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ON $\aleph_\alpha$-NOETHERIAN MODULES

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In this note we define two concepts which can be thought of as a generalization of noetherian concepts.

The main result is as follows (Corollary 4): *If R is a ring whose countably generated (left) ideals are (left) principal, then R is a (left) principal ideal ring.*

This result if obtained, more generally, for any (left) R-module and any regular cardinal $\aleph_\alpha$ (Corollary 1); a cardinal $\aleph_\alpha$ is regular whenever $W(\aleph_\alpha)=\{\text{ordinals } \gamma \mid \text{card } \gamma<\aleph_\alpha\}$ has no cofinal subset of cardinality less than $\aleph_\alpha$.

In the sequel, discrete valuation rings of finite rank (greater than 1) are shown to be genuinely $\aleph_0$-noetherian rings (this is one of the concepts herein introduced). Examples of genuinely $\aleph_\alpha$-noetherian rings (for any ordinal α) are also given.

$\aleph_\alpha$-Noetherian rings have some interest because of the results obtained by Jensen [2] who deals with a stronger concept, thus becoming able to draw important consequences about global and weak dimension of 'big' rings.

Let R be an arbitrary ring (not assumed to be commutative or to have a unity element) and let α be any ordinal.

DEFINITIONS. (i) A (left) R-module M is *$\aleph_\alpha$-generated* if it can be generated by a set of cardinality $\aleph_\alpha$; if, moreover, M cannot be generated by some set of cardinality less than $\aleph_\alpha$, it is said to be *strictly $\aleph_\alpha$-generated.*

(ii) A (left) R-module is *$\aleph_\alpha$-noetherian* if every submodule of M is $\aleph_\alpha$-generated; if, moreover, M has some strictly $\aleph_\alpha$-generated submodule, then it is called *genuinely $\aleph_\alpha$-noetherian.*

(iii) An *$\aleph_\alpha$-family* is a well-ordered strictly increasing family of submodules of a (left) R-module whose cardinality is $\aleph_\alpha$.

PROPOSITION 1. *Let M be a (left) R-module and let N be a strictly $\aleph_\alpha$-generated submodule of M; then, for every $\beta<\alpha$, there exists an $\aleph_\beta$-family $(N_\gamma)_{\gamma\in W(\aleph_\beta)}$ of submodules N_γ of M each contained in N and generated by less than $\aleph_\beta$ elements.*

Proof. We use transfinite induction to construct the desired $\aleph_\beta$-family. Given $\gamma\in W(\aleph_\beta)$, suppose a submodule $N_{\gamma'}$ of M has been obtained for every $\gamma'<\gamma$ such that $N_{\gamma'}$ can be generated by less than $\aleph_\beta$ elements and $(N_\gamma)_{\gamma'<\gamma}$ is a well-ordered strictly increasing family with $N_{\gamma'}\subset N$ for all $\gamma'<\gamma$. Clearly $\bigcup_{\gamma'<\gamma} N_{\gamma'}$ is properly contained in N (because otherwise N would be generated by $\bigcup_{\gamma'<\gamma} S_{\gamma'}=S$, where $S_{\gamma'}$ generates $N_{\gamma'}$, card $S_{\gamma'}<\aleph_\beta$; this is impossible since card $S<\aleph_\beta\aleph_\beta=\aleph_\beta$). Pick $x\in N$, $x\notin\bigcup_{\gamma'<\gamma} N_{\gamma'}$ and let N_x be the submodule of M generated by x; clearly,

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$N_\gamma = \bigcup_{\gamma' < \gamma} N_{\gamma'} + N_x$ can still be generated by less than $\aleph_\beta$ elements. The family $(N_\gamma)_{\gamma \in W(\aleph_\beta)}$ thus constructed is an $\aleph_\beta$-family since card $W(\aleph_\beta) = \aleph_\beta$.

COROLLARY 1. *Let M be a (left) R-module and $\aleph_\beta$ a regular cardinal; if M has no strictly $\aleph_\beta$-generated submodules then every submodule of M can be generated by less than $\aleph_\beta$ elements.*

Proof. Let N be a submodule of M; if N is generated by $\aleph_\beta$ elements, we are done. Thus, assume N is strictly $\aleph_\alpha$-generated for some $\alpha > \beta$. We apply the preceding proposition to get an $\aleph_\beta$-family $(N_\gamma)_{\gamma \in W(\aleph_\beta)}$ of submodules of M each contained in N. Clearly, $P = \bigcup_\gamma N_\gamma$ is a submodule of M contained in N; moreover, if S_γ is a set of generators for N_γ with card $S_\gamma < \aleph_\beta$, then P is generated by $S = \bigcup_{\gamma \in W(\aleph_\beta)} S_\gamma$ whose cardinality is at most $\aleph_\beta$. By hypothesis, P can be generated by less than $\aleph_\beta$ elements, say, $(x_i)_{i \in G}$ is a set of generators with card $G < \aleph_\beta$; for every $i \in G$, let γ_i be the smallest ordinal in $W(\aleph_\beta)$ such that $x_i \in N_{\gamma_i}$; then $P = \bigcup_{i \in G} N_{\gamma_i}$. On the other hand, the family $(\gamma_i)_{i \in G}$ has cardinality less than $\aleph_\beta$, hence it cannot be cofinal in $W(\aleph_\beta)$ since $\aleph_\beta$ is regular by assumption. This implies the existence of $\gamma' \in W(\aleph_\beta)$ such that $\gamma_i < \gamma'$ $\forall i \in G$, so $N_{\gamma_i} \subseteq N_{\gamma'}$ $\forall i \in G$. Thus $P = \bigcup_{i \in G} N_{\gamma_i} \subseteq N_{\gamma'}$, against the fact that $(N_\gamma)_{\gamma \in W(\aleph_\beta)}$ is strictly increasing.

REMARK. Corollary 1 applies whenever $\beta = 0$ or β is not a limit ordinal.

COROLLARY 2. *If M is a (left) R-module whose countably generated submodules are finitely generated, then M is (left) noetherian.*

Proof. Apply Corollary 1 with $\beta = 0$.

COROLLARY 3. *If M is a (left) R-module whose countably generated submodules are cyclic, then every submodule of M is cyclic; in particular, M is cyclic.*

Proof. By Corollary 2, M is noetherian; hence all its submodules are cyclic.

COROLLARY 4. *A ring whose countably generated (left) ideals are (left) principal is a (left) principal ideal ring.*

Examples of (commutative) genuinely $\aleph_\alpha$-noetherian rings abound as one may see from the following instances:

(1) Let $R = K[X_i]_{i \in A}$, where K is a finite field and card $A = \aleph_\alpha$.

Clearly, card $R = \aleph_\alpha$, so R is $\aleph_\alpha$-noetherian. Moreover, $(X_i)_{i \in A}$ is an ideal which cannot be generated by less than $\aleph_\alpha$ elements.

(2) Let $R = \prod_{i=1}^\infty K_i$, where $K_i = K(\forall i)$ is a countable field. Assuming the continuum hypothesis, card $R = \aleph^{\aleph_0} = \aleph_1$. On the other hand, as it is well known (cf. [3]), there is a bijection between the set of proper ideals of R and the set of filters of $\mathscr{P}(I)$, where I is the set of indices i. Precisely, if J is a proper ideal of R, then $F(J) = \{Z(f) \mid f \in J\}$ is a filter of $\mathscr{P}(I)$, where $Z(f) = \{i \in I \mid f(i) = 0\}$; conversely, if F is a filter of $\mathscr{P}(I)$, then $J(F) = \{f \in R \mid Z(f) \in F\}$ is a proper ideal of R.

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Now, let J be any nonprincipal maximal ideal of R; we show that J cannot be countably generated. For if $J=\sum_{n=0}^{\infty} Rf_n$, then $Z(f)\supseteq\bigcap_{n=0}^{m} Z(f_n)$ for every $f\in J$ and some $m\geq 0$. Thus, the collection $(Z(f_n))_{n\geq 0}$ would be a countable basis of the nonprincipal ultrafilter $F(J)$; however, this is impossible. Indeed, let U be any nonprincipal ultrafilter of $\mathscr{P}(I)$ and assume U has a countable basis $A_1, A_2, \ldots$. Clearly, $A_1, A_1, \cap A_2, A_1\cap A_2\cap A_3, \ldots$ is still a basis of U, so by dropping eventual repetitions in the chain $A_1\supseteq A_1\cap A_2\supseteq\cdots$, we may assume that U has a decreasing basis $A_1\supset A_2\supset\cdots$. Moreover, we may clearly assume that $\#(A_n\backslash A_{n+1})\geq 2$, $n=1,2,\ldots$. Let $a_n, b_n\in A_n\backslash A_{n+1}$, $a_n\neq b_n$ $(n=1,2,\ldots)$ and let $B=\{a_n, a_{n+1},\ldots\}$. Then $B_1\supset B_2\supset\ldots$. Let V be the filter generated by $B_1, B_2,\ldots$; clearly, $V\neq I$ since $\phi\notin V$. Also, $U\subseteq V$; indeed, if $X\in U$, then $A_n\subseteq X$ for some n, so $B_n\subseteq A_n\subseteq X$. On the other hand, $U\neq V$; indeed, $B_1\in V$, but $B_1\nsupseteq A_n$ $(n=1,2,\ldots)$ because $A_n\subseteq B_1\Rightarrow b_n=a_m$ for some $m\geq 1\Rightarrow n=m$ (since $b_n\in A_n\backslash A_{n+1}$) $\Rightarrow b_n=a_n$, against the assumption.

This is a contradiction since U is an ultrafilter (1).

Another important class of genuinely $\aleph_0$-noetherian rings is obtained as follows:

PROPOSITION 2. *Let R be a discrete valuation ring of finite rank; then all ideals of R are countably generated.*

Proof. We can assume that the value group of the valuation is $\Gamma=Z\times\cdots\times Z$ (lexicographically ordered). As it is well known (cf. [1]), there is a one-to-one correspondence (preserving inclusion) between the (integral) ideals of R and the upper classes of Γ contained in Γ^+; moreover, every upper class is the union of an increasing well ordered family of principal upper classes. Since Γ is countable, such a family must be countable; hence the result.

As a consequence, if R is a discrete valuation ring of finite rank greater than 1, then R is genuinely $\aleph_0$-noetherian.

It is conceivable that arbitrary valuation rings may be genuinely $\aleph_\alpha$-noetherian for some α depending only on the cardinality of the value group.

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(1) This proof was communicated to me by Professor G. Bruns.