ON THE REDUCIBILITY OF APPELL'S FUNCTION F_4*

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1. Introduction. It is a well-known fact in the theory of Appell's hypergeometric function of two variables F_4, defined by

$$(1) \qquad F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n} x^m y^n ,$$

where $|x|^{1/2} + |y|^{1/2} < 1$, that it can be expressed in terms of products of ordinary hypergeometric functions when $\gamma + \gamma' = \alpha + \beta + 1$. Bailey [1, page 306] proved this result which runs as follows:

$$(2) \qquad \begin{aligned} &F_4[\alpha, \beta; \gamma, \alpha+\beta-\gamma+1; x(1-y), y(1-x)] \\ &= {}_2F_1(\alpha, \beta; \gamma; x) \; {}_2F_1(\alpha, \beta; \alpha+\beta-\gamma+1; y), \end{aligned}$$

this formula being valid inside simply-connected regions surrounding $x = o$, $y = o$ for which

$$|x(1-y)|^{1/2} + |y(1-x)|^{1/2} < 1.$$

As a matter of fact this result was obtained by Barnes in his unpublished work about a quarter century earlier than the publication of Bailey's formula [3, page 236]. Bailey [4, page 239] has also given some cases of reducibility of Appell's hypergeometric functions of two variables F_2 and F_3 in terms of

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generalized hypergeometric function ${}_4F_3$.

The object of the present note is to obtain four different cases of reducibility of Appell's function F_4 in terms of ordinary generalized hypergeometric functions ${}_3F_2$. The results have been given in the form of four theorems.

THEOREM 1. If $|x|^{1/2} < \frac{1}{2}$, then

$$F_4(\lambda, \mu; \nu, \nu; x, x) = {}_3F_2\left(\begin{matrix}\lambda, \mu, \nu-\frac{1}{2} \\ 2\nu-1, \frac{1}{2}(\mu-\lambda)\end{matrix}; 4x\right). \tag{3}$$

The following infinite integrals will be required in the proof.

If $R(\lambda \pm \mu + \nu) > -\frac{1}{2}$ and $R[a+(b\pm c)^2] > 0$ then [8, page 174]

$$\int_0^\infty t^{\lambda-1} \exp[-\tfrac{1}{2}(a+b^2+c^2)t]\, W_{\lambda,\mu}(at)\, I_\nu(2bct)\, dt$$

$$= \frac{(bc)^\nu\, \Gamma(\frac{1}{2}+\lambda-\mu+\nu)\, \Gamma(\frac{1}{2}+\lambda+\mu+\nu)}{a^{\lambda+\nu}\, \Gamma^2(\nu+1)} \tag{4}$$

$$\times F_4\left(\tfrac{1}{2}+\lambda-\mu+\nu, \tfrac{1}{2}+\lambda+\mu+\nu; \nu+1, \nu+1; -\frac{b^2}{a}, -\frac{c^2}{a}\right).$$

If $R(a) > 0$, $R(b) > 0$, $R(\lambda+\mu) > |R(\nu)| - \frac{1}{2}$, then

$$\int_0^\infty x^{\lambda-1} \exp[-(a+2b^2)x]\, W_{\lambda,\mu}(ax)\, I_\nu(2b^2x)\, dx$$

$$= \frac{b^{2\nu}\, \Gamma(\frac{1}{2}+\lambda+\mu+\nu)\, \Gamma(\frac{1}{2}+\lambda-\mu+\nu)}{a^{\lambda+\nu}\, \Gamma^2(\nu+1)} \tag{5}$$

$$\times {}_3F_2\left(\begin{matrix}\frac{1}{2}+\nu, \frac{1}{2}+\lambda+\mu+\nu, \frac{1}{2}+\lambda-\mu+\nu \\ 2\nu+1, \nu+1\end{matrix}; -\frac{4b^2}{a}\right).$$

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(5) follows from [6, page 410].

Proof. In order to prove Theorem 1, we put $c = b$ in (4) and obtain

$$\int_0^\infty t^{\lambda-1} \exp[-\tfrac{1}{2}(a+2b^2)t]\, W_{\lambda,\mu}(at)\, I_\nu(2b^2 t)\, dt$$

$$(6) \qquad = \frac{b^{2\nu}\, \Gamma(\frac{1}{2}+\lambda-\mu+\nu)\, \Gamma(\frac{1}{2}+\lambda+\mu+\nu)}{a^{\lambda+\nu}\, \Gamma^2(\nu+1)}$$

$$\times F_4(\tfrac{1}{2}+\lambda-\mu+\nu, \tfrac{1}{2}+\lambda+\mu+\nu; \nu+1, \nu+1; -\frac{b^2}{a}, -\frac{b^2}{a})$$

which is equal to (5). The result (3) now follows on equating the right hand sides of (5) and (6) and making suitable changes in the parameters.

THEOREM 2. If $R(2\lambda-\nu) > -\frac{1}{2}$ and $|x| < |$ then

$$F_4[\lambda, \lambda+\tfrac{1}{2}; \nu, \mu; \frac{1}{(1+x)^2}, \frac{x^2}{(1+x)^2}]$$

$$= \frac{x^{\nu-2\lambda-\frac{1}{2}}(1+x)^{2\lambda}\, \Gamma(\nu)\Gamma(\mu+\nu-2\lambda-1)\Gamma(2\lambda-\nu+\frac{1}{2})}{4^{\lambda-\nu+1}\, \Gamma(\frac{1}{2})\, \Gamma(2\mu+\nu-2\lambda-\frac{3}{2})\Gamma(2\lambda)\Gamma(\mu-\frac{1}{2})}$$

$$(7) \qquad \times {}_3F_2\left(\begin{matrix}\nu-\frac{1}{2}, \mu+\nu-2\lambda-1, \frac{3}{2}-\nu \\ \frac{1}{2}+\nu-2\lambda, 2\mu+\nu-2\lambda-\frac{3}{2}\end{matrix}; -x\right)$$

$$+ \frac{(1+x)^{2\lambda}\, \Gamma(\nu-2\lambda-\frac{1}{2})\, \Gamma(\nu)}{4^{\lambda-\nu+1}\, \Gamma(\frac{1}{2})\, \Gamma(2\mu-1)\, \Gamma(2\nu-2\lambda-1)}$$

$$\times {}_3F_2\left(\begin{matrix}2\lambda, 2(\lambda-\nu+1), \mu-\frac{1}{2} \\ 2\lambda-\nu+\frac{3}{2}, 2\mu-1\end{matrix}; -x\right).$$

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THEOREM 3. If $R(\mu \pm \nu) > -1$ and $|x| < 1$ then

$$\sum_{\nu, -\nu} \frac{\Gamma(-\mu)\Gamma(\mu+\nu+1)}{(1+x)^{\mu+\nu+1}} F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+1, \mu+1; \frac{1}{(1+x)^2}, \left(\frac{x}{1+x}\right)^2\right]$$

$$(8) \quad = \frac{\Gamma(\frac{1}{2})\,\Gamma(\mu+\nu+1)\,\Gamma(\mu-\nu+1)}{2^{2\mu+1}\,\Gamma(\mu+\frac{3}{2})}$$

$$\times {}_3F_2\left(\begin{matrix}\mu+\frac{1}{2}, \mu+\nu+1, \mu-\nu+1 \\ 2\mu+1, \mu+\frac{3}{2}\end{matrix}; -x\right)$$

where the symbol $\sum_{\nu, -\nu}$ indicates that to the expression following it, a similar expression obtained by interchanging ν and $-\nu$ is to be added.

THEOREM 4. If $R(\lambda \pm \nu) > 0$ and $|x| < 1$, then

$$\sum_{\nu, -\nu} \frac{\Gamma(-\nu)\,\Gamma(\lambda+\nu)}{2^{\nu}\,(1+x)^{\lambda+\nu}}$$

$$\times F_4\left[\frac{\lambda+\nu}{2}, \frac{\lambda+\nu+1}{2}; \nu+1, \mu+1; \frac{1}{(1+x)^2}, \left(\frac{x}{1+x}\right)^2\right]$$

$$(9) \quad = \frac{\Gamma(\frac{1}{2}+\lambda+\nu)\,\Gamma(\frac{1}{2}+\lambda-\nu)\,\Gamma(\frac{1}{2})}{2^{\lambda-1}\,\Gamma(\frac{1}{2}+\lambda)}$$

$$\times {}_3F_2\left(\begin{matrix}\frac{1}{2}+\lambda+\nu, \frac{1}{2}+\lambda-\nu, \frac{1}{2}+\mu \\ 1+2\mu, \frac{1}{2}+\lambda\end{matrix}; -x\right)$$

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The following results will be found useful in the proofs.

If $R(\lambda+\mu+\nu)>0$, $R(\lambda+\mu)>0$ and $R(\alpha)>|R(\beta)|+|\operatorname{Im}\gamma|$, then

$$\int_0^\infty t^{2\lambda+2\mu-1} K_\nu(\alpha t)\, I_\nu(\beta t) \times {}_1F_2(\mu+\tfrac{1}{2};\, 2\mu+1,\ \lambda+\mu;\, -\gamma^2 t^2)\, dt$$

$$(10) \quad = \frac{2^{2(\lambda+\mu-1)} (\alpha\beta)^\nu\, \Gamma(\lambda+\mu)\, \Gamma(\lambda+\mu+\nu)}{(\alpha^2+\beta^2+2\gamma^2)^{\lambda+\mu+\nu}\, \Gamma(\nu+1)}$$

$$\times F_4\left[\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2};\, \nu+1, \mu+1;\, \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+2\gamma^2)^2}, \frac{4\gamma^4}{(\alpha^2+\beta^2+2\gamma^2)^2}\right]$$

If $R(\mu\pm\nu)>-1$, $R(\alpha+\beta)>|\operatorname{Im}\gamma|+|\operatorname{Im}\delta|$ then

$$\int_0^\infty t K_\nu(\alpha t)\, K_\nu(\beta t)\, J_\mu(\gamma t)\, J_\mu(\delta t)\, dt$$

$$(11) \quad = \frac{(\gamma\delta)^\mu}{2\Gamma(\mu+1)} \sum_{\nu,-\nu} \frac{(\alpha\beta)^\nu\, \Gamma(-\nu)\, \Gamma(\mu+\nu+1)}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^{\mu+\nu+1}}$$

$$\times F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2};\, \nu+1, \mu+1;\, \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}, \frac{4\gamma^2\delta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}\right]$$

If $R(\lambda+\mu\pm\nu)>0$ and $R(\alpha+\beta)>|\operatorname{Im}\gamma|$, then

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$$\int_0^\infty t^{2\lambda+2\mu-1} K_\nu(\alpha t) K_\nu(\beta t)\, {}_1F_2(\mu+\tfrac{1}{2}; 2\mu+1, \lambda+\mu; -\gamma^2 t^2)\, dt$$

$$(12) \quad = 2^{2\lambda+2\mu-3}\Gamma(\lambda+\mu) \sum_{\nu,-\nu} \frac{(\alpha\beta)^\nu \Gamma(-\nu)\Gamma(\lambda+\mu+\nu)}{(\alpha^2+\beta^2+2\gamma^2)^{\lambda+\mu+\nu}}$$

$$\times F_4\left[\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2}; \nu+1, \mu+1; \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+2\gamma^2)^2}, \frac{4\gamma^4}{(\alpha^2+\beta^2+2\gamma^2)^2}\right]$$

(10), (11) and (12) have been proved by the author in an earlier paper [7, pages 131-132].

If $R(\mu \pm \nu) > -1$, then [5, page 335]

$$\int_0^\infty x K_\nu^2(ax) J_\mu^2(bx)\, dx$$

$$(13) \quad = \frac{b^{2\mu}\Gamma(\mu+\nu+1)\Gamma(\mu-\nu+1)\Gamma(\mu+\frac{1}{2})}{4a^{2(\mu+1)}\Gamma(2\mu+1)\Gamma(\mu+\frac{3}{2})}$$

$$\times {}_3F_2\left(\begin{matrix}\mu+\frac{1}{2}, \ \mu+\nu+1, \ \mu-\nu+1 \\ 2\mu+1, \ \mu+\frac{3}{2}\end{matrix}; -\frac{b^2}{a^2}\right)$$

If $R(\lambda+\mu\pm\nu) > 0$, then

$$\int_0^\infty t^{2\lambda+2\mu-1} K_\nu^2(at)\, {}_1F_2(\mu+\tfrac{1}{2}; 2\mu+1, \lambda+\mu; -b^2t^2)\, dt$$

$$(14) \quad = \frac{\Gamma(\frac{1}{2})\Gamma(\lambda+\mu)\Gamma(\frac{1}{2}+\lambda+\mu+\nu)\Gamma(\frac{1}{2}+\lambda+\mu-\nu)}{4a^{2\lambda+2\mu}\Gamma(\frac{1}{2}+\lambda+\mu)}$$

$$\times {}_3F_2\left(\begin{matrix}\frac{1}{2}+\lambda+\mu+\nu, \ \frac{1}{2}+\lambda+\mu-\nu, \ \frac{1}{2}+\mu \\ 1+2\mu, \ \frac{1}{2}+\lambda+\mu\end{matrix}; -\frac{b^2}{a^2}\right)$$

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If $R(\lambda+\mu+\nu) > 0$, $R(\lambda+\mu) > 0$, then

$$\int_0^\infty t^{2\lambda+2\mu-1} K_\nu(at)\, I_\nu(at)$$

$$\times\, {}_1F_2(\mu+\tfrac{1}{2}; 2\mu+1, \lambda+\mu; -b^2t^2)\, dt$$

(15)
$$= \frac{b^{1-2\lambda-2\mu}\, \Gamma(2\mu+1)\,\Gamma(\lambda+\mu)\,\Gamma(1-\lambda)\,\Gamma(\lambda+\mu-1/2)}{4a\,\Gamma(\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})\,\Gamma(\frac{3}{2}-\lambda+\mu)}$$

$$\times\, {}_3F_2\left(\begin{matrix}\frac{1}{2}+\nu,\ \frac{1}{2}-\nu,\ 1-\lambda \\ \frac{3}{2}-\lambda-\mu,\ \frac{3}{2}-\lambda+\mu\end{matrix};\ -\frac{b^2}{a^2}\right)$$

$$+\ \frac{\Gamma(\lambda+\mu)\,\Gamma(\frac{1}{2}-\lambda-\mu)\,\Gamma(\lambda+\mu+\nu)}{4\Gamma(1+\nu-\lambda-\mu)\,\Gamma(\frac{1}{2})\,\Gamma(1+2\mu)\ a^{2\lambda+2\mu}}$$

$$\times\, {}_3F_2\left(\begin{matrix}\lambda+\mu+\nu;\ \lambda+\mu-\nu,\ \mu+\frac{1}{2} \\ \lambda+\mu+\frac{1}{2},\ 2\mu+1\end{matrix};\ -\frac{b^2}{a^2}\right)$$

(14) and (15) follow from an integral [6, page 422].

Proof of Theorem 2. Putting $\beta = \alpha = a$ and $\gamma = b$ in (10) and then equating its right hand side with that of (15) and making suitable changes in the parameters we obtain (7).

Proof of Theorem 3. On writing $\alpha = \beta = a$, $\gamma = \delta = b$ in (11) and using (13) we arrive at the result (8).

Proof of Theorem 4. It can be easily proved in a similar manner from (12) and (14).

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