Linking Number of Singular Links and the Seifert Matrix

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Abstract. We extend the notion of linking number of an ordinary link of two components to that of a singular link with transverse intersections, in which case the linking number is a half-integer. We then apply it to simplify the construction of the Seifert matrix, and therefore the Alexander polynomial, in a natural way.

1 Introduction

The notion of linking number for two non-intersecting, parametrized, closed (C^1) curves in 3-space originated from the Gauss linking integral. It is the integral over the unit torus of the pull-back of the area form of the unit sphere under the secant Gauss map. Topologically, the linking number is the degree of the secant Gauss map, and therefore the number of times the oriented unit sphere is covered by the image of the map. It is the integer representing the homology class of one curve in the first homology group of the complement of the other curve, and it is the algebraic intersection number of one curve with a 2-complex which bounds the other curve (in case of a link of two components, then the 2-complex can be chosen to be a Seifert surface). Combinatorially, the linking number is the sum of the signed crossing numbers of one specified curve crossing under the other curve in a diagram of the curves (*i.e.,* regular projection in a plane), and equivalently, it is one-half of the sum of the signed crossing numbers of any one curve crossing under the other curve in a diagram. We shall see that this last viewpoint is especially useful when we discuss the linking number of singular links. For a discussion of the above alternative and equivalent definitions of linking number for an (ordinary) link of two components, see [R, pp. 132–136] and [K, p. 14].

In this paper, we extend the notion of linking number via Gauss linking integral to the case of two parametrized, closed (C^2) curves which intersect one another transversely. In this case the linking number is no longer an integer in general, and instead it is half of an integer. We then give a combinatorial formula for computing such a linking number from a diagram of the curves. A natural question is to ask whether there is a topological explanation of this notion of linking number in terms of homology and intersection theory. We have not been able to answer it at the time of this writing.

This notion of linking number, in conjunction with the intersection form of curves in surfaces, will then be used to construct the Seifert matrix for an oriented link, which is essential in Seifert's construction of the Alexander polynomial of the link. This procedure simplifies the usual construction of the Seifert matrix via Seifert pairings by eliminating the necessity of pushing a fundamental system of curves of the Seifert surface to one side of the surface in order to avoid intersections of two distinct curves, so that the ordinary linking number can be evaluated. In a way, this procedure is also more natural than the conventional way of finding the Seifert matrix.

We work in the smooth (C^∞) category unless otherwise indicated. As usual, we denote by $lk(x, y)$ the linking number of two disjoint knots x and y in R^3 or S^3.

2   Linking Number of Singular Links

Let $x, y\colon S^1 \to R^3$ be smooth embeddings and have transverse intersections. Here transverse intersection means that at a point of intersection, the curves x and y have linearly independent tangent vectors. In this case we say that $x \cup y$ is a (parametrized) *singular link* with transverse intersections. An *isotopy* of a singular link with transverse intersections is an (ordinary) isotopy satisfying the condition that at each stage the isotopy preserves the transversality of intersections of the singular link. Consider the Gauss linking integral for $x \cup y$, which we denote by $lk(x, y)$:

$$lk(x, y) = \frac{1}{4\pi} \int_{S^1} \int_{S^1} \frac{x'(s) \times y'(t) \cdot (x(s) - y(t))}{\|x(s) - y(t)\|^3} \, dt\,ds.$$

Călugăreanu showed [C1, p. 7] that despite the blow-ups of the integrand at the intersection points of x and y, the linking integral is a well-defined finite number. We show in the following that this number is a half-integer.

Theorem 2.1   *The Gauss linking integral of a (parametrized) singular link $x \cup y$ with transverse intersections is a number in $Z[\frac{1}{2}]$ and an isotopy invariant of $x \cup y$.*

Proof   Since the Gauss linking integral is independent of the parametrization, for convenience we identify S^1 with the interval $[0, 2\pi]$, and assume that $x(0) = x(2\pi)$ and $y(0) = y(2\pi)$. Let $p_1, \ldots, p_n$ be the points of intersection of x and y, and let $(s_1, t_1), \ldots, (s_n, t_n)$ be the corresponding points in the torus $T = S^1 \times S^1$ such that $x(s_i) = y(t_i) = p_i$ for $i = 1, \ldots, n$. Let

$$w_i = \frac{x'(s_i) \times y'(t_i)}{\|x'(s_i) \times y'(t_i)\|}.$$

Then w_i is a unit vector perpendicular to the plane spanned by the tangent vectors of x and y at the point p_i. Choose a sufficiently small $\delta > 0$ so that the sets $N_i(2\delta) = \{(s, t) : |s - s_i| < 2\delta, |t - t_i| < 2\delta\}, i = 1, \ldots, n$, are disjoint square neighborhoods of the points $(s_1, t_1), \ldots, (s_n, t_n)$ in T. We will further restrict δ in the course of the proof.

Let $\lambda(t)$ be a bump function satisfying $\lambda(t) = 1$ if $|t| \leq \delta$, $0 < \lambda(t) < 1$ if $\delta < |t| < 2\delta$, and $\lambda(t) = 0$ if $|t| \geq 2\delta$. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_i = \pm 1$. For sufficiently small $\epsilon > 0$, set $y_\epsilon^\sigma(t) = y(t) + \sum_{i=1}^{n} \epsilon \lambda(t - t_i)\sigma_i w_i$. Geometrically, $\{y_\epsilon^\sigma\}$ is a family of knots obtained by pushing y off x locally at the point p_i in the direction of $\sigma_i w_i$.

Consider the Gauss integrands

$$G(s, t) = \frac{x'(s) \times y'(t) \cdot (x(s) - y(t))}{\|x(s) - y(t)\|^3}$$

and

$$G_\epsilon^\sigma(s, t) = \frac{x'(s) \times (y_\epsilon^\sigma)'(t) \cdot (x(s) - y_\epsilon^\sigma(t))}{\|x(s) - y_\epsilon^\sigma(t)\|^3}.$$

Since for each σ and for all sufficiently small $\epsilon > 0$, $\{x, y_\epsilon^\sigma\}$ is a family of isotopic nonsingular links,

$$\frac{1}{4\pi} \iint_T G_\epsilon^\sigma(s, t) \, dt ds = lk(x, y_\epsilon^\sigma)$$

is an integer which is independent of ϵ. For notational convenience let $lk(x, y^\sigma)$ denote this integer constant. Now

$$\iint_T G_\epsilon^\sigma(s, t) \, dt ds = \iint_{\widehat{T}} G_\epsilon^\sigma(s, t) \, dt ds + \sum_{i=1}^{n} \iint_{N_i(\delta)} G_\epsilon^\sigma(s, t) \, dt ds,$$

where $\widehat{T} = T - \bigcup_{i=1}^{n} N_i(\delta)$. Clearly, G_ϵ^σ converges uniformly to G on $\widehat{T}$ since the denominator does not vanish. Thus we have

$$\lim_{\epsilon \to 0} \iint_{\widehat{T}} G_\epsilon^\sigma(s, t) \, dt ds = \iint_{\widehat{T}} G(s, t) \, dt ds.$$

On the other hand, for each $i = 1, \ldots, n$, $y_\epsilon^\sigma(t) = y(t) + \epsilon \sigma_i w_i$ if $|t - t_i| < \delta$, since $\lambda(t - t_i) = 1$. Therefore for $(s, t) \in N_i(\delta)$, $G_\epsilon^\sigma(s, t) = G_\epsilon^i(s, t) - W_\epsilon^i(s, t)$, where

$$G_\epsilon^i(s, t) = \frac{x'(s) \times y'(t) \cdot (x(s) - y(t))}{\|x(s) - y(t) - \epsilon \sigma_i w_i\|^3}$$

and

$$W_\epsilon^i(s, t) = \frac{x'(s) \times y'(t) \cdot \epsilon \sigma_i w_i}{\|x(s) - y(t) - \epsilon \sigma_i w_i\|^3}.$$

The argument in [C2, p. 615] (see also [W, p. 233]) implies that

$$\lim_{\epsilon \to 0} \iint_{N_i(\delta)} W_\epsilon^i(s, t) \, dt ds = -2\pi \sigma_i.$$

In the appendix, we will show that it is possible to choose δ small enough so that there exists a constant K such that

$$(2.1) \qquad\qquad |G_\epsilon^i(s, t)| \leq \frac{K}{\sqrt{(s - s_i)^2 + (t - t_i)^2}}$$

for all i and all $(s,t) \in N_i(\delta)$.

With δ so chosen, observe that $K/\sqrt{(s-s_i)^2 + (t-t_i)^2}$ is integrable in $N_i(\delta)$, and that $G_\epsilon^\sigma(s,t) \to G(s,t)$ almost everywhere in $N_i(\delta)$. By the Lebesque dominated convergence theorem, we have

$$\lim_{\epsilon \to 0} \iint_{N_i(\delta)} G_\epsilon^\sigma(s,t)\, dt ds = \iint_{N_i(\delta)} G(s,t)\, dt ds.$$

Putting this together we have

$$\lim_{\epsilon \to 0} \iint_T G_\epsilon^\sigma(s,t)\, dt ds = \iint_{\widehat{T}} G(s,t)\, dt ds + \sum_{i=1}^n \left(\iint_{N_i(\delta)} G(s,t)\, dt ds - 2\pi\sigma_i \right)$$

$$= \iint_T G(s,t)\, dt ds - \sum_{i=1}^n 2\pi\sigma_i.$$

Dividing by 4π gives

$$lk(x, y^\sigma) = lk(x, y) - \sum_{i=1}^n \frac{\sigma_i}{2}.$$

Thus

$$lk(x, y) = lk(x, y^\sigma) + \frac{1}{2} \sum_{i=1}^n \sigma_i.$$

Since $lk(x, y^\sigma)$ is an integer, this proves the theorem. ∎

Remark Note that if we set $+ = (+1, \ldots, +1)$ and $- = (-1, \ldots, -1)$, then the proof shows that $lk(x, y) = lk(x, y^+) + n/2$, and $lk(x, y) = lk(x, y^-) - n/2$. Averaging implies $lk(x, y) = \frac{1}{2}[lk(x, y^+) + lk(x, y^-)]$. Note also that Theorem 2.1 holds for $x, y \colon S^1 \to R^3$ that are C^2-immersions with transverse intersections.

We may also interpret the linking integral of a singular link by means of a generalized Brouwer degree of the Gauss secant map. Observe that for a singular link $x \cup y$, the Gauss secant map

$$\gamma(s,t) = \frac{x(s) - y(t)}{\|x(s) - y(t)\|}$$

maps the finitely punctured torus $\{(s,t) \in S^1 \times S^1 \mid x(s) \neq y(t)\}$ into the sphere S^2. The puncture points correspond to the points of intersections of x and y. We compactify this punctured torus by attaching the circle of directions in the tangent space of a puncture point to form an ideal circle boundary component. To describe this, we put polar coordinates (r, θ) about a puncture point. The polar coordinates define coordinates in a neighborhood of the boundary, in which the boundary is coordinatized by $\{(0, \theta) \mid \theta \in S^1\}$. In this way the punctured torus is compactified into a surface M with boundary, and the Gauss secant map extends to a map $\gamma \colon M \to S^2$

that carries each boundary component diffeomorphically onto a great circle of S^2. In
the coordinates of the ideal boundary we have

$$\gamma(0, \theta) = \frac{x'(s_0) \cos \theta + y'(t_0) \sin \theta}{\|x'(s_0) \cos \theta + y'(t_0) \sin \theta\|},$$

where (s_0, t_0) is the puncture point. This formula shows that the image of the bound-
ary component under γ is the great circle lying in the plane through the origin
which is parallel to the plane spanned by $x'(s_0)$ and $y'(t_0)$ at the point of intersec-
tion. If dA is the area form on S^2 (oriented by the inward normal), then $lk(x, y) =
\frac{1}{4\pi} \int_M \gamma^*(dA)$. This integral measures the signed area of M covering S^2 under γ.

   More generally, let M be a compact oriented surface with boundary. Consider a
smooth mapping $f: M \to S^2$ that carries each boundary component of M diffeo-
morphically onto a great circle of S^2. If p is a regular point of f, then $f_*: T_pM \to
T_{f(p)}S^2$ is an isomorphism. Define as usual the local degree of f at p by $\deg(f, p) =
\pm 1$ depending upon whether f_* preserves or reverses orientation. If $C \subset \partial M$ is a
boundary component of M and $q \in S^2 \setminus f(C)$, then $f(C)$ is the boundary of a hemi-
sphere $H_{C,q}$ that does not contain q. Using the induced orientations on $C \subset \partial M$
and $\partial H_{C,q}$, define the winding number of $f(C)$ around q by $w(f, C, q) = \pm 1$ de-
pending upon whether the diffeomorphism $f|C: C \to \partial H_{C,q}$ preserves or reverses
orientation.

Theorem 2.2   *Let $q \in S^2$ be a regular value of f which is not in $f(\partial M)$. Then*

$$\frac{1}{4\pi} \int_M f^*(dA) = \sum_{p \in f^{-1}(q)} \deg(f, p) + \frac{1}{2} \sum_{C \subset \partial M} w(f, C, q).$$

Proof   Cap each boundary component C by a disk D_C to produce an oriented closed
surface $\widehat{M} = M \cup \bigcup_{C \subset \partial M} D_C$. Extend f to a piecewise smooth map $\widehat{f}: \widehat{M} \to S^2$ so
that $\widehat{f}|D_C : D_C \cong H_{C,q}$. Here $\widehat{f}$ may fail to be smooth at points of ∂M. Note that
$\widehat{f}|D_C$ preserves orientation if and only if $f|C: C \to \partial H_{C,q}$ reverses orientation (and
vice versa). Thus $\widehat{f}|D_C$ preserves orientation if and only if $w(f, C, q) = -1$. Now by
degree theory,

$$\deg(\widehat{f}) = \frac{1}{4\pi} \int_{\widehat{M}} \widehat{f}^*(dA) = \frac{1}{4\pi} \int_M f^*(dA) - \frac{1}{4\pi} \sum_{C \subset \partial M} 2\pi w(f, C, q),$$

since $\widehat{f}$ maps each D_C diffeomorphically onto a hemisphere. On the other hand,

$$\deg(\widehat{f}) = \sum_{p \in \widehat{f}^{-1}(q)} \deg(\widehat{f}, p) = \sum_{p \in f^{-1}(q)} \deg(f, p).$$

Eliminating $\deg(\widehat{f})$ between the two equations gives the result.   ■

Corollary 2.3 *Let $\pm q \in S^2$ be antipodal regular values of f which are not in $f(\partial M)$. Then*

$$\frac{1}{4\pi} \int_M f^* (dA) = \frac{1}{2} \Big[\sum_{p \in f^{-1}(q)} \deg(f, p) + \sum_{p \in f^{-1}(-q)} \deg(f, p) \Big].$$

Proof Since $w(f, C, -q) = -w(f, C, q)$, the result follows by averaging the formulas in Theorem 2.2 with q and $-q$. ∎

Remark These formulas generalize to maps $f \colon M \to S^n$ of oriented compact n-manifolds M with spherical boundary which carry boundary components diffeomorphically onto great hyperspheres of S^n.

We shall henceforth define the *linking number* of a (parametrized) singular link $x \cup y$ with transverse intersections to be its Gauss linking integral $lk(x, y)$.

Given a singular link $x \cup y$ with transverse intersections, a combinatorial formula for the linking number $lk(x, y)$ from a diagram of the singular link can be obtained. By Theorem 2.1, we may assume that the diagram is the regular projection of the singular link $x \cup y$, which lies in the plane of projection except at crossings (not intersections) where one of x and y crosses over or under the other in a sufficiently small ϵ-neighborhood of the plane of projection. As $\epsilon \to 0$, we see that the only nonzero portion in the Gauss linking integral of $x \cup y$ are those around crossing points (note that the linking integral around an intersection point is 0 in this case), and by the proof of Theorem 2.1, each crossing gives rise to $\pm \frac{1}{2}$ according to whether it is a positive or negative crossing. This can also be obtained by Corollary 2.3 by taking $\pm q \in S^2$ to be vectors normal to the plane of projection. We therefore obtain an algorithm for computing $lk(x, y)$ combinatorially as follows: *In a given diagram of $x \cup y$, ignore intersections and assign $\pm \frac{1}{2}$ to every positive/negative crossing correspondingly, and then sum up these assigned values to obtain $lk(x, y)$.*

Now consider a parametrized embedding $x \colon S^1 \to R^3$. Its image is an oriented knot in R^3, and for convenience we shall still denote the image by x. Every oriented knot in R^3 is the image of such a parametric embedding. Two parametrizations of x give the same oriented knot in R^3 if there is an orientation-preserving diffeomorphism on S^1 taking one parametrization to the other. For oriented knots x and y in R^3 which intersect transversely, define the *linking number* of x and y to be the linking number of their representative parametric embeddings.

3 Seifert Matrix and Linking Number of Singular Links

The Alexander polynomial [A] of a knot K in S^3 was originally defined to be the determinant of a presentation matrix, called the Alexander matrix, of the first homology group of the universal abelian cover of the complement of K in S^3, considered as a $Z[t^{-1}, t]$-module. The polynomial is well defined up to a unit $\pm t^i$ in $Z[t^{-1}, t]$. Given a fixed orientation of S^3, if L is an oriented link in S^3 and F a (oriented and smooth) Seifert surface for L, then F has a product neighborhood $F \times [-1, 1]$ in S^3, where $F^{\pm} = F \times \{\pm 1\}$ corresponds to the positive/negative side of F in S^3 determined by

the orientations of F and S^3. Let $H_1(F) = \langle \alpha_1, \ldots, \alpha_m \rangle$, where each α_i is a smooth closed curve immersed in F, and for $i \neq j$, if α_i intersects α_j, then they intersect transversely. Then the Seifert matrix of L [S] (determined by the Seifert surface F and ordered basis $\{\alpha_1, \ldots, \alpha_m\}$ of $H_1(F)$) is the matrix $V = [v_{ij}]$, where $v_{ij} = lk(\alpha_i, \alpha_j^+)$ for $1 \leq i, j \leq m$, in which α_j^+ is $\alpha_j \subset F = F \times \{0\}$ pushed into F^+ so that $\alpha_i \cap \alpha_j^+ = \phi$. By the stable equivalence of Seifert surfaces for a given oriented link, and the S-equivalence of the corresponding Seifert matrices, the (Conway-normalized) Alexander polynomial $\Delta_L(t) \in Z[t^{\frac{1}{2}}, t^{\frac{-1}{2}}]$ defined by $\Delta_L(t) = \det(t^{\frac{1}{2}}V - t^{\frac{-1}{2}}V^T)$ is a well-defined isotopy invariant of L. See for example [L, M].

By Theorem 2.1, $lk(\alpha_i, \alpha_j)$ is a number in $Z[\frac{1}{2}]$ for $i \neq j$. We define $lk(\alpha_i, \alpha_i)$ to be $lk(\alpha_i, \widetilde{\alpha}_i)$, where $\widetilde{\alpha}_i$ is a parallel copy of α_i in the surface F. Thus we have $lk(\alpha_i, \alpha_i) = lk(\alpha_i, \alpha_i^+) = lk(\alpha_i, \alpha_i^-)$.

Lemma 3.1 $lk(\alpha_i, \alpha_j) = \frac{1}{2}\left[lk(\alpha_i, \alpha_j^+) + lk(\alpha_i, \alpha_j^-) \right]$ for $1 \leq i, j \leq m$.

Proof   By an extension of the argument used in proving Theorem 2.2, one can show that if the number of intersection points of α_i and α_j is n, then $lk(\alpha_i, \alpha_j) - lk(\alpha_i, \alpha_j^+) = \frac{k}{2}$, and $lk(\alpha_i, \alpha_j) - lk(\alpha_i, \alpha_j^-) = \frac{-k}{2}$ for some $-n \leq k \leq n$, and the result follows.   ∎

Theorem 3.2   *With the notations given in the above, the (i, j)-th entry of the Seifert matrix V is the linking number $lk(\alpha_i, \alpha_j)$ of a diagram of the singular link $\alpha_i \cup \alpha_j$, computed by ignoring all the intersection points of α_i and α_j and counting $\pm\frac{1}{2}$ at each positive/negative crossing correspondingly, plus the $Z[\frac{1}{2}]$-valued algebraic intersection number of α_i and α_j computed by counting $\pm\frac{1}{2}$ at each positive/negative intersection correspondingly.*

Proof   Lemma 3.1 gives rise to a symmetric bilinear form $lk \colon H_1(F) \times H_1(F) \to Z[\frac{1}{2}]$ which is represented by the matrix $A = [a_{ij}]$, where $a_{ij} = lk(\alpha_i, \alpha_j)$ for $1 \leq i, j \leq m$, with respect to the basis $\{\alpha_1, \ldots, \alpha_m\}$. Since $v_{ij} = lk(\alpha_j, \alpha_i^+) = lk(\alpha_i^-, \alpha_i) = lk(\alpha_i, \alpha_j^-)$, we have $a_{ij} = \frac{1}{2}(v_{ij} + v_{ji})$ for $1 \leq i, j \leq m$, hence $A = \frac{1}{2}(V + V^T)$. Recall the intersection form ι on $H_1(F)$ is a skew-symmetric bilinear form whose matrix representation with respect to the ordered basis $\{\alpha_1, \ldots, \alpha_m\}$ is $W = [w_{ij}]$, where $w_{ij} = \iota(\alpha_i, \alpha_j)$ for $1 \leq i, j \leq m$. The matrix W is related to the Seifert matrix V by $W = V - V^T$, see for example [R, p. 202]. Let $B = \frac{1}{2}W = \frac{1}{2}(V - V^T)$. The matrix B may be considered as the matrix representing the $Z[\frac{1}{2}]$-valued intersection form on $H_1(F)$ where the value assigned at each transverse intersection is $\pm\frac{1}{2}$ (with the same sign convention), instead of ± 1. It follows that $V = A + B$, and so the Seifert matrix V may be recovered from A and B (algebraically this is the well-known fact that every square matrix is the sum of a symmetric and a skew-symmetric matrix.)   ∎

Theorem 3.2 gives rise to a procedure of finding the Seifert matrix for an oriented link by using a fundamental system of curves in the Seifert surface for the link alone.

A   Appendix

To prove (2.1), first fix i. Without loss of generality we may assume $(s_i, t_i) = (0,0)$. Integration by parts implies that

$$x(s) = x(0) + x'(s)s - \int_0^s u x''(u)\, du \quad \text{and} \quad y(t) = y(0) + y'(t)t - \int_0^t v y''(v)\, dv.$$

Since $x(0) = y(0)$,

$$x'(s) \times y'(t) \cdot (x(s) - y(t)) = x'(s) \times y'(t) \cdot \left(x'(s)s - y'(t)t - \int_0^s u x''(u)\, du \right.$$

$$+ \int_0^t v y''(v)\, dv \Big)$$

$$= x'(s) \times y'(t) \cdot \left(- \int_0^s u x''(u)\, du + \int_0^t v y''(v)\, dv \right)$$

Thus $\| x'(s) \times y'(t) \cdot (x(s) - y(t)) \| \le K_1 (s^2 + t^2)$, where the constant K_1 can be taken to be $\frac{1}{2} \cdot \| x' \|_\infty \cdot \| y' \|_\infty \cdot \max\{ \| x'' \|_\infty, \| y'' \|_\infty \}$. Here $\| \cdot \|_\infty$ denotes the sup norm on S^1.

Now we may choose $\delta' > 0$ so that $x'(s)$, $y'(t)$, and w_i are linearly independent on the closure $\overline{N}_i(\delta')$ of $N_i(\delta')$. Let

$$K_2 = \frac{1}{2} \inf \left\{ \| u x'(s) + v y'(t) + w w_i \| : u^2 + v^2 + w^2 = 1, (s,t) \in \overline{N}_i(\delta') \right\}.$$

Then $K_2 > 0$. Let $\delta = \min\{ \delta', \sqrt{2} K_2 / c \}$, where $c = \max\{ \| x'' \|_\infty, \| y'' \|_\infty \}$. Then if $(s,t) \in N_i(\delta)$, we have

$$\| x(s) - y(t) - \epsilon \sigma_i w_i \| \ge \| x'(s)s - y'(t)t - \epsilon \sigma_i w_i \|$$

$$- \| - \int_0^s u x''(u)\, du + \int_0^t v y''(v)\, dv \|$$

$$\ge 2 K_2 \sqrt{s^2 + t^2 + \epsilon^2} - \frac{c}{2}(s^2 + t^2)$$

$$\ge \left(2 K_2 - \frac{c}{2} \sqrt{s^2 + t^2} \right) \sqrt{s^2 + t^2}$$

$$\ge K_2 \sqrt{s^2 + t^2}.$$

Thus with this δ, we have for $(s,t) \in N_i(\delta)$

$$|G_\epsilon^i(s,t)| \le \frac{\| x'(s) \times y'(t) \cdot (x(s) - y(t)) \|}{\| x(s) - y(t) - \epsilon \sigma_i w_i \|^3} \le \frac{K_1 (s^2 + t^2)}{K_2 (\sqrt{s^2 + t^2})^3} = \frac{K}{\sqrt{s^2 + t^2}},$$

where $K = K_1 / K_2$.

Inequality (2.1) holds uniformly for all i by taking the smallest of the δ's and the largest of K's.

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