo79**, Theorem 2.4]), no action of a countable amenable group by measure-preserving transformations on a non-atomic probability space has a spectral gap.

Let us look more closely to the case $n = 1$. We have $Sp_2(\mathbf{Z}[1/S]) = SL_2(\mathbf{Z}[1/S])$ and the stabilizer of every element in $\widehat{\mathbf{Sol}_S} \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$ is conjugate to the group of unipotent matrices in $SL_2(\mathbf{Z}[1/S])$ and hence amenable. This implies that κ_1 is weakly contained in λ_Γ (see the decomposition of κ_1 appearing before Proposition 15); so, we have

$$\|\kappa_1(\nu)\| < 1 \Longleftrightarrow \Gamma \text{ is not amenable.}$$

As a consequence, we see that the action of Γ on $\mathbf{Nil}_S$ has a spectral gap if and only if Γ is not amenable.

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