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# ON THE NUMBER OF BINOMIAL COEFFICIENTS WHICH ARE DIVISIBLE BY THEIR ROW NUMBER: II

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ABSTRACT. If $n$ is a natural number, let $A(n)$ denote the number of integers, $k$, such that $0 < k < n$ and $n$ divides $\binom{n}{k}$. Let $\phi(n)$ denote Euler's totient function. Necessary and sufficient conditions are given so that $A(n) = \phi(n)$ when $n$ is square-free.

**Introduction**. Let $n$ be a non-negative integer. In Pascal's triangle, we find $\binom{n}{k}$ in the $k^{\text{th}}$ position of row $n$, with $0 \leq k \leq n$. We therefore say that $n$ is the row number of $\binom{n}{k}$. Following [3], Definition 1, if $n$ is positive, let $A(n)$ denote the number of integers, $k$, such that $0 < k < n$ and $n \mid \binom{n}{k}$. Let $p$ denote a prime. $\phi(n)$ denotes Euler's totient function.

In [3], we showed that (i) $A(n) \geq \phi(n)$ for all $n$; (ii) $A(n) = \phi(n)$ if $n = p^e$, $e \geq 1$, or $n$ is twice a Mersenne prime. In this note, we find necessary and sufficient conditions that $A(n) = \phi(n)$ when $n$ is square-free. We then consider in some detail the cases where $n$ is a product of 2 or 3 distinct primes. In the former case, a number of solutions of $A(n) = \phi(n)$ are presented in Table 1 below; in the latter case, one solution is obtained, thus disproving a conjecture of P. Erdös [2].

Let $\omega(n)$ be the number of distinct prime factors of $n$, while $d(n)$ is the number of divisors of n.

DEFINITION 2. *$O_n(m) = k$ if $n^k \mid m$ but $n^{k+1} \nmid m$, where $k \geq 0$.*

DEFINITION 3. *$t_p(n) = \sum_{i=0}^{r} a_i$ if $n = \sum_{i=0}^{r} a_i p^i$, where $0 \leq a_i < p$ for each $i$.*

REMARK. Definitions 2 and 3 above correct errors which appeared in Definitions 2 and 3 of [3].

PRELIMINARIES.

(1) $\quad [a] + [b] \leq [a + b] \leq [a] + [b] + 1$

(2) $\quad O_p(ab) = O_p(a) + O_p(b)$

(3) $\quad O_p(n!) = \sum_{k=1}^{\infty} [n/p^k]$

(4) $\quad O_p(\binom{n}{k}) = [\{t_p(k) + t_p(n - k) - t_p(n)\}/(p - 1)]$

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TABLE 1

| $p$ | $j$ | $a$ | $q$ | $p$ | $j$ | $a$ | $q$ |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 1 | 3 | 13 | 2 | 12 | 2027 |
| 2 | 3 | 1 | 7 | 17 | 1 | 4 | 67 |
| 2 | 5 | 1 | 31 | 17 | 1 | 6 | 101 |
| 2 | 7 | 1 | 127 | 17 | 1 | 16 | 271 |
| 2 | 13 | 1 | 8191 | 17 | 2 | 2 | 577 |
| 3 | 1 | 2 | 5 | 17 | 2 | 6 | 1733 |
| 3 | 2 | 2 | 17 | 17 | 2 | 8 | 2311 |
| 3 | 3 | 2 | 53 | 17 | 2 | 12 | 3467 |
| 3 | 7 | 2 | 4373 | 17 | 3 | 16 | 78607 |
| 3 | 8 | 2 | 13121 | 19 | 1 | 2 | 37 |
| 5 | 1 | 4 | 19 | 19 | 1 | 6 | 113 |
| 5 | 4 | 2 | 1249 | 19 | 1 | 8 | 151 |
| 5 | 6 | 2 | 31249 | 19 | 1 | 12 | 227 |
| 7 | 1 | 2 | 13 | 19 | 2 | 8 | 2887 |
| 7 | 1 | 6 | 41 | 19 | 3 | 12 | 82307 |
| 7 | 2 | 2 | 97 | 23 | 1 | 6 | 137 |
| 7 | 2 | 6 | 293 | 23 | 1 | 10 | 229 |
| 7 | 4 | 2 | 4801 | 23 | 1 | 16 | 367 |
| 7 | 5 | 2 | 33613 | 23 | 2 | 8 | 4231 |
| 11 | 1 | 4 | 43 | 23 | 2 | 18 | 9521 |
| 11 | 1 | 10 | 109 | 29 | 1 | 6 | 173 |
| 11 | 2 | 2 | 241 | 29 | 1 | 12 | 347 |
| 11 | 2 | 8 | 967 | 29 | 1 | 16 | 463 |
| 11 | 3 | 4 | 5323 | 29 | 1 | 18 | 521 |
| 11 | 3 | 10 | 13309 | 29 | 1 | 28 | 811 |
| 13 | 1 | 8 | 103 | 29 | 2 | 12 | 10091 |
| 13 | 2 | 2 | 337 | 29 | 2 | 18 | 15137 |
| 13 | 2 | 6 | 1013 | 29 | 2 | 24 | 20183 |

(5) $\quad O_p\left(\binom{bp^k}{ap^j}\right) = O_p\left(\binom{bp^{k-j}}{a}\right)$ if $p \nmid ab$, $j \le k$, and $0 < a < bp^{k-j}$

(6) $\quad A(n) \geq \phi(n)$ for all $n$

(7) $\quad \omega(2^j - 1) \geq d(j) - 1$ if $j \neq 6$

REMARKS. (1) through (4) are elementary; (5) is [3], Theorem 3; (6) is [3], Corollary 1. (7) follows from Theorem V in [1].

**The main results.**

LEMMA 1. (i) $A(n) = \phi(n)$ *if and only if* (ii) $n \nmid \binom{n}{k}$ *for all $k$ such that* $0 < k < n$ *and* $(k, n) > 1$.

PROOF. Follows from (6) and from the fact that (ii) is equivalent to: $A(n) \leq \phi(n)$.

LEMMA 2. *Let $p < t$. Then* (i) $p \nmid \binom{t}{m}$ *for all $m$ such that* $0 < m < t$ *if and only if* (ii) $t = ap^j - 1$, *where* $0 < a < p$ *and* $0 < j$.

PROOF. (ii) implies (i): hypothesis implies $t = (a-1)p^j + \sum_{i=0}^{j-1}(p-1)p^i$. Let $m = \sum_{i=0}^{j} b_i p^i$, with $0 \le b_i \le p-1$ for all $i$. Now $m < t$ implies $b_j \le a-1$. Therefore $t - m = (a-1-b_j)p^j + \sum_{i=0}^{j-1}(p-1-b_i)p^i$, so that $t_p(t-m) = t_p(t) - t_p(m)$. Therefore (4) implies $p \nmid \binom{t}{m}$.

(i) implies (ii): Let $t = \sum_{i=0}^{j} c_i p^i \neq ap^j - 1$. Let $r$ be the least integer such that $0 \le r \le j-1$ and $c_r < p-1$. (By hypothesis, $r$ exists.) Let $m = (c_r + 1)p_r$. Therefore

$$t - m = \sum_{i=r+2}^{j} c_i p^i + (c_{r+1} - 1)p^{r+1} + (p-1)p^r + \sum_{i=0}^{r-1} c_i p^i,$$

so that $t_p(t-m) + t_p(m) - t_p(t) = p - 1$, and therefore (4) implies $p \mid \binom{t}{m}$.

LEMMA 3. *If $p$ is prime, $p \nmid mt$, and $s$ is arbitrary, then $p \mid \binom{pst}{sm}$.*

PROOF. First suppose $p \nmid s$. Applying (2), it suffices to show that $O_p((pst)!) > O_p((sm)!) + O_p((pst - sm)!)$. By virtue of (1) and (3), we need merely find one $k \ge 1$ such that

$$[pst/p^k] > [sm/p^k] + [(pst - sm)/p^k].$$

The required value of $k$ is 1. If $p \mid s$, the same conclusion holds by appeal to (5).

DEFINITION 4. *If $n = \prod_{i=1}^{r} p_i$, where $r \ge 2$ and the $p_i$ are primes such that $p_j < p_k$ whenever $j < k$, let $n_i = n/p_i$ for each $i$.*

LEMMA 4. *If $n$ and $n_i$ are as in Definition 4 above, then $p_r^{r-1} > 1 + n_r$ unless $n =$ 6.*

PROOF. If $r = 2$, then hypothesis implies $p_2 > p_1 + 1$. But $n_2 = p_1$, so $p_2 > n_2 + 1$. If $r \ge 3$, then since $p_i < p_r$ for all $i < r$, we have $p_r^{r-2} > n/p_{r-1}p_r$. Let $p_r = d + p_{r-1}$. Therefore $p_r^{r-1} > (d + p_{r-1})n/p_{r-1}p_r = dn/p_{r-1}p_r + n/p_r > 1 + n_r$.

THEOREM 1. *Let $n$ and $n_i$ be as in Definition 4 above. Then $A(n) = \phi(n)$ if and only if* (i) *$n_i = a_i p_i^{j_i} - 1$, $0 < a_i < p_i$, $0 < j_i$ for all $i < r$, and* (ii) *$p_r > n_r$ or $n_r = a_r p_r^{j_r} - 1$, $0 < a_r < p_r$, $0 < j_r < r - 1$.*

PROOF. By Lemma 1, (I) $A(n) = \phi(n)$ if and only if for all $i$, (II) $n \nmid \binom{n}{p_i m_i}$ for all $m_i$ such that $0 < m_i < n_i$. If $k \neq i$, then Lemma 3 implies $p_k \mid \binom{n}{p_i}$, so that (II) simplifies to: $p_i \nmid \binom{n}{p_i m_i}$ for all $m_i$ such that $0 < m_i < n_i$. Using (5), (II) becomes: $p_i \nmid \binom{n_i}{m_i}$ for all $m_i$ such that $0 < m_i < n_i$. If $i < r$, then by Lemma 2, (II) holds precisely when $n_i = a_i p_i^{j_i} - 1$, with $0 < a_i < p_i$ and $0 < j_i$. If $i = r$, then (II) holds if and only if $p_r > n_r$ (which is trivially true if $r = 2$) or by Lemma 2, $n_r = a_r p_r^{j_r} - 1$, with $0 < a_r < p_r$ and $0 < j_r$. In the latter case, we may assume $r \ge 3$. Therefore Lemma 4 implies $p_r^{r-1} > 1 + n_r$. But $n_r \ge p_r^{j_r} - 1$, so $p_r^{r-1} > p_r^{j_r}$, and therefore $j_r < r - 1$.

COROLLARY 1. If $p, q$ are primes with $p < q$, then $A(pq) = \phi(pq)$ if and only if $q = ap^j - 1$, with $0 < a < p$, $0 < j$.

PROOF. Follows from Theorem 1 with $p = p_1 = n_2$, $q = p_2$, $r = 2$.

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Table 1 below lists all pairs of primes $p, q$ such that $p < q$, $p < 30$, $q < 10^5$, $q = ap^j - 1$, $0 < a < p$, $0 < j$.

THEOREM 2. *If* $A(pqr) = \phi(pqr)$ *where* $p$, $q$, $r$ *are primes with* $p < q < r$, *then* (i) $qr = ap^j - 1$, $0 < a < p$, $0 < j$; (ii) $pr = bq^k - 1$, $0 < b < q$, $0 < k$; (iii) $pq < r$.

PROOF. By Theorem 1, it suffices to show that (**) $pq = cr - 1$ is impossible. If (**) holds, then $r = (pq + 1)/c > q$ implies (***) $c \leq p$. (ii) implies $p(pq + 1) = cbq^k - c$, so that $p + c = uq$. (**) implies $p + c$ is odd, so $u$ is odd. If $u \geq 3$, then $3p < 3q \leq p + c$, so that $c \geq 2p$, contradicting (***). Therefore $u = 1$, so $p + c = q$, and we have $p(p + c) = cr - 1$, which implies $c(r - p) = p^2 + 1$. If $c = 1$, then $p = 2$, so $q = 3$ and $r = 7$, which contradicts (i). If $p = 2$, then $c(r - 2) = 5$ implies $c = 1$, $r = 7$, $q = 3$, again contradicting (i). If $c > 1$, then $p$ is odd, so $c$ and $r - p$ are even. But then $p^2 + 1 \equiv 0 \pmod 4$, an impossibility.

Now consider (*) $A(pqr) = \phi(pqr)$ where $p$, $q$, $r$ are primes with $p < q < r$.

CASE 1. *Let* $p = 2$. *Then Theorem* 2 *implies:* (i) $qr = 2^j - 1$; (ii) $2r = bq^k - 1$, $0 < b < q$, $0 < k$; (iii) $2q < r$.

Now (i) implies $\omega(2^j - 1) = 2$. Since $2^6 - 1 = 3^2{*}7 \neq qr$, we know $j \neq 6$. Therefore (7) implies $d(j) \leq 3$, which implies $j = t^m$, where $t$ is prime and $m = 1$ or 2. If $m = 1$, then there is no $j < 100$ such that (i) and (ii) are compatible.

If $m = 2$, we have $qr = 2^{t^2} - 1 = (2^t - 1)\,((2^{t^2} - 1)/(2^t - 1))$. Since $q < r$, we have $q = 2^t - 1$, $r = (2^{t^2} - 1)/(2^t - 1) = ((q + 1)^t - 1)/q = \sum_{i=1}^{t} \binom{t}{i} q^{i-1} = t + \sum_{i=2}^{t} \binom{t}{i} q^{i-1}$, which implies $q \mid (r - t)$, hence $q \mid (2r - 2t)$. But (ii) implies $q \mid (2r + 1)$, so that $q \mid (2t + 1)$. Now $2^t - 1 > 2t + 1$ for $t > 3$, and $(2^2 - 1) \nmid (2{*}2 + 1)$, so we must have $t = 3$, $q = 7$, $r = 73$. (ii) holds with $b = 3$, $k = 2$, and (iii) holds since $14 < 73$. Therefore $1022 = 2{*}7{*}73$ is a solution of (*).

Now suppose $p \geq 3$. By Theorem 2, we have:

(i) $qr = 2ap^j - 1$, $1 \leq a \leq \frac{1}{2}(p - 1)$, $0 < j$

(ii) $pr = 2bq^k - 1$, $1 \leq b \leq \frac{1}{2}(q - 1)$, $0 < k$

(iii) $pq < r$

Eliminating $r$ between (i) and (ii), one obtains:

(iv) $ap^{j+1} + \frac{1}{2}(q - p) = bq^{k+1}$.

LEMMA 5. *In* (ii), *if* $2b = c^2$, *then* $k$ *is odd.*

PROOF. If $k = 2m$, then hypothesis and (ii) imply $pr = c^2q^{2m} - 1 = (cq^m - 1)(cq^m + 1)$, so that $p = cq^m - 1$, $r = cq^m + 1$. But then $r = p + 2$, an impossibility, since $r \geq p + 4$.

CASE 2. *Let* $p = 3$. *Therefore* $a = 1$ *in* (i) *and we have:*

(i) $qr = 2(3^j) - 1$, $0 < j$

(ii) $3r = 2bq^k - 1$, $1 \leq b \leq \frac{1}{2}(q - 1)$, $0 < k$

(iii) $3q < r$

(iv) $3^{j+1} + \frac{1}{2}(q - 3) = bq^{k+1}$.

LEMMA 6. *If $p = 3$ in $(*)$, then $3^{j+1} \equiv \frac{1}{2}(q + 3) \pmod{q}$.*

PROOF. Follows from hypothesis and (iv).

THEOREM 3. *If $p = 3$ in $(*)$, then $q \notin \{11, 13, 37, 41, 59, 61, 67, 73, 83$, etc.$\}$*

PROOF. Follows from Lemma 6.

LEMMA 7. *If $p = 3$ in $(*)$, then $b \equiv \frac{1}{2}(q - 1) \pmod{2}$.*

PROOF. Follows from (iv).

LEMMA 8. *Let $p = 3$ in $(*)$. If $q \equiv 1 \pmod{3}$, then $b \equiv 2 \pmod{3}$; if $q \equiv -1 \pmod{3}$, then $b \equiv (-1)^{k+1} \pmod{3}$.*

PROOF. Follows from (ii).

LEMMA 9. *Let $p = 3$ in $(*)$. If $q \equiv 1 \pmod{12}$, then $b \equiv 2 \pmod{6}$; if $q \equiv 7 \pmod{12}$, then $b \equiv 5 \pmod{6}$.*

PROOF. Follows from Lemmas 7 and 8.

CASE 2.1. *Let $p = 3$, $q = 5$ in $(*)$. (ii) implies $1 \leq b \leq 2$; Lemma 7 implies $b$ is even, so $b = 2$. Now Lemma 5 implies $k$ is odd, but then Lemma 8 implies $b \equiv 1 \pmod{3}$, an impossibility.*

CASE 2.2. *Let $p = 3$, $q = 7$ in $(*)$. (ii) implies $1 \leq b \leq 3$. Lemma 9 implies $b \equiv 5 \pmod{6}$ so that $b \geq 5$, an impossibility.*

CASE 2.3. *Let $p = 3$, $q = 17$ in $(*)$. (ii) implies $1 \leq b \leq 8$; Lemmas 7 and 8 imply $b \in \{2, 4, 8\}$. (i) implies $2(3^j) \equiv 1 \pmod{17}$ so that $j \equiv 2 \pmod{16}$, and $2 \mid j$. (iv) implies $3^{j+1} + 7 = b(17^{k+1})$ so that $3^{j-k} \equiv b \pmod{7}$. Since $b \in \{2, 4, 8\}$, we have $2 \mid (j - k)$, so $2 \mid k$. Now Lemma 8 implies $b \equiv 2 \pmod{3}$, so that $b = 2$ or $8$, contradicting Lemma 5.*

CASE 2.4. *Let $p = 3$, $q = 19$ in $(*)$. (ii) implies $1 \leq b \leq 9$; Lemma 9 implies $b \equiv 5 \pmod{6}$, so $b = 5$. (iv) implies $3^{j+1} + 8 = 5(19^{k+1})$, so that $3^{j-k} \equiv 5 \pmod{8}$, an impossibility.*

**Concluding Remarks**. We have disposed of the cases $3 = p < q \leq 19$ by generating congruence incompatibilities. This method seems to fail for the case $p = 3$, $q = 23$, but also works for the cases $5 \leq p < q \leq 19$. Therefore any solution of $(*)$ such that $p$ is odd must have $q \geq 23$.

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