COMPOSITIO MATHEMATICA

A generalization of the b-function lemma

Sam Raskin

Compositio Math. **157** (2021), 2199–2214.

doi:10.1112/S0010437X21007491

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

Compositio Math. 157 (2021) 2199–2214
doi:10.1112/S0010437X21007491

A generalization of the b-function lemma

Sam Raskin

Abstract

We establish some cohomological bounds in D-module theory that are known in the holonomic case and folklore in general. The method rests on a generalization of the b-function lemma for non-holonomic D-modules.

1. Introduction

1.1 This article studies how D-module operations interact with singular support. The main technical result, Theorem 2.5.1, shows that D-module operations preserve a certain numerical obstruction to holonomicity. This result generalizes the usual preservation of holonomic D-modules under such operations, which is essentially equivalent to the b-function lemma: see [Kas76] or [Ber].

1.2 Exactness properties

Our main application is Theorem 3.4.1, which states that, inasmuch as the assertion makes sense (see § 1.3), $f_!$ is left t-exact for an affine morphism $f : X \to Y$.

This is certainly an old folklore result. It is standard for holonomic D-modules, where it is a consequence of the usual b-function lemma. It is also easy to show for $Y = \mathrm{Spec}(k)$, or for a map of curves. Otherwise, it does not seem to follow from existing foundational results in the literature, which is surprising for something so basic.

1.3 We remark that, as just stated, this result does not quite make sense.

Let, for example, $D(X)$ denote the derived category of D-modules on X. There is a standard pullback functor $f^! : D(Y) \to D(X)$ for D-modules. For $\mathcal{F} \in D(X)$, $f_!(\mathcal{F}) \in D(Y)$ should corepresent the functor $\mathcal{G} \mapsto \mathrm{Hom}_{D(X)}(\mathcal{F}, f^!(\mathcal{G}))$, but this functor is *not* necessarily corepresentable. We say that $f_!(\mathcal{F})$ *is defined* whenever this functor is corepresentable.

Recall that standard D-module theory shows that $f_!(\mathcal{F})$ is defined when $\mathcal{F}$ is holonomic, or when f is proper (in which case it coincides with the D-module pushforward $f_{*,\mathrm{dR}}(\mathcal{F})$).

One version of Theorem 3.4.1 states that if f is affine, $\mathcal{F} \in D(X)^{\geq 0}$, and $f_!(\mathcal{F})$ is defined, then $f_!(\mathcal{F}) \in D(Y)^{\geq 0}$; for technical reasons, we also need to assume that $\mathcal{F}$ is coherent (in the applications we have in mind, one can readily remove this hypothesis). We remark that for coherent $\mathcal{F}$, $f_!(\mathcal{F})$ being defined is equivalent to $f_{*,\mathrm{dR}}(\mathcal{F})$ being coherent on Y; in this case, $f_!(\mathcal{F}) = \mathbb{D} f_{*,\mathrm{dR}} \mathbb{D}(\mathcal{F})$ for $\mathbb{D}$ the relevant Verdier duality functors.

Received 11 October 2018, accepted in final form 25 February 2021, published online 7 September 2021.
2020 Mathematics Subject Classification 14F10, 32C38 (primary), 17B55 (secondary).
Keywords: D-modules, singular support, and holonomic defect.

More generally, we prove a somewhat more technical version of this theorem using pro-categories, which allows us to remove the hypothesis that $f_!(\mathcal{F})$ is defined. The author finds this to be a more satisfying formulation, but we remark that the use of pro-categories in this paper is not so significant.

1.4 Finally, there is the question of applications of this left t-exactness.

In fact, one commonly finds affine morphisms f and non-holonomic D-modules $\mathcal{F}$ with $f_!(\mathcal{F})$ being defined in geometric representation theory. Moreover, in such situations, $f_!(\mathcal{F})$ typically coincides with $f_{*,\mathrm{dR}}(\mathcal{F})$, and therefore one can deduce t-exactness results from our result (for affine f, it is easy to see $f_{*,\mathrm{dR}}$ is right t-exact).

For example, this situation occurs for the Fourier–Deligne transform, and the results here can be used to show its t-exactness in a conceptual way.[1]

We include another example, due to [BBM04], in Appendix A. A related example in infinite-dimensional Lie theory is given in [Ras21, Theorem 2.7.1].[2]

Finally, see [BY18, Theorem 5.4] for another instance of this phenomenon.

1.5 Categorical conventions

As was already remarked above, we use derived categories of pro-D-modules in § 3. It is well known that it is inconvenient to use triangulated categories when working with pro-complexes, and better to use more sophisticated homotopical methods.

In § 2, which is the core of the paper, it is manifestly adequate to merely consider the triangulated category of D-modules on X. Indeed, the key point is Lemma 2.7.1, and this lemma is not about complexes; the reductions used in the remainder of the section only use standard homological algebra methods.

To keep consistent conventions in this paper, we will formally use the language of differential graded (DG) categories throughout this paper. Because this language is not relevant until § 3, we refer to § 3.2 for more details and references on our precise formalism.

1.6 Notation

We let k denote a field of characteristic zero.

By a *variety*, we mean a reduced, separated, finite type k-scheme.

For X a variety over k, we let $D(X)$ denote the DG category of D-modules on X. For $f : X \to Y$, we let $f^! : D(Y) \to D(X)$ and $f_{*,\mathrm{dR}} : D(X) \to D(Y)$ denote the standard D-module pullback and pushforward operations. We let $f_!$ and $f^{*,\mathrm{dR}}$ denote their left adjoints (where appropriate).

Remark 1.6.1. We refer the reader to [GR17, § III.4] for a detailed development of D-module theory in the DG formalism. For the reader's convenience, we remark that in [GR17], our category $D(X)$ is denoted $\mathrm{Crys}(X)$, our functor $f^!$ is denoted $f^!_{\mathrm{dR}}$, and our functor $f_{*,\mathrm{dR}}$ is denoted $f_{\mathrm{dR},*}$. The comparison with the classical theory of D-modules (as developed, for example, in [Ber] or [Kas95, § 2]) is given in [GR17, § III.4.4].

We highlight that [GR17, Theorem III.4.2.1.2] encodes base-change properties of D-module functors, as is extensively discussed in [GR17]. In its symmetric monoidal form,

[1] Compare [Gai16, § 1.8]. Note that [Gai16] implicitly assumes the left t-exactness of $f_!$ for affine f.

[2] Our main result, Theorem 3.4.1, plays a key role in one of two proofs of (an asymptotic form of) Lemma 5.3.1 in [Ras21], which is a key technical result in that paper that is used in the proof of its main theorem, Theorem 5.1.1. See [Ras21, Appendix B] for the application.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

which is a special case of [GR17, Corollary III.3.6.1.3], this result also encodes the projection formula.

Remark 1.6.2. As in [GR17], with caveats, the forgetful functor from D-modules to quasi-coherent sheaves intertwines the upper-! functor for D-modules with Grothendieck's upper-! functor for quasi-coherent sheaves. The two caveats are that one should consider the forgetful functor taking the underlying right (as opposed to left) D_X-module, and that for singular X, it is better to work with ind-coherent sheaves than quasi-coherent sheaves (as is always the case when considering Grothendieck's functor).

By contrast, the quasi-coherent sheaf underlying $f_{*,\mathrm{dR}}(\mathcal{F})$ is quite sensitive to the connection on $\mathcal{F}$, leading to the asymmetric notation.

We use $-\overset{!}{\otimes}-$ to denote the standard symmetric monoidal product on $D(X)$. We recall that $\mathcal{F}\overset{!}{\otimes}\mathcal{G}=\Delta^!(\mathcal{F}\boxtimes\mathcal{G})$ for $\mathcal{F},\mathcal{G}\in D(X)$. We refer to [GR17, § III.3.6.1] for the construction in the DG formalism.

We let $D(X)^{\geq i}$ and $D(X)^{\leq i}$ respectively denote the subcategories[3] of complexes $\mathcal{F}\in D(X)$ with $H^j(\mathcal{F})=0$ for $j<i$ and $j>i$, respectively. We let $D(X)^{\heartsuit}=D(X)^{\geq 0}\cap D(X)^{\leq 0}$ denote the heart of the t-structure, that is, the abelian category of D-modules. We let $\tau^{\geq i}$ and $\tau^{\leq i}$ denote the corresponding truncation functors.

We let $D(X)^c\subseteq D(X)$ denote the DG subcategory of *coherent* complexes, that is, bounded complexes with locally coherent[4] cohomologies. Recall that $D(X)$ is compactly generated[5] with compact objects $D(X)^c$, that is, $D(X)=\mathsf{Ind}(D(X)^c)$ for Ind denoting the ind-category;[6] cf. [GR17, Corollary III.4.1.6.4].[7]

We let $\mathbb{D}: D(X)^c\xrightarrow{\simeq}D(X)^{c,\mathrm{op}}$ denote the Verdier duality functor. We refer to [GR17, § III.4.2.2.5] for its construction in the DG formalism.

We let Vect denote the DG category of chain complexes of vector spaces. For $\mathcal{C}$ a DG category and $\mathcal{F},\mathcal{G}\in\mathcal{C}$, we let $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F},\mathcal{G})\in\mathsf{Vect}$ denote the chain complex of maps between $\mathcal{F}$ and $\mathcal{G}$.

[3] Formally, these are sub-∞-categories: cf. the conventions of § 3.2. Note that these are not DG subcategories as they are not closed under shifts.

For the reader who prefers triangulated categories to DG categories, the homotopy categories of these categories are the (non-triangulated) subcategories of the (triangulated) homotopy (1-)category of $D(X)$. As we emphasized already, the distinction between DG and triangulated only becomes relevant in § 3 (and even there it is quite mild).

[4] Recall that $\mathcal{F}\in D(X)^{\heartsuit}$ has locally coherent cohomologies if for every $j:U\hookrightarrow X$ an open affine and every $i:U\to Z$ a closed embedding with Z smooth affine, $\Gamma(Z,i_{*,\mathrm{dR}}j^!(\mathcal{F}))$ is finitely generated over the ring of global differential operators on Z. It is enough to check this for a Zariski cover $\{U_i\to X\}$ and any choice of $U_i\hookrightarrow Z_i$ as above.

[5] See, for example, [Lur09, § 5.3] or [GR17, § I.1.7] for the relevant definition.

[6] See [Lur09, § 5.3] for the construction of the ind-category.

[7] This result technically only states that $D(X)$ is compactly generated and does not give the description of compact objects as coherent complexes. For completeness, let us give the argument here.

To check this, first note that if X is smooth affine, [GR17, III.4.1.2] gives an equivalence $D(X)\simeq\Gamma(X,D_X)\text{–}\mathsf{mod}$ for $\Gamma(X,D_X)$ denoting the ring of global differential operators. Then the result is standard in this case: it is true for modules over any ring of finite cohomological dimension.

Then, for any X, the fact that a compact object $\mathcal{F}\in D(X)$ is coherent in our sense follows by adjointness: $j^{*,\mathrm{dR}}$ for an open embedding j and $i_{*,\mathrm{dR}}$ for a closed embedding i admit continuous right adjoints $j_{*,\mathrm{dR}}$ and $i^!$ respectively, and therefore preserve compact objects.

The converse follows immediately from the following two observations. First, if X has an open cover $U_1,\ldots,U_n$ and $\mathcal{F}\in D(X)$ has $\mathcal{F}|_{U_i}$ compact for every i, then $\mathcal{F}$ is compact: this follows by Zariski descent and by using commutation between finite limits and filtered colimits in the (∞-)category of (∞-)groupoids. Second, if $i:X\to Z$ is a closed embedding and $i_{*,\mathrm{dR}}(\mathcal{F})$ is compact, then $\mathcal{F}$ is compact by Kashiwara's lemma.

2. Holonomic defect

2.1 In this section we introduce a generalization of the holonomic condition on a D-module and show that it is preserved under D-module operations.

The method is standard. The main point is Lemma 2.7.1, which is a generalization of the fact that pushforward along an open embedding preserves holonomic objects, which is essentially equivalent to the usual b-function lemma. The main difference is that we cannot use finite-length methods. See also Remark 2.7.2 for indications on a different approach.

The presentation is based on [Kas76, Gin86].

2.2 Gabber–Kashiwara–Sato (GKS) filtration

We begin by reviewing some material from [Gin86, § 1].

Definition 2.2.1. Let X be a variety and let $\mathcal{F} \in D(X)^{\heartsuit}$ be a coherent D-module. For an integer i, we let

$$F_i^{\mathrm{GKS}}\mathcal{F} := \mathrm{Image}(H^0(\mathbb{D}\tau^{\geq -i}\mathbb{D}\mathcal{F}) \to \mathcal{F}). \tag{2.2.1}$$

We extend this definition to general $\mathcal{F}$ by ind-extension. That is, if we write $\mathcal{F}$ as a filtered colimit $\mathrm{colim}_j\, \mathcal{F}_j$ with each $\mathcal{F}_j$ coherent (as can always be done), we take $F_i^{\mathrm{GKS}}\mathcal{F} := \mathrm{colim}_j\, F_i^{\mathrm{GKS}}\mathcal{F}_j$.

Remark 2.2.2. Formula (2.2.1) is valid for general $\mathcal{F}$ if one interprets $\mathbb{D}$ as an equivalence between $D(X)$ and the DG category of pro-coherent D-modules equipped with the t-structure of § 3.3.

Note that $F_\bullet^{\mathrm{GKS}}$ is an increasing filtration on $\mathcal{F}$. Because $\mathbb{D}\mathcal{F}$ is in cohomological degrees $[-\dim X, 0]$, we have $F_i^{\mathrm{GKS}}\mathcal{F} = 0$ for $i < 0$, and $F_i^{\mathrm{GKS}}\mathcal{F} = \mathcal{F}$ for $i \geq \dim X$. Formation of the GKS filtration is functorial for D-module morphisms, that is, a map $\mathcal{F}_1 \to \mathcal{F}_2 \in D(X)^{\heartsuit}$ sends $F_i^{\mathrm{GKS}}\mathcal{F}_1$ to $F_i^{\mathrm{GKS}}\mathcal{F}_2$.

Note that if $\mathcal{F} = \mathrm{colim}_j\, \mathcal{F}_j$ is a filtered colimit in $D(X)^{\heartsuit}$, then $F_i^{\mathrm{GKS}}\mathcal{F} = \mathrm{colim}_j\, F_i^{\mathrm{GKS}}\mathcal{F}_j$.

Lemma 2.2.3. *Formation of $F_\bullet^{\mathrm{GKS}}$ commutes with open restriction and pushforwards along closed embeddings.*

Proof. Each of these functors is t-exact and commutes with Verdier duality. □

Therefore, many results about this filtration reduce to the case of smooth X by taking Zariski local closed embeddings into affine space. The key property in the smooth case is given in the following theorem.

Theorem 2.2.4. *If X is smooth, then a local section s of $\mathcal{F}$ lies in $F_i^{\mathrm{GKS}}\mathcal{F}$ if and only if the D-module generated by it has singular support with dimension $\leq \dim X + i$.*[8]

See [Gin86, Proposition V.14]. Note that it is equivalent to say that $F_i^{\mathrm{GKS}}\mathcal{F}$ is the maximal submodule of $\mathcal{F}$ with singular support of dimension $\leq \dim X + i$.

2.3 Holonomic defect

For $\delta \in \mathbb{Z}^{\geq 0}$, we say that $\mathcal{F} \in D(X)^{\heartsuit}$ has *holonomic defect* $\leq \delta$ if $F_\delta^{\mathrm{GKS}}\mathcal{F} = \mathcal{F}$.

Example 2.3.1. A coherent D-module $\mathcal{F}$ has holonomic defect ≤ 0 if and only if $\mathcal{F}$ is holonomic. Indeed, this follows by reduction to the smooth case and Theorem 2.2.4.

Example 2.3.2. Every $\mathcal{F}$ has holonomic defect $\leq \dim X$.

[8] We regard $\dim X$ as a locally constant function on X if X is not equidimensional.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

Example 2.3.3. If X is smooth and $\mathcal{F}$ is coherent, then by Theorem 2.2.4, $\mathcal{F}$ has holonomic defect $\leq \delta$ if and only if $\mathcal{F}$ has singular support with dimension $\leq \dim X + \delta$.

LEMMA 2.3.4. *The subcategory of $D(X)^{\heartsuit}$ consisting of objects with holonomic defect $\leq \delta$ is closed under submodules, quotient modules, and extensions.*

Proof. The argument reduces to the case of X smooth, and then follows from Theorem 2.2.4 and standard facts about singular support. □

LEMMA 2.3.5. *Holonomic defect is preserved under filtered colimits, and $\mathcal{F} \in D(X)^{\heartsuit}$ has holonomic defect $\leq \delta$ if and only if $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i$ coherent of holonomic defect $\leq \delta$.*

Proof. The first part is clear since formation of $F_{\bullet}^{\mathrm{GKS}}$ commutes with filtered colimits. For the second part, write $\mathcal{F} = \operatorname{colim}_i \mathcal{F}'_i$ with $\mathcal{F}'_i$ coherent, and then set $\mathcal{F}_i = F_{\delta}^{\mathrm{GKS}} \mathcal{F}'_i$. □

2.4 More generally, for $\mathcal{F} \in D(X)$ a complex of D-modules, we say that $\mathcal{F}$ has *holonomic defect* $\leq \delta$ if all of its cohomology groups do. By Lemmas 2.3.4 and 2.3.5, this defines a DG subcategory of $D(X)$ closed under (homotopy) colimits (equivalently, under cones, shifts, and arbitrary direct sums).

2.5 The following theorem is the main result of this section.

THEOREM 2.5.1. *If $f : X \to Y$ is a morphism and $\mathcal{F} \in D(X)$ (respectively, $\mathcal{G} \in D(Y)$) has holonomic defect $\leq \delta$, then $f_{*,\mathrm{dR}}(\mathcal{F})$ (respectively, $f^!(\mathcal{G})$) does as well. Moreover, for $\mathcal{F}$ coherent as above, $\mathbb{D}\mathcal{F}$ has holonomic defect $\leq \delta$ as well.*

This theorem generalizes the preservation of holonomic objects under D-module operations, so the proof must follow similar lines. It is given below.

2.6 Verdier duality

The compatibility with Verdier duality in Theorem 2.5.1 is well known. Indeed, the result immediately reduces to X being smooth, and then we have the following proposition.

PROPOSITION 2.6.1. *For coherent $\mathcal{F} \in D(X)^{\heartsuit}$ with singular support of dimension $\leq \dim X + i$, we have $H^{-j}\mathbb{D}\mathcal{F} = 0$ unless $0 \leq j \leq i$. Moreover, $H^{-j}\mathbb{D}\mathcal{F}$ has holonomic defect $\leq j$.*

See, for example, [Kas76, Theorem 2.3].

2.7 Affine open embeddings

The main case of Theorem 2.5.1 is pushforward along an open embedding.

LEMMA 2.7.1. *Let X be a smooth variety and let $f : X \to \mathbb{A}^1$ be a function. Let $U = \{f \neq 0\}$ be the corresponding basic open and let $j : U \hookrightarrow X$ denote the corresponding affine open embedding.*

Then $j_{*,\mathrm{dR}}$ preserves the property of having holonomic defect $\leq \delta$.

Remark 2.7.2. The argument that follows is a version of the standard proof in the holonomic setting via b-functions. Victor Ginzburg communicated to us that the proof of the holonomic version of Lemma 2.7.1 via the Bernstein filtration and Hilbert polynomials has a more straightforward generalization. This argument, which follows early work [Ber72] in the subject, is easily extracted from [HTT08, § 3.2.2].

Proof of Lemma 2.7.1.

Step 1. We may obviously assume X is connected and affine and that f is non-constant.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

We let D_X and D_U denote the respective rings of global differential operators (as opposed to the sheaves of differential operators). Because X and U are affine, we may consider D-modules on either of these schemes as modules over these rings.

Let $\mathcal{F} \in D_U\text{–}\mathsf{mod}^{\heartsuit} \simeq D(U)^{\heartsuit}$. Because we are working with modules rather than sheaves, considering $\mathcal{F}$ as a D_X-module by restriction is the same as considering the sheaf $j_{*,\mathrm{dR}}(\mathcal{F}) \in D(X)^{\heartsuit}$.

For $s \in \mathcal{F}$, we write $\mathrm{SS}_U(s) \subseteq T^*U$ for the singular support of $D_U \cdot s$ and $\mathrm{SS}_X(s) \subseteq T^*X$ for the singular support of $D_X \cdot s$. Note that $\mathrm{SS}_X(s)|_{T^*U} = \mathrm{SS}_U(s)$. We always understand singular support as a *reduced* subscheme.

We want to show that if every section $s \in \mathcal{F}$ has $\dim \mathrm{SS}_U(s) \leq \dim U + \delta = \dim X + \delta$, then the same is true of $\dim SS_X(s)$.

Step 2. First, we observe that there is a D_X-submodule $\mathcal{G} \subseteq \mathcal{F}$ such that every section of $\mathcal{G}$ has singular support with dimension $\leq \dim X + \delta$, and which is a *lattice*, that is, $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_U \xrightarrow{\simeq} \mathcal{F}$.

Indeed, we can take $\mathcal{G} = F_\delta^{\mathrm{GKS}}\mathcal{F}$, where the GKS filtration is with $\mathcal{F}$ considered as a D_X-module. Because the GKS filtration commutes with open restriction, we must have $\mathcal{G}|_U = \mathcal{F}$.

(Note that by Theorem 2.2.4, we are trying to show that $\mathcal{G} = \mathcal{F}$.)

Step 3. We begin with some preliminary constructions.

Let λ be an indeterminate. We write $\mathbb{A}^1_\lambda$ for $\mathrm{Spec}(k[\lambda])$. We let k' denote the fraction field $k(\lambda)$ of $k[\lambda]$. We use similar notation for a base-change to k'; for example, X' or $\mathcal{F}'$, etc. We always consider X' and U' as schemes over k', so, for example, their cotangent bundles are understood relative to k', and $D'_X = D_{X'}$.

We now recall that there is certain standard object $\mathcal{O}_{U'}f^\lambda \in D(U')^{\heartsuit}$ which is $(\mathcal{O}_{U'}, \nabla = d + \lambda(df/f))$. We denote the basis vector of $\Gamma(U', \mathcal{O}_{U'}f^\lambda) \in D'_U\text{–}\mathsf{mod}^{\heartsuit}$ also by f^λ.

We similarly let $\mathcal{F}'f^\lambda$ denote the tensor product of $\mathcal{F}'$ with $\mathcal{O}_{U'}f^\lambda$ over $\mathcal{O}_{U'}$, equipped with its standard $D_{U'}$-module structure. Explicitly, this D'_U-module has the same underlying $\Gamma(U', \mathcal{O}_{U'})$-module as $\mathcal{F}'$, except that we write this identification as

$$(s \in \mathcal{F}')) \mapsto (sf^\lambda \in \mathcal{F}'f^\lambda).$$

Then the action of vector fields is modified: for ξ a vector field and $s \in \mathcal{F}'$, we have

$$\xi \cdot (sf^\lambda) = (\xi s + \lambda \xi(f) f^{-1} s) f^\lambda.$$

Step 4. We first show that the result is true for $\mathcal{F}'f^\lambda$, that is, that every section has $SS_{X'}$ with dimension $\leq \dim X + \delta$.

First, note that the singular support in U' of any section has dimension $\leq \dim X + \delta$: this follows because $\mathcal{O}_{U'}f^\lambda$ is lisse on U' (that is, a vector bundle when considered as a mere quasi-coherent sheaf).

We have a canonical field automorphism $\gamma \in \mathrm{Aut}(k'/k)$ sending $\lambda \mapsto \lambda + 1$. Of course, anything obtained by extension of scalars from k to k' also carries such an automorphism γ; in particular, D'_X does (it sends a differential operator $P(\lambda)$ to $P(\lambda+1)$).

Similarly, $\mathcal{F}'$ has such an automorphism: note that this is not an automorphism as a D'_X-module, but rather intertwines the standard action with the one obtained by twisting by the automorphism γ of D'_X. That is, $\gamma(P \cdot s) = \gamma(P) \cdot \gamma(s)$ for $P \in D'_X$ and $s \in \mathcal{F}'$.

Define γ on the D'_X-module $\mathcal{O}_{U'}f^\lambda$ by setting

$$\gamma(gf^\lambda) = \gamma(g) f^{\lambda+1} := (\gamma(g) \cdot f) f^\lambda$$

for g a function on U'. Again, this morphism intertwines the actions of D'_X up to the automorphism γ of D'_X.

Tensoring, we obtain an automorphism γ of $\mathcal{F}' f^\lambda$ with similar semi-linearity.

By the semi-linearity, we have

$$\mathrm{SS}_{X'}(sf^\lambda) = \gamma \cdot \mathrm{SS}_{X'}(\gamma(sf^\lambda)),$$

where we are using γ to indicate the induced automorphism of T^*X'. In particular, we find that

$$\dim \mathrm{SS}_{X'}(sf^\lambda) = \dim \mathrm{SS}_{X'}(\gamma(sf^\lambda)).$$

Now let $\mathcal{G} = F_\delta^{\mathrm{GKS}}(\mathcal{F}' f^\lambda)$,[9] where the GKS filtration is taken with $\mathcal{F}' f^\lambda$ considered as a D'_X-module. By the above and Theorem 2.2.4, $sf^\lambda \in \mathcal{G}$ if and only if $\gamma(sf^\lambda) \in \mathcal{G}$.

For any $s \in \mathcal{F}$ (as opposed to $\mathcal{F}'$), we clearly have $\gamma(sf^\lambda) = sf^{\lambda+1}$. Since $\mathcal{G}$ is a lattice (by Step 2), $\gamma^N(s) = sf^{\lambda+N} \in \mathcal{G}$ for $N \gg 0$. But by the above, this means that $s \in \mathcal{G}$. Since $\mathcal{F}' f^\lambda$ is k'-spanned by such vectors, this means that $\mathcal{G} = \mathcal{F}' f^\lambda$, as desired.

Step 5. We now show that the result is true for our original $\mathcal{F}$. Let $s \in \mathcal{F}$; we want to show $\dim \mathrm{SS}_X(s) \leq \dim X + \delta$.

We now write $\mathcal{F} f^\lambda$ for the corresponding $D_X[\lambda]$-module (as opposed to the fiber over the generic point in $\mathbb{A}^1_\lambda$, which is what we called by this name previously). Note that $\mathcal{F} f^\lambda = \mathcal{F} \otimes_k k[\lambda]$ as a $\mathcal{O}_X[\lambda]$-module.

Let $\mathcal{F}_0$ be the $D_X[\lambda]$ submodule generated by sf^λ. Equip $\mathcal{F}_0$ with the filtration $F_i \mathcal{F}_0 = D_X^{\leq i}[\lambda] sf^\lambda$, where $D_X^{\leq i}$ are differential operators of order $\leq i$.

Then $\mathrm{gr}_\bullet(\mathcal{F}_0)$ is the structure sheaf of some closed subscheme $Z \subseteq T^*X \times \mathbb{A}^1_\lambda$. We have seen that the base-change of Z to the generic point of $\mathbb{A}^1_\lambda$ has dimension $\leq \dim X + \delta$, so the same is true for its fibers at closed points with only finitely many possible exceptions.

Choose a negative integer $-N$ not among this finite number of exceptions. Then the coherent D_X-module $\mathcal{F}_0/(\lambda + N)$ has singular support contained in $Z \times_{\mathbb{A}^1_\lambda} \{-N\}$, so has dimension $\leq \dim X + \delta$. We have the obvious morphism of D_U-modules (in particular, of D_X-modules):

$$(\mathcal{F} f^\lambda)/(\lambda + N) \to \mathcal{F}$$

$$\sum_{i=0}^r \sigma_i \lambda^i f^\lambda \mapsto \sum_{i=0}^r f^{-N} \cdot \sigma_i \cdot (-N)^i, \quad \sigma_i \in \mathcal{F},$$

inducing a map $\mathcal{F}_0/(\lambda + N)$ to $\mathcal{F}$ sending the generator to $f^{-N}s$. By functoriality of the GKS filtration (or standard singular support analysis), this means that $f^{-N}s \in F_\delta^{\mathrm{GKS}}\mathcal{F}$, and since $F_\delta^{\mathrm{GKS}}\mathcal{F}$ is a D_X-module, this means that $s \in F_\delta^{\mathrm{GKS}}\mathcal{F}$ as well. □

2.8 Preservation of holonomic defect

We now proceed to prove Theorem 2.5.1. The argument is straightforward at this point, and we proceed in cases.

2.9 First, we treat pushforwards along an open embedding $j : U \to X$.

Suppose X is smooth and affine. We proceed by induction on the number n of basic affine opens required to cover U. For $n = 1$, the result is treated by Lemma 2.7.1.

The inductive step is a standard Čech argument.

For $n > 1$, U admits a cover by some U_1 and U_2 where U_1 is a basic affine open and U_2 admits a cover by $n - 1$ basic affine opens. Let $j_i : U_i \to U$ denote the embeddings and let $j_{12} : U_1 \cap U_2 \to U$.

[9] So this is a different $\mathcal{G}$ from Step 2, that is, we are applying the same construction to a different D-module.

Fix $\mathcal{F} \in D(U)$ with holonomic defect $\leq \delta$. Note that

$$\mathcal{F} = j_{1,*,\mathrm{dR}} j_1^!(\mathcal{F}) \underset{j_{12,*,\mathrm{dR}} j_{12}^!(\mathcal{F})}{\times} j_{2,*,\mathrm{dR}} j_2^!(\mathcal{F})$$

(where this fiber product is a homotopy[10] fiber product). By induction, $j_{?,*,\mathrm{dR}} j_?^!(\mathcal{F})$ has holonomic defect $\leq \delta$. Now recall from § 2.4 that D-modules with holonomic defect $\leq \delta$ are a DG subcategory and therefore closed under finite limits.

Finally, for general X, note that the problem is Zariski local, so we may assume X is affine. Take a closed embedding $X \subseteq \mathbb{A}^N$. If $U = X \setminus Z$, then we have $U \hookrightarrow \mathbb{A}^N \setminus Z \hookrightarrow \mathbb{A}^N$, with the first map being closed and the second being open between smooth affine schemes. Therefore, this pushforward preserves having holonomic defect $\leq \delta$. Clearly this implies the result for the pushforward along $U \hookrightarrow X$.

2.10 Next, we treat restrictions to closed subschemes.

Let $i : Z \hookrightarrow X$ be closed and let $j : U = X \setminus Z \hookrightarrow X$. Then we have an exact triangle:

$$i_{*,\mathrm{dR}} i^!(\mathcal{F}) \to \mathcal{F} \to j_{*,\mathrm{dR}} j^!(\mathcal{F}) \xrightarrow{+1} .$$

If $\mathcal{F}$ has holonomic defect $\leq \delta$, we have shown the same for $j_{*,\mathrm{dR}} j^!(\mathcal{F})$, so $i_{*,\mathrm{dR}} i^!(\mathcal{F})$ has holonomic defect $\leq \delta$, which is equivalent to $i^!(\mathcal{F})$ having holonomic defect $\leq \delta$.

2.11 We can now show the result for restrictions in general.

If $f : X \to Y$ is smooth of relative dimension d, then $f^{*,\mathrm{dR}}[d] = f^![-d]$ commutes with Verdier duality and is t-exact. Therefore, it commutes with formation of the GKS filtration, and therefore preserves the property of having holonomic defect $\leq \delta$.

The case of general $f : X \to Y$ is immediately reduced to the case of affine varieties (since holonomic defect is Zariski local). We can then find a commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \mathbb{A}^N \arrow[d, "g"] \\ Y \arrow[r] & \mathbb{A}^M \end{tikzcd} \tag{2.11.1}$$

with the horizontal arrows being closed embeddings. This reduces to the case where X and Y are smooth.

Then we can factor f through the graph as $X \to X \times Y \xrightarrow{p_1} Y$. The former map is a closed embedding, and the latter is smooth because X is. We have treated each of these cases, so we obtain the result.

2.12 Next, we treat pushforwards along a proper morphism $f : X \to Y$ between smooth varieties.

This case does not need the work we have done so far. Let $\mathcal{F} \in D(X)^\heartsuit$ with holonomic defect $\leq \delta$ be given. By Lemma 2.3.5, we may assume $\mathcal{F}$ is coherent, so the hypothesis is that $\mathcal{F}$ has singular support $\mathrm{SS}_X(\mathcal{F})$ with dimension $\dim X + \delta$.

[10] We remind the reader that if $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xleftarrow{\beta} \mathcal{H}$ is a diagram in a DG category $\mathcal{C}$, the (homotopy) fiber product of this diagram may be calculated as $\mathrm{Cone}(\mathcal{F} \oplus \mathcal{H} \xrightarrow{(\alpha,-\beta)} \mathcal{G})[-1]$. In particular, homotopy fiber products may be calculated in the underlying triangulated category of a DG category.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

Recall that $\mathrm{SS}_Y(H^i(f_{*,\mathrm{dR}}(\mathcal{F})))$ is bounded in terms of $\mathrm{SS}_X(\mathcal{F})$. More precisely, if we take the diagram

$$\begin{tikzcd} T^*Y \underset{Y}{\times} X \arrow[r, "\alpha"] \arrow[d, "\beta"] & T^*X \\ T^*Y & \end{tikzcd}$$

then the singular support of these cohomologies is contained in $\alpha(\beta^{-1}\,\mathrm{SS}_X(\mathcal{F}))$ (see, for example, [Kas76, Theorem 4.2]).

Because $\mathrm{SS}_X(\mathcal{F})$ is coisotropic by [Gab81], we have

$$\dim \alpha(\beta^{-1}\,\mathrm{SS}_X(\mathcal{F})) \leq \dim(\mathrm{SS}_X(\mathcal{F})) + \dim Y - \dim X$$

by usual symplectic geometry. This immediately gives the claim.

2.13 Now observe that preservation of holonomic defect under pushforward along a general morphism $f : X \to Y$ between smooth varieties follows: by Nagata's compactification theorem (see, for example, [Con07]), there exist a scheme $\overline{X}$ and a factorization

$$X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y$$

of f with $\overline{f}$ proper and j an open embedding. Moreover, by resolution of singularities [Hir64, Main Theorem I], we may take $\overline{X}$ to be smooth.[11] Then the result follows in this case from our earlier work.

2.14 We can now treat a general pushforward along $f : X \to Y$ a morphism between possibly singular varieties.

The problem is Zariski local on Y, so we may assume Y is affine.

Next, we reduce to the case where X is affine. It suffices to observe that the subcategory of $D(X)$ of D-modules of holonomic defect $\leq \delta$ is generated under colimits[12] by objects of the form $j_{*,\mathrm{dR}}(\mathcal{F})$ for $j : U \to X$ an affine open and $\mathcal{F}$ of holonomic defect $\leq \delta$. This observation follows by induction on the number of affines required to cover X by a Čech argument exactly as in § 2.9, and the fact (shown above) that pushforward along an open embedding preserves holonomic defect.

When X and Y are affine, we can find a commutative diagram (2.11.1) as before. This reduces to the case with X and Y smooth, which we have already treated.

3. Cohomological bounds

3.1 The main result of this section says that $f_!$ is left t-exact for an affine morphism f. We also show that for $i : X \to Y$ a closed embedding, $i^{*,\mathrm{dR}}$ has cohomological amplitude $\geq -\dim(Y) + \dim(X)$, that is, $i^{*,\mathrm{dR}}[-\dim(Y) + \dim(X)]$ is left t-exact. Since $f_!$ and $i^{*,\mathrm{dR}}$ are not necessarily defined on non-holonomic D-modules, we use the language of pro-categories to formulate this result.

[11] One may slightly modify this argument to use de Jong's alterations [DeJ96], which are more elementary than resolution of singularities.

[12] Equivalently, under direct sums and cones. (In particular, this makes sense as the triangulated level.)

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

3.2 As was remarked in the introduction, we use some language from higher category theory in order to discuss pro-categories.

We use the theory of ∞-categories developed in [Lur09]. In particular, we refer to [Lur09, § 5.3] for a discussion of ind-categories in this formalism and we refer to [Lur12, § 1.4.4] for the theory in the stable setting. There is a dual notion of pro-category: see [Lur09, Remark 7.1.6.2].

We apply this theory in the setting of DG categories. We follow [GR17, § I.1.10.3] in our terminology here, and in particular in viewing DG categories as ∞-categories with additional structure.

Remark 3.2.1. The reader may alter the treatment that follows in many possible ways to avoid the theory of ∞-categories.

First, the reader may use 'strict' DG categories, that is, usual categories enriched over chain complexes, and standard models for the relevant ind/pro-categories (see, for example, [Dri04, § 4]).

Alternatively, the reader may simply assume that we are given D-modules on which !-pushforwards are defined and proceed accordingly.

Finally, the reader may note that, for example, in Theorem 3.4.1, the functor in question takes values in $\mathsf{Pro}(D(Y)^c) \simeq D(Y)^{\mathrm{op}}$. Therefore, the use of pro-categories is inessential, and can be replaced by the use of a non-standard t-structure on $D(Y)$. We spell out this reformulation in Remark 3.4.3 below.

In what follows, we omit the prefix ∞ to avoid clutter, so, for example, 'category' means '∞-category'.

3.3 Pro-categories

We refer to [Lur11, § 3.1] and [GR17, § III.1.3.1] for the material that follows.

For $\mathcal{C}$ an accessible[13] category, we have the corresponding pro-category $\mathsf{Pro}(\mathcal{C})$. If $\mathcal{C}$ is a DG category, $\mathsf{Pro}(\mathcal{C})$ is as well. If $\mathcal{C}$ admits small colimits, then so does $\mathsf{Pro}(\mathcal{C})$. For $F : \mathcal{C} \to \mathcal{D}$, there is an induced functor $\mathsf{Pro}(\mathcal{C}) \to \mathsf{Pro}(\mathcal{D})$, which we denote again by F where there is no risk for confusion.

For any functor $G : \mathcal{D} \to \mathcal{C}$ commuting with finite colimits (for example, a DG functor), the induced functor $\mathsf{Pro}(\mathcal{D}) \to \mathsf{Pro}(\mathcal{C})$ admits a left adjoint F. We say that F *is defined* on an object $\mathcal{F} \in \mathcal{C}$ if $F(\mathcal{F}) \in \mathcal{D} \subseteq \mathsf{Pro}(\mathcal{D})$. (This coincides with the usual notion of a left adjoint being defined on some object.)

If $\mathcal{C}$ is a DG category equipped with a t-structure, then $\mathsf{Pro}(\mathcal{C})$ inherits one as well. It is characterized by the equality $\mathsf{Pro}(\mathcal{C})^{\leq 0} = \mathsf{Pro}(\mathcal{C}^{\leq 0})$. Truncation functors are the pro-extensions of the truncation functors on $\mathcal{C}$. In particular, we find that $\mathcal{C}$ is closed under truncations and inherits its given t-structure. We also find that $\mathsf{Pro}(\mathcal{C})^{\geq 0} = \mathsf{Pro}(\mathcal{C}^{\geq 0})$: if $\mathcal{F} = \lim_i \mathcal{F}_i \in \mathsf{Pro}(\mathcal{C})^{\geq 0}$, then $\mathcal{F} = \tau^{\geq 0}\mathcal{F} = \lim_i \tau^{\geq 0}\mathcal{F}_i$.

3.4 Affine morphisms

For $f : X \to Y$, we have the functor $f_! : \mathsf{Pro}(D(X)) \to \mathsf{Pro}(D(Y))$ left adjoint to $f^!$.

Theorem 3.4.1. *For f affine, the induced functor $f_! : D(X)^c \to \mathsf{Pro}(D(Y))$ is left t-exact.*

[13] We remind the reader that this is a robust set-theoretic condition satisfied by any idempotent-complete essentially small category and by any compactly generated category. One should be aware that $\mathsf{Pro}(\mathcal{C})$ is typically not accessible.

Remark 3.4.2. Before giving the argument, we spell out the meaning of this result without the language of pro-categories. Suppose $\mathcal{F} \in D(X)^{c,\geq 0}$ is coherent and in cohomological degrees ≥ 0. Let $\mathcal{G} \in D(Y)$ be arbitrary, and suppose we are given a morphism $\alpha : \mathcal{F} \to f^!(\mathcal{G})$. Then the theorem is equivalent to asserting that when f is affine, there necessarily exist $\mathcal{G}_0 \in D(Y)^{\geq 0}$, a morphism $\beta : \mathcal{G}_0 \to \mathcal{G} \in D(Y)$, and a factorization

$$\mathcal{F} \to f^!(\mathcal{G}_0) \xrightarrow{f^!(\beta)} f^!(\mathcal{G})$$

of α. In particular, the assertion of Theorem 3.4.1 can be reformulated using only triangulated categories.

Remark 3.4.3. Here is another, somewhat more classical reformation of the assertion. Define the *non-standard* t-structure on $D(X)$ by defining $D(X)^{\overset{ns}{\leq 0}}$ to be generated under filtered colimits by coherent objects of $D(X)$ whose Verdier duals lie in usual cohomological degrees ≥ 0. In these terms, Theorem 3.4.1 is equivalent (essentially by (3.4.1) below) to the assertion that for f affine, $f_{*,\mathrm{dR}}$ is right t-exact for these non-standard t-structures, that is, $f_{*,\mathrm{dR}}$ maps $D(X)^{\overset{ns}{\leq 0}}$ to $D(Y)^{\overset{ns}{\leq 0}}$.

Proof of Theorem 3.4.1. The problem is[14] Zariski local on Y, so we may assume X and Y are affine.

Note that D-module pushforward along closed embeddings remains fully faithful on pro-categories: the identity $i^! i_{*,\mathrm{dR}} = \mathrm{id}$ induces the same for the pro-functors. Therefore, the same argument as in § 2.11 allows us to assume X and Y are smooth.

Recall that we have a Verdier duality equivalence $\mathbb{D} : D(X)^{\mathrm{op}} \xrightarrow{\simeq} \mathsf{Pro}(D(X)^c)$ induced by the usual Verdier duality equivalence $\mathbb{D} : D(X)^c \xrightarrow{\simeq} D(X)^{c,\mathrm{op}}$, and similarly for Y.

We then claim that

$$f_!(\mathcal{F}) = \mathbb{D} f_{*,\mathrm{dR}} \mathbb{D}(\mathcal{F}). \tag{3.4.1}$$

This follows formally from the fact that $f_{*,\mathrm{dR}}$ and $f^!$ are *dual* functors in the sense of [Gai12], but we give a direct proof below. Note that in this formula, $f_{*,\mathrm{dR}}\mathbb{D}(\mathcal{F}) \in D(Y)$, and we are then using $\mathbb{D}$ to convert it to a pro-coherent object. Since this object is pro-coherent, it suffices to observe that for $\mathcal{G} \in D(Y)^c$, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathsf{Pro}(D(Y)^c)}(\mathbb{D} f_{*,\mathrm{dR}} \mathbb{D}(\mathcal{F}), \mathcal{G}) = \underline{\mathrm{Hom}}_{D(Y)}(\mathbb{D}\mathcal{G}, f_{*,\mathrm{dR}}\mathbb{D}(\mathcal{F})) = \Gamma_{\mathrm{dR}}(Y, f_{*,\mathrm{dR}}\mathbb{D}(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}) \\ = \Gamma_{\mathrm{dR}}(X, \mathbb{D}(\mathcal{F}) \overset{!}{\otimes} f^!(\mathcal{G})) = \underline{\mathrm{Hom}}_{D(X)}(\mathcal{F}, f^!(\mathcal{G})).\end{aligned}$$

Here Γ_{dR} is the complex of de Rham cochains of a D-module, and we are repeatedly using the formula[15] that if $\mathcal{F}_1$ is coherent, then

$$\underline{\mathrm{Hom}}_{D(X)}(\mathcal{F}_1, \mathcal{F}_2) = \Gamma_{\mathrm{dR}}(X, \mathbb{D}(\mathcal{F}_1) \overset{!}{\otimes} \mathcal{F}_2).$$

[14] Indeed, if $Y = U_1 \cup U_2$ with embeddings $j_i : U_i \hookrightarrow Y$ and $j_{12} : U_1 \cap U_2 \hookrightarrow Y$, then for $\mathcal{G} \in \mathsf{Pro}(D(Y))$ with $j_i^!(\mathcal{G}) \in \mathsf{Pro}(D(U_i))^{\geq 0}$, we want to see that $\mathcal{G} \in \mathsf{Pro}(D(Y))^{\geq 0}$. Note that

$$\mathcal{G} = j_{1,*,\mathrm{dR}} j_1^!(\mathcal{G}) \underset{j_{12,*,\mathrm{dR}} j_{12}^!(\mathcal{G})}{\times} j_{2,*,\mathrm{dR}} j_2^!(\mathcal{G}).$$

Indeed, this follows by pro-extension from the corresponding fact for usual D-modules. Since t-exact functors induce t-exact functors on pro-categories as well, we obviously obtain the claim.

[15] We remind the reader that this formula characterizes Verdier duality uniquely and is essentially taken as the definition in [GR17]. The reader may find this formula in the more conventional approach to D-module theory in [HTT08] as Corollary 2.6.15.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

Now suppose that $\mathcal{F} \in D(X)^{c,\heartsuit}$ is coherent and in cohomological degrees ≥ 0. Note that $\mathbb{D}\mathcal{F}$ carries the canonical filtration with subquotients $(H^{-j}\mathbb{D}\mathcal{F})[j]$. By Proposition 2.6.1, $H^{-j}\mathbb{D}\mathcal{F}$ has holonomic defect $\leq j$. By Theorem 2.5.1, $f_{*,\mathrm{dR}}H^{-j}\mathbb{D}\mathcal{F}$ has holonomic defect $\leq j$ as well. Moreover, by affineness of f, the latter complex is in cohomological degrees ≤ 0.

Note that by Proposition 2.6.1, if $\mathcal{G} \in D(Y)^{\heartsuit}$ has holonomic defect $\leq \delta$, then $\mathbb{D}\mathcal{G} \in \mathsf{Pro}(D(Y)^c)$ is in cohomological degrees $[-\delta, 0]$: indeed, this immediately reduces to the coherent case.

Therefore, $\mathbb{D}H^{-k}f_{*,\mathrm{dR}}H^{-j}\mathbb{D}\mathcal{F}$ is in cohomological degrees $[-j, 0]$ for every k, which means $\mathbb{D}(H^{-k}(f_{*,\mathrm{dR}}H^{-j}\mathbb{D}\mathcal{F})[k])$ is in cohomological degrees $[-j+k, k]$. This complex vanishes unless $k \geq 0$, so $\mathbb{D}f_{*,\mathrm{dR}}H^{-j}\mathbb{D}\mathcal{F}$ is in cohomological degrees $\geq -j$. Finally, this means that $\mathbb{D}((f_{*,\mathrm{dR}}H^{-j}\mathbb{D}\mathcal{F})[j])$ is in cohomological degrees ≥ 0, so the same follows for $\mathbb{D}f_{*,\mathrm{dR}}\mathbb{D}(\mathcal{F}) = f_!(\mathcal{F})$. □

Remark 3.4.4. More generally, this argument shows that if f is a possibly non-affine morphism between varieties X and Y such that $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ has amplitude $\leq n$, then $f_! : D(X)^c \to \mathsf{Pro}(D(Y))$ has amplitude $\geq -n$. (Here QCoh denotes the DG category of quasi-coherent sheaves and f_* is the standard pushforward functor for such.)

3.5 Closed embeddings

Similarly, we have the following theorem.

Theorem 3.5.1. *For $i : X \to Y$ a closed embedding, $i^{*,\mathrm{dR}} : D(Y)^c \to \mathsf{Pro}(D(X))$ has cohomological amplitude $\geq -\dim(Y) + \dim(X)$.*

Proof. The argument is the same as the above: one writes $i^{*,\mathrm{dR}} = \mathbb{D}i^!\mathbb{D}$ and applies Theorem 2.5.1 and Proposition 2.6.1, plus the fact that $i^!$ has amplitude $\leq \dim(Y) - \dim(X)$. □

Remark 3.5.2. As in Remark 3.4.4, one can generalize Theorem 3.5.1 to arbitrary morphisms $f : X \to Y$: if $f^!$ has amplitude $\leq r$, $f^{*,\mathrm{dR}} : D(Y)^c \to \mathsf{Pro}(D(X))$ has amplitude $\geq -r$.

Appendix A. Exactness properties of Kostant's functor

A.1 In this appendix, we briefly give an application of Theorem 3.4.1 to representation theory. The result below is a toy model of the applications in [Ras21].

A.2 Let $\mathfrak{g}$ be a semi-simple Lie algebra, let $\mathfrak{b}, \mathfrak{b}^- \subseteq \mathfrak{g}$ be opposed Borels with radicals $\mathfrak{n}$ and $\mathfrak{n}^-$. Let $\psi : \mathfrak{n}^- \to k$ be a non-degenerate character, that is, ψ is non-zero on weight spaces corresponding to negative simple roots.

Let $\mathfrak{g}\text{–}\mathsf{mod}$ denote the DG category of $\mathfrak{g}$-modules. Let $\Psi^{\mathrm{fin}} : \mathfrak{g}\text{–}\mathsf{mod} \to \mathsf{Vect}$ be the functor computing Lie algebra homology of $\mathfrak{n}^-$ twisted by ψ. That is, $M \in \mathfrak{g}\text{–}\mathsf{mod}$ maps to $C_\bullet(\mathfrak{n}^-, M \otimes \psi)$, where $M \otimes \psi$ indicates the $\mathfrak{n}^-$-module with the same underlying vector space as M but action twisted as $x \overset{\mathrm{new}}{\cdot} v := x \overset{\mathrm{old}}{\cdot} v + \psi(x)$ for $x \in \mathfrak{n}^-$ and $v \in M$.

Let $\mathfrak{g}\text{–}\mathsf{mod}^N \subseteq \mathfrak{g}\text{–}\mathsf{mod}$ be the full subcategory consisting of complexes such that $\mathfrak{n}$ acts locally nilpotently on cohomology.

Theorem A.2.1. *The functor Ψ^{fin} is exact when restricted to $\mathfrak{g}\text{–}\mathsf{mod}^N$.*

Remark A.2.2. This theorem is well known (cf. [Kos78]) when we replace $\mathfrak{g}\text{–}\mathsf{mod}^N$ by the Bernstein–Gelfand–Gelfand category $\mathcal{O}$. Indeed, because the latter category is Artinian, it suffices to verify that $\Psi^{\mathrm{fin}}(L_\lambda) \in \mathsf{Vect}^{\heartsuit}$ for any λ, where L_λ is the simple of highest weight λ.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

But this is easy to see: either L_λ is a Verma module and therefore free over $U(\mathfrak{n}^-)$, or L_λ is partially integrable, in which case $\Psi^{\mathrm{fin}}(L_\lambda) = 0$.

However, this method does not work in the above setting, since $\mathfrak{g}\text{–}\mathsf{mod}^N$ is not Artinian (the Cartan subalgebra may not act locally finitely). I do not know another reference for this result in this generality.

Example A.2.3. As a very simple case of Theorem A.2.1, note that the universal Verma module $M^{\mathrm{univ}} := U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k$ is an object of $\mathfrak{g}\text{–}\mathsf{mod}^{N,\heartsuit}$. As M^{univ} is free over $U(\mathfrak{n}^-)$ by the triangular decomposition and the Poincaré–Birkhoff–Witt theorem, $\Psi^{\mathrm{fin}}(M^{\mathrm{univ}})$ is concentrated in degree 0.

In fact, by functoriality, for $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$, $\Psi^{\mathrm{fin}}(M^{\mathrm{univ}})$ is canonically a $(Z(\mathfrak{g}), \mathrm{Sym}(\mathfrak{t}))$-bimodule. Moreover, the above reasoning shows that it is free of rank one for the $\mathrm{Sym}(\mathfrak{t})$-action. By construction, the induced homomorphism $Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{t})$ is the Harish-Chandra morphism.

Proof of Theorem A.2.1.

Step 1. We will use some notions from group actions on categories. The reader may skip this background material and refer back to it as needed. We refer to [Ber17, § 2] for details on what follows.

Let $\mathsf{DGCat}_{\mathrm{cont}}$ denote the symmetric monoidal category of presentable[16] DG categories under colimit-preserving DG functors; the symmetric monoidal structure is discussed in [GR17, § I.10.4].

Let $\mathsf{Sch}^{\mathrm{f.t.}}$ denote the category of finite type k-schemes. The functor $D(-) : \mathsf{Sch}^{\mathrm{f.t.}} \to \mathsf{DGCat}_{\mathrm{cont}}$ from [GR17, § III.4] carries a canonical symmetric monoidal structure. Therefore, $D(G) \in \mathsf{Alg}(\mathsf{DGCat}_{\mathrm{cont}})$ has a canonical monoidal structure; the underlying monoidal product sends $\mathcal{F}, \mathcal{G} \in D(G)$ to $m_{*,\mathrm{dR}}(\mathcal{F} \boxtimes \mathcal{G})$ for $m : G \times G \to G$ the group multiplication.

By definition, a *(strong) action* of G on $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ is an action of $D(G)$ on $\mathcal{C}$, where $D(G)$ is thought of as an algebra object in $\mathsf{DGCat}_{\mathrm{cont}}$.

For G acting on $\mathcal{C}$, recall from [Ber17] that there is a full subcategory $\mathcal{C}^N \subseteq \mathcal{C}$ of N-*equivariant objects*. (In general, equivariant objects are not a full subcategory, but this is the case for unipotent groups.) Moreover, this embedding functor admits a right adjoint $\mathrm{Av}_*^N : \mathcal{C} \to \mathcal{C}^N$ that is continuous.

As in [Ber17, § 2.5], the character $N^- \to \mathbb{G}_a$ obtained by exponentiating ψ also defines a *twisted* notion of invariants for N^-, which we denote $\mathcal{C}^{N^-,\psi}$. Again, $\mathcal{C}^{N^-,\psi}$ is a certain full subcategory of $\mathcal{C}$, and the embedding admits a right adjoint that commutes with colimits, which we denote $\mathrm{Av}_*^{N^-,\psi}$.

In what follows, we will be particularly interested in the functor $\mathrm{Av}_*^{N^-,\psi} : \mathcal{C}^N \to \mathcal{C}^{N^-,\psi}$. (The notation is slightly abusive; we are also letting $\mathrm{Av}_*^{N^-,\psi}$ denote the composition $\mathcal{C}^N \to \mathcal{C} \xrightarrow{\mathrm{Av}_*^{N^-,\psi}} \mathcal{C}^{N^-,\psi}$.)

Step 2. Let $\mathrm{Loc} : \mathfrak{g}\text{–}\mathsf{mod} \to D(G)$ denote the functor $\mathcal{F} \mapsto D_G \otimes_{U(\mathfrak{g})} \mathcal{F}$ for $D_G \in D(G)^\heartsuit$ the D-module of differential operators. We remark that if we think of $\mathfrak{g}\text{–}\mathsf{mod}$ as the category of weakly equivariant D-modules on G, Loc is the functor of forgetting the weak equivariance structure. The notation indicates that Loc may also be thought of as the *localization functor* for G acting on itself.

[16] We refer to [Lur09] for the definition. Roughly, a category is presentable if it admits colimits and satisfies a mild set-theoretic condition. Any compactly generated DG category is presentable.

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press

Below, we will upgrade this functor to a map of categories acted on strongly by G.

Step 3. Recall that when G acts on $\mathcal{C}$, there is a DG category $\mathcal{C}^{G,w} \in \mathsf{DGCat}_{\text{cont}}$ of *weakly* equivariant objects in $\mathcal{C}$; see [Ber17, § 2.2]. We remind the reader that $\mathcal{C}^{G,w}$ is equipped with a canonical functor $\mathrm{Oblv} : \mathcal{C}^{G,w} \to \mathcal{C}$.

Consider $D(G)$ as a bimodule over itself and form $D(G)^{G,w}$ the weak invariants for the right action of G; the left action of G on itself equips $D(G)^{G,w}$ with a canonical G-action.

By [GR17, § IV.3 Proposition-Construction 5.1.2] and the definitions, there is a canonical equivalence $D(G)^{G,w} \simeq \mathfrak{g}\text{–}\mathsf{mod}$ normalized by having $\mathrm{Oblv}(U(\mathfrak{g})) = D_G \in D(G)^{\heartsuit}$. Under this equivalence, Loc constructed above corresponds to the forgetful functor $\mathrm{Oblv} : D(G)^{G,w} \to D(G)$, making it clear that Loc is canonically G-equivariant.

Step 4. Note that the subcategory we denoted $\mathfrak{g}\text{–}\mathsf{mod}^N \subseteq \mathfrak{g}\text{–}\mathsf{mod}$ is in fact the N-equivariant category for our G-action on $\mathfrak{g}\text{–}\mathsf{mod}$. Indeed, this N-equivariant category embeds fully faithfully into $\mathfrak{g}\text{–}\mathsf{mod}$ and is closed under truncations, so this claim may be checked at the abelian categorical level where it is standard (see, for example, [FG06, § 20.4]).

Similarly, $\mathfrak{g}\text{–}\mathsf{mod}^{N^-,\psi}$ is the full subcategory of objects $M \in \mathfrak{g}\text{–}\mathsf{mod}$ such that $\mathfrak{n}$ acts locally nilpotently on the cohomology of $M \otimes -\psi$. (We use $- \otimes -\psi$ similarly to $\otimes\psi$, but for the inverse character.)

Step 5. We now have a commutative diagram

$$\begin{tikzcd}
\mathfrak{g}\text{–}\mathsf{mod}^N \arrow[r, "\mathrm{Loc}^N"] \arrow[d, "\mathrm{Av}_*^{N^-,\psi}"] & D(G/N) = D(G)^N \arrow[d, "\mathrm{Av}_*^{N^-,\psi}"] \\
\mathfrak{g}\text{–}\mathsf{mod}^{N^-,\psi} \arrow[r, "\mathrm{Loc}^{N^-,\psi}"] & D(G)^{N^-,\psi}
\end{tikzcd}$$

where the horizontal arrows are induced by the functor Loc considered above and the vertical arrows are the $*$-averaging functors discussed above.

We claim the left vertical arrow is t-exact up to shift by the dimension, that is, it maps objects in $\mathfrak{g}\text{–}\mathsf{mod}^{N,\heartsuit}$ to objects in cohomological degree $\dim(N)$.

The localization functors Loc^N and $\mathrm{Loc}^{N^-,\psi}$ are clearly t-exact, since this is evident for Loc itself. Moreover, the $*$-averaging functor on the right coincides with the !-averaging functor up to cohomological shift $2 \dim N$ by [BBM04, Theorem 1.5 (1)].[17]

[17] Formally, [BBM04] is written in the language of perverse sheaves. However, its arguments apply essentially verbatim to the setting of D-modules. Regarding this point we reproduce here the discussion of [Ras21, § 2.11], written in a different, but similar, context.

Suppose X is a variety with a G-action. Let $\mathrm{act} : N^- \times X \to X$ denote the action map for the N^--action. Define $\overline{\mathrm{act}} : G \overset{B}{\times} X \to X$; here the superscript B indicates that we quotient by the diagonal B-action (acting on the right on the B-factor).

By the Bruhat decomposition, the evident map $j : N^- \times X = N^-B \overset{B}{\times} X \to G \overset{B}{\times} X$ is an open embedding. Clearly $\overline{\mathrm{act}} \circ j = \mathrm{act}$. Finally, $\overline{\mathrm{act}}$ is proper.

We claim that for any $\mathcal{F} \in D(X)^N$, the map $j_!(\psi \boxtimes \mathcal{F}) \to j_{*,\mathrm{dR}}(\psi \boxtimes \mathcal{F})$ is an isomorphism; in particular, the left-hand side is defined. In other words, we claim that $j_{*,\mathrm{dR}}(\psi \boxtimes \mathcal{F})$ satisfies the universal property of $j_!(\psi \boxtimes \mathcal{F})$.

Assuming this *cleanness* result, we see that $\mathrm{act}_!(\psi \boxtimes \mathcal{F}) = \overline{\mathrm{act}}_{*,\mathrm{dR}} j_!(\psi \boxtimes \mathcal{F})$ is defined; by standard reasoning, this object computes $\mathrm{Av}_!^{\psi}(\mathcal{F})$, while $\mathrm{act}_{*,\mathrm{dR}}(\psi \boxtimes \mathcal{F}) = \overline{\mathrm{act}}_{*,\mathrm{dR}} j_{*,\mathrm{dR}}(\psi \boxtimes \mathcal{F})$ computes $\mathrm{Av}_*^{\psi}(\mathcal{F})[2 \dim N^-]$, showing the [BBM04] result.

It remains to verify cleanness. It is equivalent to check it after pullback along the (smooth) projection map $G \times X \to G \overset{B}{\times} X$.

The $*$-averaging functor has cohomological amplitude $\leq \dim N$ because N^- is affine, and the !-averaging functor has amplitude $\geq -\dim N$ by Theorem 3.4.1. We remark that because $D(G/N)$ is compactly generated with compact objects closed under truncations and mapping to compact objects in $D(G)$, we are justified in passing from coherent objects to arbitrary ones here. (Note that the D-modules arising by localization here are not necessarily holonomic!) Identifying the functors after shift then gives the result.

Step 6. Let $\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi) \in \mathfrak{g}\text{–}\mathsf{mod}^{N^-,\psi,\heartsuit}$ be the standard induced module.

Note that for $M \in \mathfrak{g}\text{–}\mathsf{mod}$, we have

$$C^\bullet(\mathfrak{n}^-, M \otimes -\psi) = \underline{\mathrm{Hom}}_{\mathfrak{g}\text{–}\mathsf{mod}}(\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi), M) = \underline{\mathrm{Hom}}_{\mathfrak{g}\text{–}\mathsf{mod}^{N^-,\psi}}(\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi), \mathrm{Av}_*^{N^-,\psi}(M)) \in \mathsf{Vect}.$$

Here $C^\bullet(\mathfrak{n}^-, -)$ is the cohomological Chevalley complex.

By Skryabin's theorem,[18] the functor $C^\bullet(\mathfrak{n}^-, (-) \otimes -\psi)$ is t-exact on $\mathfrak{g}\text{–}\mathsf{mod}^{N^-,\psi}$.

For $M \in \mathfrak{g}\text{–}\mathsf{mod}^{N,\heartsuit}$, $\mathrm{Av}_*^{N^-,\psi}(M)[\dim N] \in \mathfrak{g}\text{–}\mathsf{mod}^{N^-,\psi,\heartsuit}$ by Step 5.

Noting that Lie algebra homology and cohomology coincide up to shift by $\dim N$, and switching the role of ψ and $-\psi$, we obtain the desired result. $\square$

Acknowledgements

We are grateful to Dennis Gaitsgory, Victor Ginzburg, and Masaki Kashiwara for useful correspondence on the subject of this article. The methods owe a great deal to [Ber, Gin86, Kas76].

References

Ber17 D. Beraldo, *Loop group actions on categories and Whittaker invariants*, Adv. Math. **322** (2017), 565–636.

Ber J. Bernstein, *Algebraic theory of D-modules*, http://math.uchicago.edu/~drinfeld/langlands/Bernstein/Bernstein-dmod.ps.

Ber72 J. Bernstein, *The analytic continuation of generalized functions with respect to a parameter*, Funct. Anal. Appl. **6** (1972), 273–285.

BBM04 R. Bezrukavnikov, A. Braverman and I. Mirkovic, *Some results about geometric Whittaker model*, Adv. Math. **186** (2004), 143–152.

BY18 R. Bezrukavnikov and A. Yom Din, *On parabolic restriction of perverse sheaves*, Preprint (2018), arXiv:1810.03297.

The pullback of $\psi \boxtimes \mathcal{F}$ is a D-module on $N^-B \times X$. It is manifestly (N^-,ψ)-equivariant for the left N^--action on the first factor. It is straightforward (cf. [BBM04, § 2.4]) to see that it is N-equivariant for the right N-action on the first factor; here we use N-equivariance of $\mathcal{F}$.

Now observe that for any $w \neq 1$ in the Weyl group, $D(B^-wN/N)^{N^-,\psi} = 0$. Indeed, the restriction functor from here to $D(T \cdot w)$ is clearly conservative. Now choose a simple root α_i with $w^{-1} \cdot \alpha_i$ negative: it exists because $w \neq 1$. Then the corresponding embedded copy $\mathbb{G}_a \subseteq N^-$ stabilizes $T \cdot w \subseteq B^-wN/N$ and acts trivially on it. Therefore, if we take an object of $D(B^-wN/N)^{N^-,\psi}$, its restriction to $T \cdot w$ is $(\mathbb{G}_a, \psi|_{\mathbb{G}_a})$-equivariant for the trivial action of $\mathbb{G}_a$. As $\psi|_{\mathbb{G}_a}$ is non-trivial, this is only possible for the zero object.

By the Bruhat decomposition, we obtain that $D(G/N)^{N^-,\psi} \xrightarrow{\simeq} D(B^-N/N)^{N^-,\psi} (= D(T))$. The same logic applies with the factor X, that is, the restriction map $D(G/N \times X)^{N^-,\psi} \to D(B^-N/N \times X)^{N^-,\psi}$ is an isomorphism, where the N^--action is on the first factor alone.

Using $*$-averaging functors, we deduce that any object of $D(B^-N/N \times X)^{N^-,\psi}$ extends cleanly to $D(G/N \times X)^{N^-,\psi}$. This reasoning applies in particular to the object we were considering, completing the argument.

[18] The original version of this theorem is proved in [Pre02], but, for example, in the derived setting in which we use the result, this is proved in [Ras21] as Theorem 1.5.1.

Con07 B. Conrad, *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), 205.

DeJ96 A. J. De Jong, *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51–93.

Dri04 V. Drinfeld, *DG quotients of DG categories*, J. Algebra **272** (2004), 643–691.

FG06 E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence and affine Kac-Moody algebras*, in *Algebraic geometry and number theory* (Springer, 2006), 69–260.

Gab81 O. Gabber, *The integrability of the characteristic variety*, Amer. J. Math. **103** (1981), 445–468.

Gai12 D. Gaitsgory, *Generalities on DG categories* (2012), http://math.harvard.edu/gaitsgde/GL/textDG.pdf.

Gai16 D. Gaitsgory, *Functors given by kernels, adjunctions and duality*, J. Algebraic Geom. **25** (2016), 461–548.

GR17 D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, Mathematical Surveys and Monographs, vol. 221 (American Mathematical Society, Providence, RI, 2017).

Gin86 V. Ginzburg, *Characteristic varieties and vanishing cycles*, Invent. Math. **84** (1986), 327–402.

Hir64 H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Ann. Math. **79** (1964), 109–203.

HTT08 R. Hotta, K. Takeuchi and T. Tanisaki, *D-Modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236 (Birkhäuser, Boston, MA, 2008); translated from the 1995 Japanese edition by Takeuchi.

Kas76 M. Kashiwara, *b-functions and holonomic systems*, Invent. Math. **38** (1976), 33–53.

Kas95 M. Kashiwara, *Algebraic study of systems of partial differential equations*, in *Memoires de la Société Mathématique de France*, vol. 63 (Société Mathématique de France, 1995); translation of 1970 master's thesis.

Kos78 B. Kostant, *On Whittaker vectors and representation theory*, Invent. Math. **48** (1978), 101–184.

Lur09 J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170 (Princeton University Press, Princeton, NJ, 2009).

Lur11 J. Lurie, *DAG XIII: Rational and p-adic homotopy theory* (2011), https://www.math.ias.edu/~lurie/papers/DAG-XIII.pdf.

Lur12 J. Lurie, *Higher algebra* (2012), http://math.harvard.edu/lurie/papers/HigherAlgebra.pdf.

Pre02 A. Premet, *Special transverse slices and their enveloping algebras*, Adv. Math. **170** (2002), 1–55.

Ras21 S. Raskin, *$\mathcal{W}$-algebras and Whittaker categories*, Selecta Math. (N.S.) **27** (2021), 46.

Sam Raskin sraskin@math.utexas.edu
Department of Mathematics, University of Texas at Austin, PMA 8.100,
2515 Speedway Stop C1200, Austin, TX 78712, USA

https://doi.org/10.1112/S0010437X21007491 Published online by Cambridge University Press