Ergod. Th. & Dynam. Sys., page 1 of 31
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On the Hausdorff dimension of invariant measures of piecewise smooth circle homeomorphisms

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Abstract. We show that, generically, the unique invariant measure of a sufficiently regular piecewise smooth circle homeomorphism with irrational rotation number and zero mean nonlinearity (e.g. piecewise linear) has zero Hausdorff dimension. To encode this generic condition, we consider piecewise smooth homeomorphisms as *generalized interval exchange transformations* (GIETs) of the interval and rely on the notion of *combinatorial rotation number* for GIETs, which can be seen as an extension of the classical notion of rotation number for circle homeomorphisms to the GIET setting.

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1. *Introduction*

An *irrational circle homeomorphism* $f : \mathbb{T} \to \mathbb{T}$, that is, a continuous bijection on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with no periodic points, is uniquely ergodic [13]. Moreover, this map is topologically conjugated to an irrational rotation $x \mapsto x + \alpha$ on $\mathbb{T}$ if and only if it does not admit wandering intervals, the latter condition being guaranteed if, for example, the map is piecewise smooth and its derivative has bounded variation. In this case, its unique invariant probability measure μ_f can be expressed as the pushforward $\mu_f = h_*\mathrm{Leb}$ of the Lebesgue measure on $\mathbb{T}$ by the conjugacy map $h : \mathbb{T} \to \mathbb{T}$.

Many recent works have aimed to understand more deeply this unique invariant probability measure under different assumptions on the map. Let us point out that the fine statistical properties of μ_f are closely related to the (lack of) regularity of h, but, in general, it is not possible to study the conjugacy map directly to understand dimensional properties of the invariant measure. Nevertheless, the existence of this conjugacy allows the transport of valuable techniques, such as *dynamical partitions* (see §2.1.2), from rigid rotations to more general maps, and to use these tools to study geometric properties of the associated invariant measures.

Of particular interest to us will be the notion of *Hausdorff dimension* of a probability measure μ on $\mathbb{T}$, defined as

$$\dim_H(\mu) := \inf\{\dim_H(X) \mid \mu(X) = 1\},$$

where $\dim_H(X)$ denotes the Hausdorff dimension of the set X (see §2.4). Intuitively, this is a way to assess the 'size' of the set where the measure is concentrated whenever the measure is singular with respect to Lebesgue.

For circle diffeomorphisms, it follows from the works of Herman [16] and Yoccoz [30] that sufficiently regular circle diffeomorphisms are smoothly conjugated to a rigid rotation provided its rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is *Diophantine*, that is, if it verifies

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\gamma}{q^\tau} \quad \text{for all } p, q \in \mathbb{Z},\ q \neq 0,$$

for some $\gamma, \tau > 0$. Hence, for any smooth circle diffeomorphism with a Diophantine rotation number, its unique invariant measure is equivalent to the Lebesgue measure and, therefore, its Hausdorff dimension is equal to one. However, for any $0 \leq \beta \leq 1$ and any *Liouville* number α, that is, any non-Diophantine irrational number, Sadovskaya [26] constructed examples of smooth diffeomorphisms with rotation number α whose unique invariant measure has Hausdorff dimension β. In the analytic category, V. Arnold showed the existence of analytic circle maps with Liouville rotation number whose conjugacy to the circle rotation is non-differentiable.

As for *critical circle maps with power-law criticalities*, that is, smooth diffeomorphisms with a finite number of singular points where the derivative vanishes and where the map behaves as $x \mapsto x|x|^p + c$ in the neighborhood of the critical point (in a suitable coordinate system) for some $p > 0$ and $c \in \mathbb{R}$ which may depend on the critical point, Khanin [20] proved that the unique invariant measure of any sufficiently regular irrational critical circle map is singular with respect to the Lebesgue measure. If, in addition, the rotation number of the map is of bounded type, Graczyk and Świątek [15] showed that the Hausdorff dimension of the unique invariant measure is bounded away from 0 and 1. More recently, the author [27] provided explicit bounds, depending only on the arithmetic properties of the rotation number, for the Hausdorff dimension of these maps.

In this work, we study the unique invariant probability measures of certain piecewise smooth circle homeomorphisms known as *P-homeomorphisms* or *circle diffeomorphisms with breaks*. These are smooth orientation-preserving homeomorphisms, differentiable away from countable many points, so-called *break points*, at which left and right derivatives exist but do not coincide, and such that $\log Df$ has bounded variation.

For P-homeomorphisms with exactly one break point and irrational rotation number, Dzhalilov and Khanin [11] showed that the associated invariant probability measure is singular with respect to Lebesgue. The case of two break points has been studied by Dzahlilov and Liousse [9] in the bounded rotation number case, and by Dzahlilov, Liousse, and Mayer [10] for any irrational rotation numbers. In both works, the authors conclude the singularity of the associated invariant probability measure.

More recently, for P-homeomorphisms of class $C^{2+\epsilon}$ with a finite number of break points and non-zero *mean nonlinearity*, Khanin and Kocić [19] showed that the Hausdorff dimension of their unique invariant measure is equal to 0, provided that their rotation number belongs to a specific (explicit) full-measure set of irrational numbers. In the same work, the authors show that this result cannot be extended to all irrational rotation numbers. Recall that the mean nonlinearity of a piecewise C^2 circle homeomorphism f such that $D \log Df \in L^1$ is given by

$$\mathcal{N}(f) = \int_{\mathbb{T}} D \log Df(x)\, dx,$$

where, as an abuse of notation, we denote by Df the derivative of any lift $F : \mathbb{R} \to \mathbb{R}$ of f.

The aim of this work is to show that for 'typical' P-homeomorphisms without periodic points and zero mean nonlinearity, the conclusions in [19] remain valid, that is, the associated unique invariant probability measure has zero Hausdorff dimension. For the sake of clarity, we postpone the exact formulation of this result to §3, but let us mention that the meaning of 'typical' in the previous statement relies on the notion of *combinatorial rotation number* (see §2.2.3), which is widely used in the theory of *interval exchange transformations* (see §2.2) and can be seen as an extension of the classical notion of rotation number for circle maps. Throughout this work, a *typical* map will stand for a map having a typical combinatorial rotation number.

We stress the fact that the non-zero mean nonlinearity assumption in [19] plays a crucial role in their proof, which relies on renormalization techniques for circle maps, as the behavior of successive renormalizations of these transformations in this class is well understood (they approach in a very specific way the space of Möbius maps, see [18]). However, the behavior for renormalizations of maps with zero mean nonlinearity is very different (they approach the space of piecewise affine homeomorphisms, see [14]), and thus a different approach is required. A more in-depth discussion about this is given at the beginning of §3.

A recent result by Berk and the author [3] shows that a typical sufficiently smooth P-homeomorphism without periodic points and zero mean nonlinearity is smoothly conjugated to a piecewise linear P-homeomorphism, or *PL-homeomorphism* for short. Thus, to prove our main result (Theorem 3.1) concerning the Hausdorff dimension of typical P-homeomorphisms with zero mean nonlinearity, it suffices to consider the piecewise linear case (Theorem 3.2), since diffeomorphisms preserve the Hausdorff dimension of the unique invariant measure.

The invariant measures of PL-homeomorphisms were first studied by Herman [16], who showed that a PL-homeomorphism with exactly two break points and irrational rotation number has an invariant measure absolutely continuous with respect to Lebesgue if and only if its break points lie on the same orbit. More generally, Liousse [21] showed that the invariant measure of a generic PL-homeomorphism with a finite number of break points and irrational rotation number of bounded type is singular with respect to Lebesgue. The generic condition in [21] is explicit and appears as an arithmetic condition on the logarithm of the slopes of the PL-homeomorphism.

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Theorem 3.2 provides an improvement of the results in [16, 21] under a stronger generic condition (namely, typical combinatorial rotation number) by showing that the associated invariant measures are not only singular but have zero Hausdorff dimension.

To prove our main results, we consider P-homeomorphisms as *generalized interval exchange transformations*, or GIETs for short (see §2.2.3). We provide a canonical way to make this identification in §2.3. The renormalization for these maps, known as *Rauzy–Veech renormalization*, allows for finer control than those for circle maps. In fact, when seen as a circle map, the successive renormalizations of a P-homeomorphism appear as a subsequence of the successive renormalizations of the same map when considered a GIET. Let us mention that this approach has been successfully used to study renormalizations and rigidity properties for an exceptional class of P-homeomorphisms in [6–8], and to study rigidity properties of P-homeomorphisms with zero mean nonlinearity in [3].

We will obtain a good control for the renormalizations of PL-homeomorphisms, along a subsequence, by using ergodicity properties of an accelerated version of the Rauzy–Veech renormalization, known as *Zorich map*, together with a 'Borel–Cantelli lemma' for the Zorich map, due to Aimino, Nicol, and Todd [1, Theorem 2.18]. The desired properties will hold for a full-measure set of combinatorial rotation numbers.

We finish this introduction with a brief outline of the article. Section 2 will introduce the core notions used throughout this work and recall some well-known facts concerning circle maps, IETs, and P-homeomorphisms. In §3, we state our main results (Theorems 3.1 and 3.2) and discuss the strategy of proof, which relies on a general criterion for zero Hausdorff dimension using Rohlin towers (Proposition 3.3). The proof of this criterion is given in §4. Finally, in §5, we prove our main result by using renormalization techniques for GIETs to build appropriate Rohlin towers, fulfilling the hypotheses of the criterion.

2. *Preliminaries*

2.1. *Circle maps.*

Let us quickly recall some of the properties of circle maps that will be used throughout this work.

2.1.1. *Rotation number.*

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving circle homeomorphism and let $F : \mathbb{R} \to \mathbb{R}$ be a *lift* of *f*, that is, a continuous homeomorphism of $\mathbb{R}$ such that $F(x+1) = F(x) + 1$ and $F(x) (\text{mod } 1) = f(x)$ for all $x \in \mathbb{R}$. By a classical result of Poincaré, the limit

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \mod 1$$

is well defined and independent of the value $x \in \mathbb{R}$ initially chosen. This limit is called the *rotation number* of *f*. By Poincaré's classification theorem for circle maps, any minimal orientation-preserving circle homeomorphism with irrational rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is conjugate to the *rigid rotation* $R_\alpha : \mathbb{T} \to \mathbb{T}$ given by $R_\alpha(x) = x + \alpha$ for any $x \in \mathbb{T}$. In particular, such a map is uniquely ergodic.

2.1.2. *Dynamical partitions.*

Given a circle homeomorphism $f : \mathbb{T} \to \mathbb{T}$ topologically conjugated to an irrational rotation R_α, a classical way to study fine statistical properties

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of its unique invariant measure μ_f is to consider the so-called *dynamical partitions* of f, which are defined through the denominators $(q_n)_{n\geq 0}$ of the *convergents*

$$\frac{p_n}{q_n} = [a_1, a_2, a_3, \ldots, a_n]$$

associated to the *continued fraction* of α

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}} = [a_1, a_2, a_3, \ldots],$$

as follows. Given $x_0 \in \mathbb{T}$ and $n \geq 1$, the nth *dynamical partition* $\mathcal{P}_n(x_0)$ of f, with base point x_0, is given by

$$\mathcal{P}_n(x_0) = \{I_{n-1}^0(x_0), \ldots, I_{n-1}^{q_n-1}(x_0)\} \cup \{I_n^0(x_0), \ldots, I_n^{q_{n-1}-1}(x_0)\},$$

where $I_m^i(x_0) = f^i(I_m(x_0))$ is the ith iterate of the circle arc given by

$$I_m(x_0) = \begin{cases} [x_0, f^{q_m}(x_0)) & \text{if } m \text{ is even,} \\ [f^{q_m}(x_0), x_0) & \text{if } m \text{ is odd.} \end{cases}$$

These partitions form a refining sequence. In fact, it is easy to see that $I_{n+1}(x_0) \subseteq I_{n-1}(x_0)$ for any $n \geq 1$. Moreover, they verify

$$I_{n-1}^i(x_0) \setminus I_{n+1}^i(x_0) = \bigcup_{j=0}^{a_{n+1}-1} I_n^{i+q_{n-1}+jq_n}(x_0) \tag{1}$$

for all $0 \leq i < q_n$ and all $n \geq 1$. Notice that the right-hand side of equation (1) is a disjoint union of a_{n+1} different iterates of I_n. Furthermore, each iterate is *adjacent* to the next one when seen as arcs in the circle; that is, they share a common endpoint.

Geometric properties of these partitions are closely related to dimensional properties of the subjacent invariant measure, see, for example, [**19**, **20**, **27**].

2.1.3. *Renormalization.* The renormalization maps of f are closely related to the dynamical partitions and are often used to study their geometric properties. The *nth renormalization of f*, with base point x_0, is the map $f_n : I_{n-1}(x_0) \cup I_n(x_0) \longrightarrow I_{n-1}(x_0) \cup I_n(x_0)$ given by

$$f_n(x) = \begin{cases} f^{q_{n-1}}(x) & \text{if } x \in I_n(x_0), \\ f^{q_n}(x) & \text{if } x \in I_{n-1}(x_0). \end{cases}$$

It is not difficult to check that the map f_n is the first return map of f to $I_{n-1}(x_0) \cup I_n(x_0)$.

2.2. *Interval exchange transformations.* Let $I = [0, 1)$ be the unit interval. An *interval exchange transformation* (IET) is a bijective, right-continuous function $T : I \longrightarrow I$, with a finite number of discontinuities, whose restriction to any subinterval of continuity is given by a translation. We say that T is an *IET on d intervals* if there exists a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, where the indexes belong to some finite alphabet $\mathcal{A}$ with $d \geq 2$ symbols, such that T is continuous when restricted to I_α for each $\alpha \in \mathcal{A}$. Notice that T is simply exchanging the order of the intervals in the partition.

An IET T of d intervals can be encoded by a pair (λ, π) corresponding to a *combinatorial datum* $\pi = (\pi_0, \pi_1)$, consisting of two bijections $\pi_0, \pi_1 : \mathcal{A} \to \{1, \ldots, d\}$ describing the order of the intervals before and after T is applied (the numbers 0 and 1 in the previous notation are usually referred to *top* and *bottom*, see §2.2.1), and a *lengths vector* $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}}$ in the simplex $\Delta_d = \{\nu \in \mathbb{R}_+^{\mathcal{A}} \mid \sum_{\alpha\in\mathcal{A}} \nu_\alpha = 1\}$, which corresponds to the lengths of the intervals in the partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ associated to T. We call $\pi_1 \circ \pi_0^{-1} : \{1, \ldots, d\} \to \{1, \ldots, d\}$ the *monodromy invariant* of π.

A combinatorial datum $\pi = (\pi_0, \pi_1)$ is said to be of *rotation type* if its monodromy invariant verifies

$$\pi_1 \circ \pi_0^{-1}(i) - 1 = i + k \ (\mathrm{mod}\ d)$$

for some $k \in \{0, \ldots, d-1\}$ and all $i \in \{1, \ldots, d\}$. Similarly, we say that an IET is of *rotation type* if its combinatorial datum is of rotation type. Notice that any IET of rotation type induces a well-defined circle rotation on the circle $\mathbb{T}$.

Given an IET T with associated partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, we can obtain an explicit expression for the intervals I_α as $[l_\alpha, r_\alpha)$, where

$$l_\alpha = \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta, \quad r_\alpha = l_\alpha + \lambda_\alpha$$

for any $\alpha \in \mathcal{A}$. Notice that $\{l_\alpha\}_{\pi_0(\alpha)\neq 1}$ are the only possible discontinuity points of T. With this notation,

$$T(x) = x + w_\alpha$$

for any $x \in I_\alpha$ and any $\alpha \in \mathcal{A}$, where

$$w_\alpha = \sum_{\pi_1(\beta)<\pi_1(\alpha)} \lambda_\beta - \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta.$$

We denote by $w = (w_\alpha)_{\alpha\in\mathcal{A}}$ the *translation vector* of the IET T. Notice that w_α can be expressed as a linear transformation on $\mathbb{R}^{\mathcal{A}}$ as $w_\alpha = \Omega_\pi(\lambda)$, where $\Omega_\pi : \mathbb{R}^A \to \mathbb{R}^A$ is given by

$$\Omega_{\alpha,\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta) \text{ and } \pi_0(\alpha) < \pi_0(\beta), \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta) \text{ and } \pi_0(\alpha) > \pi_0(\beta), \\ 0 & \text{in other cases.} \end{cases} \tag{2}$$

We say that the pair $\pi = (\pi_0, \pi_1)$ is *reducible* if there exists $1 \leq k < d$ such that

$$\pi_1 \circ \pi_0^{-1}(\{1, \ldots, k\}) = \{1, \ldots, k\}.$$

Otherwise, it is said to be *irreducible*. An IET is said to be reducible (respectively irreducible) if the associated combinatorial datum π is reducible (respectively irreducible). We say that an IET T satisfies the *Keane condition* if $T^m_{(\pi,\lambda)}(l_\alpha) \neq l_\beta$ for all $m \geq 1$ and

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all $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$. In particular, any IET verifying the previous condition is irreducible. Recall that a transformation on a metric space is said to be *minimal* if the orbit of all points is dense. By [17], any IET satisfying Keane's condition is minimal.

2.2.1. *Rauzy–Veech renormalization.* Let $T = (\lambda, \pi)$ be an IET on d intervals. Denote

$$\alpha_\epsilon = \pi_\epsilon^{-1}(d) \tag{3}$$

for $\epsilon = 0, 1$. The letters α_0 and α_1 correspond to the 'last' intervals (that is, those having 1 as their right endpoint) in the partitions $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{T(I_\alpha)\}_{\alpha\in\mathcal{A}}$, respectively.

If $\lambda_{\alpha_0} \neq \lambda_{\alpha_1}$, by comparing the lengths of these intervals, we define the *type* of T as

$$\epsilon(\lambda, \pi) = \begin{cases} 0 & \text{if } \lambda_{\alpha_0} > \lambda_{\alpha_1}, \\ 1 & \text{if } \lambda_{\alpha_0} < \lambda_{\alpha_1}. \end{cases}$$

The longest of these two intervals is sometimes referred to as the *winner* and the shortest as the *loser*. Notice that $\alpha_{\epsilon(\lambda,\pi)}$ and $\alpha_{1-\epsilon(\lambda,\pi)}$ correspond to the symbols of the winner and the loser intervals, respectively. If there is no risk of confusion, we will denote these letters simply by α_ϵ and $\alpha_{1-\epsilon}$. We will sometimes refer to types 0 and 1 as *top* and *bottom*, respectively.

The *Rauzy–Veech induction* of T, which we denote by $\widehat{T}$, is defined as the first return map of T to the subinterval

$$\widehat{I} = \begin{cases} I \setminus T(I_{\alpha_1}) & \text{if } T \text{ is of type top}, \\ I \setminus I_{\alpha_0} & \text{if } T \text{ is of type bottom}. \end{cases}$$

We define the *Rauzy–Veech renormalization* of T, which we denote by $\mathcal{R}(T)$, by rescaling linearly $\widehat{T}$ to the interval I. The renormalized map $\mathcal{R}(T)$ is an IET with the same number of subintervals as T. This induction/renormalization procedure can be iterated infinitely many times if and only if T verifies the Keane condition. Also, it follows quickly from the definition that each infinitely renormalizable pair (λ, π) will admit exactly two preimages. We refer the interested reader to [29] for a proof of these facts.

We denote by $\mathfrak{G}_d$ the *set of combinatorial data* $\pi = (\pi_0, \pi_1)$ with d symbols and let $\mathfrak{G}_d^0 \subseteq \mathfrak{G}_d$ be the subset of *irreducible combinatorial data*, which we equip with the counting probability measure $d\pi$. We will sometimes refer to a combinatorial datum in $\mathfrak{G}_d$ simply as a *permutation*. We denote by X_d the set of IETs verifying Keane's condition.

Let us point out that for a fixed $\pi \in \mathfrak{G}_d^0$ and for any $\lambda \in \Delta_d$ such that (λ, π) is renormalizable, there are only two possibilities for the combinatorial datum of $\mathcal{R}(\lambda, \pi)$, depending on whether the IET is of top or bottom type. For notational simplicity, for any $\pi \in \mathfrak{G}_d^0$ and for any $\epsilon \in \{0, 1\}$, we let $\Delta_\pi = X_d \cap (\Delta_d \times \{\pi\})$ and denote by $\Delta_{\pi,\epsilon}$ the set of IETs in Δ_π of type ϵ.

Notice that the set X_d of pairs (λ, π) verifying Keane's condition is $\mathcal{R}$-invariant and of full measure in $\Delta_d \times \mathfrak{G}_d^0$, with respect to Leb $\times\, d\pi$. Hence,

$$\mathcal{R} : X_d \to X_d$$

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is a well-defined 2-to-1 map defined in a full measure subset $X_d \subseteq \Delta_d \times \mathfrak{S}_d^0$. In fact, it is easy to see that any $(\lambda, \pi) \in X_d$ has exactly two preimages, one of type top and one of type bottom. Moreover, for any $\pi \in \mathfrak{S}_d^0$ and for any $\epsilon \in \{0, 1\}$, the map $\mathcal{R}\,|_{\Delta_{\pi,\epsilon}} \colon \Delta_{\pi,\epsilon} \to \Delta_{\pi(\epsilon)}$ is bijective, where we denote by $\pi(\epsilon)$ the permutation obtained from π after a Rauzy–Veech renormalization of type ϵ.

It was shown independently by Masur [25] and Veech [28] that $\mathcal{R}$ admits an infinite invariant measure $\mu_{\mathcal{R}}$, absolutely continuous with respect to $\mathrm{Leb} \times d\pi$. Moreover, the measure $\mu_{\mathcal{R}}$ is unique up to product by a scalar.

2.2.2. *Rauzy classes.* Given $\pi, \pi' \in \mathfrak{S}_d$, the permutation π' is said to be a *successor* of π if there exist $\lambda, \lambda' \in \Delta_d$ such that $(\lambda', \pi') = \mathcal{R}(\lambda, \pi)$. We denote this relation by $\pi \to \pi'$. Notice that any successor of an irreducible permutation is also irreducible.

The relation '$\to$' defines an oriented graph structure on the set of irreducible permutations $\mathfrak{S}_d^0$. We call *Rauzy classes* the connected components of the oriented graph $\mathfrak{S}_d^0$ with respect to the successor relation. It follows from the discussion in the previous section that any irreducible permutation π has at least one and at most two successors, namely $\pi(0), \pi(1) \in \mathfrak{S}_d^0$, and that it is itself the successor of at least one and at most two permutations $\pi^0, \pi^1 \in \mathfrak{S}_d^0$ satisfying $\pi^0(0) = \pi = \pi^1(1)$.

Let us point out that if π, π' belong to the same Rauzy class, then an oriented path exists in $\mathfrak{S}_d^0$ from π to π'. For combinatorial data of rotation type, we have the following.

PROPOSITION 2.1. *For any $d \geq 2$, the permutations of rotation type belong to the same Rauzy class in $\mathfrak{S}_d^0$.*

2.2.3. *GIETs and combinatorial rotation number.* A *generalized interval exchange transformation* (GIET) is a piecewise smooth bijective, right-continuous function $f \colon I \to I$ with a finite number of discontinuities, whose derivative is non-negative and extends to the closure of any subinterval where the function is smooth. Similar to the case of IETs, to any GIET f, we can associate a partition $\{I_\alpha(f)\}_{\alpha \in \mathcal{A}}$ of I such that, for every $\alpha \in \mathcal{A}$, the restriction $f\,|_{I_\alpha} \colon I_\alpha \to f(I_\alpha)$ is smooth, as well as a permutation π describing the order in which these intervals are exchanged.

Rauzy–Veech renormalization and Keane's condition, initially defined only for IETs, extend trivially to GIETs. Given a GIET f with associated permutation π and exchanged intervals $\{I_\alpha(f)\}_{\alpha \in \mathcal{A}}$, we denote by $\mathcal{R}(f)$ its Rauzy–Veech renormalization whenever it is well defined, that is, if $|I_{\alpha_0}| \neq |f(I_{\alpha_1})|$, where α_0, α_1 are given by equation (3) and $|\cdot|$ denotes the length of an interval. We say that f is *infinitely renormalizable* if and only if $\mathcal{R}^n(f)$ is well defined for all $n \in \mathbb{N}$. Similar to the IET setting, if f verifies Keane's condition, then it is infinitely renormalizable. An infinitely renormalizable GIET f defines a unique path $\gamma(f)$ on the Rauzy diagram, which we call the *combinatorial rotation number* or simply the *rotation number* of f. The notion of rotation number for GIETs is now classical and goes back to the works of Marmi, Moussa, and Yoccoz [23, 24]. Let us point out that the combinatorial rotation number is a topological invariant for GIETs.

An infinite path on the Rauzy diagram is called ∞-*complete* if each letter in $\mathcal{A}$ wins infinitely many times. We say that a GIET is *irrational* if it is infinitely renormalizable and its rotation number is ∞-complete.

Let us point out that in the case of GIETs, the role played by IETs and combinatorial rotation numbers are analogous to that of rigid rotations and rotation numbers in the case of circle homeomorphisms. Indeed, an infinite path in a Rauzy diagram is associated with some infinitely renormalizable IET if and only if it is ∞-complete. Also, two infinitely renormalizable IETs are conjugated if and only if they have the same rotation number. Moreover, any irrational GIET is semi-conjugated to a unique IET with the same rotation number. Analogously to the circle case, this semi-conjugacy is a conjugacy if the GIET does not admit wandering intervals. We refer the interested reader to [31] for proof of these facts.

Given the previous discussion, we can consider the combinatorial rotation number for irrational GIETs on d intervals as taking values on $\Delta_d \times \mathfrak{G}_d^0$. This will allow us to speak in the following of *almost every combinatorial rotation number* for GIETs. As an abuse of notation, we will sometimes write $\gamma(T) = (\lambda, \pi)$ to state that a GIET T and a standard IET T_0, associated with some $(\lambda, \pi) \in X_d$, have the same rotation number.

2.2.4. *The lengths cocycle.* Given $T = (\lambda, \pi)$ such that $\lambda_{\alpha_0} \neq \lambda_{\alpha_1}$, we define the *Rauzy–Veech matrix* $A(T) : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ associated to T as

$$A(T) = I_{\mathcal{A}} + E_{\alpha_\epsilon, \alpha_{1-\epsilon}}, \tag{4}$$

where $I_{\mathcal{A}}$ denotes the identity matrix on $\mathbb{R}^{\mathcal{A}}$ and $E_{\alpha,\beta}$ is the matrix whose entries are 1 at the position (α, β) and 0 otherwise. Notice that $\det(A(T)) = 1$.

The Rauzy–Veech matrices depend only on the IET's combinatorial datum and type. For any $\pi \in \mathfrak{G}_d^0$ and any $\epsilon \in \{0, 1\}$, we denote

$$A_{\pi,\epsilon} = I_{\mathcal{A}} + E_{\pi_\epsilon^{-1}(d), \pi_{1-\epsilon}^{-1}(d)}. \tag{5}$$

Given $T = (\lambda, \pi)$ verifying Keane's condition, denote

$$A^n(T) = A(T) \cdots A(\mathcal{R}^{n-1}(T))$$

for any $n \geq 0$. Then the lengths vector of $\mathcal{R}^n(T)$ is given by

$$\frac{A^n(T)^{-1}\lambda}{|A^n(T)^{-1}\lambda|_1}.$$

The map

$$\begin{array}{rccc} A^{-1} : & X_d & \to & SL(d, \mathbb{Z}) \\ & T & \mapsto & A(T)^{-1} \end{array}$$

is a cocycle over $\mathcal{R}$, known as the *Rauzy–Veech cocycle* or *lengths cocycle*.

The following observation will be of fundamental importance. For a proof, see [32, Proposition 7.6].

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PROPOSITION 2.2. *The set*

$$\bigcup_{\pi \in \mathfrak{G}_d^0} \Delta_\pi \times \mathrm{Ker}(\Omega_\pi),$$

where $\Omega_\pi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ *is given by equation (2), is invariant by the lengths cocycle*

$$\begin{array}{rccc} \mathcal{L}: & X_d \times \mathbb{R}^{\mathcal{A}} & \to & X_d \times \mathbb{R}^{\mathcal{A}} \\ & (T, v) & \mapsto & (\mathcal{R}(T), A(T)^{-1} v) \end{array}$$

and its action is trivial on it. More precisely, one can choose a basis of row vectors of $\mathrm{Ker}(\Omega_\pi)$ *for every* $\pi \in \mathfrak{G}_d^0$*, such that for any* $(\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ *infinitely renormalizable and for any* $n \in \mathbb{N}$*, the transformation*

$$A^n(\lambda, \pi)^{-1}|_{\mathrm{Ker}(\Omega_\pi)} : \mathrm{Ker}(\Omega_\pi) \to \mathrm{Ker}(\Omega_{\pi'})$$

is the identity with respect to the selected bases, where $\pi' \in \mathfrak{G}_d^0$ *is such that* $\mathcal{R}^n(\lambda, \pi) \in \Delta_{\pi'}$*. In particular, if* $\pi' = \pi$*, then* $A^n(\lambda, \pi)^{-1}$ *acts as the identity on* $\mathrm{Ker}(\Omega_\pi)$.

2.2.5. *The heights cocycle.* Let $T = (\lambda, \pi)$ satisfying Keane's condition. Using the Rauzy–Veech matrix $A(T)$ given by equation (4), we define a cocycle over $\mathcal{R}$, known as the *heights cocycle*, by

$$\begin{array}{rccc} A^T : & X_d & \to & SL(d, \mathbb{Z}) \\ & T & \mapsto & A(T)^T \end{array}.$$

This cocycle allows us to describe the return times, or *heights*, if we think of the induced map as a system of Rohlin towers associated with the iterates of $\mathcal{R}$. Indeed, given $n \in \mathbb{N}$, the transformation $\mathcal{R}^n(T)$ is defined as the linear rescaling of the first return map of T to some subinterval $I^n \subseteq I$. Moreover, this interval admits a decomposition $I^n = \bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^n$ such that the return time to I^n on each subinterval I_α^n is constant. The vector of return times to I^n is given by

$$h^n = A^n(T)^T \overline{1},$$

where $\overline{1} \in \mathbb{N}^{\mathcal{A}}$ is the vector whose entries are all equal to 1. We refer to the d distinct Rohlin towers $\{I_\alpha^n, T(I_\alpha^n), \ldots, T^{h_\alpha^n - 1}(I_\alpha^n)\}$, with $\alpha \in \mathcal{A}$, as *Rauzy–Veech towers*.

2.2.6. *Affine IETs.* An *affine interval exchange transformation* (AIET) is a GIET for which the restriction to each subinterval of continuity is a linear map. As for IETs, given an AIET f, we can decompose the interval I into intervals of continuity of f, which we denote by $\{I_\alpha\}_{\alpha \in \mathcal{A}}$, where $\mathcal{A}$ is a finite alphabet. Notice that f will change the order of these intervals and linearly modify their lengths. If the permutation associated with f is in the Rauzy class of rotations, we say that f is of *rotation type*.

Given an AIET f on d intervals with associated partition $\{I_\alpha\}_{\alpha \in \mathcal{A}}$, we define its *log-slope* as the logarithm of the slope of f in each interval of continuity, namely, the vector $\omega = (\log Df|_{I_\alpha})_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$.

The following relation between the log-slope of Rauzy–Veech renormalizations of an AIET and the heights cocycle will play a crucial role in this work.

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PROPOSITION 2.3. *Let f be an irrational AIET on d intervals with combinatorial rotation number $\gamma(f) = (\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ and log-slope $\omega \in \mathbb{R}^{\mathcal{A}}$. Then, the log-slope of $\mathcal{R}^n(f)$ is given by $A^n(\lambda, \pi)^T \omega$ for any $n \geq 0$.*

Let us point out that, by a result of Camelier and Gutierrez [4, Lemma 3.3], for almost every (a.e.) (λ, π), and any AIET f with $\gamma(f) = (\lambda, \pi)$ and log-slope $\omega \in \mathbb{R}^{\mathcal{A}}$, we have $\langle \omega, \lambda \rangle = 0$.

2.2.7. *Zorich acceleration.* Zorich [33] showed that one can 'accelerate the dynamics' of $\mathcal{R}$ to define a map $\mathcal{Z} : X_d \to X_d$ admitting an invariant probability measure $\mu_{\mathcal{Z}}$ which is absolutely continuous with respect to the Leb $\times\, d\pi$ and whose density is a rational function on Δ_d uniformly bounded away from the boundary of Δ_d (see, e.g. [29, Proposition 21.4]). For every Rauzy class $\mathfrak{R}$, the restriction to the $\mathcal{Z}$-invariant set $\bigcup_{\pi \in \mathfrak{R}} \Delta_\pi$ is ergodic. This map is given by

$$\mathcal{Z}(T) = \mathcal{R}^{z(T)}(T),$$

where $z(T)$ is the smallest $n > 0$ such that $\mathcal{R}^{n-1}(T)$ and $\mathcal{R}^n(T)$ have different type.

Similarly, using $z : X_d \to \mathbb{N}$ as the *accelerating map*, we define the *accelerated lengths and heights cocycles*

$$B^{-1} : X_d \to SL(d, \mathbb{Z}), \quad B^T : X_d \to SL(d, \mathbb{Z}),$$

by setting

$$B^{-1}(T) = A^{z(T)}(T)^{-1}, \quad B^T(T) = A^{z(T)}(T)^T.$$

As before, these cocycles are related to the transformation of lengths and heights under the action of $\mathcal{Z}$. Moreover, these cocycles are integrable with respect to the invariant probability measure $\mu_{\mathcal{Z}}$.

2.2.8. *Notation for iterates.* In the following, given $T = (\lambda, \pi) \in X_d$, we denote by

$$(\lambda^{(n)}, \pi^{(n)}) = \mathcal{Z}^n(T)$$

its orbit under $\mathcal{Z}$ and by $T^{(n)}$ the map associated to $(\lambda^{(n)}, \pi^{(n)})$. We denote the type of $\mathcal{Z}^n(T)$ by $\epsilon^{(n)}$, and the letters associated with its winner and loser intervals by $\alpha^{(n)}$ and $\beta^{(n)}$, respectively. We denote by $I^{(n)}$ the subinterval $I^{(n)} \subseteq I$, given by the Rauzy–Veech algorithm, such that $T^{(n)}$ coincides with the linear rescaling of T when induced to $I^{(n)}$. We denote its associated decomposition by $\{I_\alpha^{(n)}\}_{\alpha \in \mathcal{A}}$. We denote by $h_\alpha^{(n)}$ the return time of T to $I^{(n)}$ for any $x \in I_\alpha^{(n)}$. We define

$$B^n(T) = B(T) \cdots B(\mathcal{Z}^n(T))$$

for any $n \geq 0$. Then, by definition of the accelerated lengths and heights cocycles, we have

$$\lambda^{(n)} = \frac{B^n(T)^{-1}\lambda^{(0)}}{|B^n(T)^{-1}\lambda^{(0)}|}, \quad h^{(n)} = B^n(T)^T h^{(0)},$$

where $h^{(0)} := (1, \ldots, 1) \in \mathbb{N}^{\mathcal{A}}$.

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2.2.9. *Oseledet's splitting.* By Oseledet's theorem and the combination of several classical works [2, 12, 28, 33], for a.e. (λ, π), there exist Lyapunov exponents

$$\theta_1 > \theta_2 > \cdots > \theta_g > 0 > -\theta_g > \cdots - \theta_2 > -\theta_1$$

for some $1 \leq g \leq (d)/(2)$, and Oseledets' filtrations

$$\mathbb{R}^d = E_g \supsetneq E_{g-1} \cdots \supsetneq E_1 \supsetneq E_0 \supseteq E_{-1} \supsetneq \cdots \supsetneq E_{-g+1} \supsetneq E_{-g} \supsetneq \{0\},$$

$$\mathbb{R}^d = F_g \supsetneq F_{g-1} \cdots \supsetneq F_1 \supsetneq F_0 \supseteq F_{-1} \supsetneq \cdots \supsetneq F_{-g+1} \supsetneq F_{-g} \supsetneq \{0\},$$

invariant by the cocycles $(\mathcal{Z}, B^T)$ and $(\mathcal{Z}, B^{-1})$, respectively, where $F_i \setminus F_{i-1}$ (respectively $E_i \setminus E_{i-1}$) is associated to the Lyapunov exponent θ_{g-i+1} for $1 \leq |i| \leq g$, and vectors in $E_0 \backslash E_{-1}$ (respectively $F_0 \setminus F_{-1}$) are associated to a zero Lyapunov exponent.

We denote

$$E^u = \mathbb{R}^{|\mathcal{A}|} = F^u, \quad E^{cs} = E_0, \quad E^s = E_{-1}, \quad F^{cs} = F_0, \quad F^s = F_{-1}.$$

We have

$$E^u \supseteq E^{cs} \supseteq E^s, \quad F^u \supseteq F^{cs} \supseteq F^s.$$

For a.e. (λ, π), the following hold:
- $\lambda \in F^s(\lambda, \pi)$;
- $d - 2g = \dim(\mathrm{Ker}(\Omega_\pi))$ and $\mathrm{Ker}(\Omega_\pi) \subseteq F^{cs}(\lambda, \pi) \setminus F^s(\lambda, \pi)$;
- $E^s = (F^{cs})^\perp$ and $F^s = (E^{cs})^\perp$.

We refer the interested reader to [32, 34] for details.

If π is of rotation type, then $\dim(\mathrm{Ker}(\Omega_\pi)) = d - 2$. Hence, it follows from the previous relations that

$$\dim(E^u) = \dim(E^s) = 1 = \dim(F^s) = \dim(F^u),$$

$$F^s(\lambda, \pi) = \langle \lambda \rangle, \quad E^{cs} = \lambda^\perp, \quad E^s = \mathrm{Ker}(\Omega_\pi)^\perp \cap \lambda^\perp.$$

Since for a.e. (λ, π) and any AIET f with $\gamma(f) = (\lambda, \pi)$ and log-slope $\omega \in \mathbb{R}^A$ we have $\langle \omega, \lambda \rangle = 0$ (see §2.2.6), it follows from the equation above that if π is of rotation type, then $\omega \in E^{cs}(\lambda, \pi)$.

2.3. *P-homeomorphisms as GIETs.* Recall that a circle homeomorphism f is called a *P-homeomorphism* if it is a smooth orientation-preserving homeomorphism, differentiable away from countable many points, so-called *break points*, at which left and right derivatives, denoted by Df_-, Df_+, respectively, exist but do not coincide, and such that Df (which is defined away from break points) coincides with a function uniformly bounded from below and of bounded variation. A P-homeomorphism that is linear in each domain of differentiability is called a *PL-homeomorphism*. We denote the set of break points of a P-homeomorphism f by

$$BP(f) = \{x \in \mathbb{T} \mid Df_-(x) \neq Df_+(x)\}.$$

Since we often require P-homeomorphisms to have additional properties, we introduce the following notation. Define

$$\begin{array}{cccc} \varphi : & [0,1) & \to & \mathbb{T} \\ & x & \mapsto & e^{2\pi i x}. \end{array}$$

For any $d \geq 1$ and any $r \in [0, +\infty)$, let $P_d^r(\mathbb{T})$ (respectively $PL_d(\mathbb{T})$) be the space P-homeomorphisms (respectively PL-homeomorphisms) $f : \mathbb{T} \to \mathbb{T}$ such that:

(i) f is piecewise C^r (respectively piecewise linear);

(ii) $\varphi(0) \in BP(f)$;

(iii) $|BP(f)| = d$;

(iv) $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$;

(v) $f^n(x) \neq f^m(y)$ for any $n, m \in \mathbb{Z}$ and any $x, y \in BP(f)$, $x \neq y$.

We treat P-homeomorphisms (respectively PL-homeomorphisms) as GIETs (respectively AIETs) using the circle parameterization given by the map φ. For any $f \in P_d^r(\mathbb{T})$, the map

$$T_f = \varphi^{-1} \circ f \circ \varphi$$

is a well-defined GIET (respectively AIET) on $d+1$ intervals. Since f has exactly d break points lying in different orbits, T_f defines an irrational GIET (respectively AIET). Moreover, T_f cannot be seen as a GIET (respectively AIET) on a smaller number of intervals. In the following, for any $f \in P_d^r(\mathbb{T})$, we define its *combinatorial rotation number* as $\gamma(f) = \gamma(T_f)$.

By Denjoy's theorem, a P-homeomorphism with an irrational rotation number is topologically conjugated to a rigid rotation (see [16, Theorem 6.5.5]). In particular, given $f \in P_d^r(\mathbb{T})$, the associated GIET T_f has no wandering intervals. Hence, if $\gamma(f) = (\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$, recalling that by [31, Proposition 7], T_f is semi-conjugated to the IET associated with (λ, π) and that this is actually a conjugacy if T_f has no wandering intervals, it follows that T_f is topologically conjugated to the IET T associated with (λ, π). We summarize this in the following proposition.

PROPOSITION 2.4. *Let $d \geq 1$ and $r \in [0, +\infty)$. Let $f \in P_d^r(\mathbb{T})$ with rotation number $\alpha = \rho(f) \in \mathbb{R} \setminus \mathbb{Q}$ and combinatorial rotation number $\gamma(f) \in \Delta_d \times \mathfrak{G}_d^0$. Then:*

- *f is topologically conjugated to R_α (as circle maps);*
- *T_f is topologically conjugated to $T = (\lambda, \pi)$ (as GIETs).*

2.4. *Hausdorff dimension.* For a subset X of a metric space M, we define its *d-dimensional Hausdorff content* by

$$C_H^d(X) := \lim_{\epsilon \to 0} \inf_{(U_i)} \sum_i (\mathrm{diam}(U_i))^d,$$

where the infimum is taken over all countable covers (U_i) of X satisfying $\mathrm{diam}(U_i) < \epsilon$. The *Hausdorff dimension* of X is given by

$$\dim_H(X) := \inf\{d \geq 0 \mid C_H^d(X) = 0\}.$$

We recall that the *Hausdorff dimension of a probability measure* μ over M is given by

$$\dim_H(\mu) := \inf\{\dim_H(X) \mid \mu(X) = 1\}.$$

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3. *Statement of the main results*

In this work, we aim to complement the results of Khanin and Kocić [19], which concerns P-homeomorphisms with a finite number of breaks and non-zero mean nonlinearity, by considering the zero mean nonlinearity case. We will show that, typically, the unique invariant probability measure of P-homeomorphisms with a finite number of breaks, irrational rotation number, and zero mean nonlinearity has zero Hausdorff dimension. To encode this generic condition, we consider P-homeomorphisms as generalized interval exchange transformations (GIETs) of the interval (see §2.3) and rely on the notion of *combinatorial rotation number*, which can be seen as an extension of the classical notion of rotation number for circle homeomorphisms to the GIET setting. Notice that a P-homeomorphism with zero mean nonlinearity has either none or at least two break points.

Our main result is the following.

THEOREM 3.1. *Let $d \geq 2$. There exists a full-measure set of combinatorial rotation numbers $C_d \subseteq \Delta_{d+1} \times \mathfrak{G}^0_{d+1}$ such that, for any $f \in P^3_d(\mathbb{T})$ with zero mean nonlinearity and $\gamma(f) \in C_d$, the unique invariant probability measure μ_f of f verifies $\dim_H(\mu_f) = 0$.*

Let us point out that the non-zero mean nonlinearity hypothesis plays an essential role in the argument of [19] (which proves a similar result in the non-zero mean nonlinearity case) as the proof relies heavily on the behavior of renormalizations of P-homeomorphisms with a finite number of break points and non-zero mean nonlinearity. In fact, for a given map f in this class, its renormalizations converge, in the C^2 norm, to a class of Möbius transformations whose *second derivative is negative and uniformly bounded away from zero*. Exploiting this convergence, the authors show that the union of adjacent intervals in the right-hand side of equation (1) accumulates 'geometrically' near the boundary of $I^i_{n-1} \setminus I^i_{n+1}$, that is, their lengths decrease geometrically with respect to the length of I^i_{n-1}.

Since all of the intervals in the right-hand side of equation (1) have the same measure with respect to the unique invariant measure μ_f of f, the observation above allows to construct sets (more precisely, Rohlin towers) with small Lebesgue measure (in fact, small Hausdorff content) but whose measure with respect to μ_f tend to 1. Refining this argument, the authors in [19] show that if an appropriate full-measure condition in the rotation number is satisfied, then the Hausdorff dimension of the unique invariant measure μ_f is zero.

However, for circle diffeomorphisms with breaks and zero mean nonlinearity, the renormalizations exhibit very different behavior. For example, for piecewise affine circle homeomorphisms, the second derivative of any of their renormalizations is equal to 0 everywhere. Furthermore, it follows from a recent result by Ghazouani and Ulcigrai [14] that the renormalizations of any circle diffeomorphism with breaks of class $C^{2+\epsilon}$ and zero mean nonlinearity converge, in C^2 norm, to the space of piecewise affine circle homeomorphisms. In particular, the second derivative of their renormalizations converges to 0, and thus, the argument in [19] cannot be extended to the zero mean nonlinearity case.

As mentioned in §1, it follows from a recent work by Berk and the author [3] that, for $d \geq 2$, a typical P-homeomorphism $f \in PL^3_d(\mathbb{T})$ with zero mean nonlinearity is C^1 conjugated to a PL-homeomorphism. Moreover, following §2.3, this PL-homeomorphism

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induces an AIET with exactly $d+1$ intervals, which cannot be reduced to an AIET on a smaller number of intervals. Hence, Theorem 3.1 is a direct consequence of the following.

THEOREM 3.2. *Let $d \geq 2$. For a.e. $(\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ of rotation type, and for any AIET f on d intervals with log-slope $\omega \in \mathbb{R}^d$ and $\gamma(f) = (\lambda, \pi)$, we have the following dichotomy, either:*

(1) *f is C^∞ conjugate to a standard IET; or*

(2) *$\dim_H(\mu_f) = 0$, where μ_f denotes the unique invariant probability measure of f.*

Moreover, the first assertion is verified if and only if f can be seen as an AIET on two intervals, which corresponds to $\omega \in E^s(\lambda, \pi)$.

3.1. *Strategy of proof.* For $T = (\lambda, \pi)$, f and ω as in the statement of Theorem 3.2, the existence of a smooth conjugacy between f and T if $\omega \in E^s(\lambda, \pi)$ is a direct consequence of a result by Cobo [**5**, Theorem 1]. Thus, it suffices to show that if $\omega \notin E^s(\lambda, \pi)$, then $\dim_H(\mu) = 0$.

Extracting the main elements of the strategy of [**19**] described in the previous section, and refining the argument therein, yields the following criterion. For an ergodic piecewise continuous orientation-preserving bijection (T, μ) on an interval, the existence of a sequence of 'sufficiently rigid' Rohlin towers $\mathcal{F}_k$ (see equations (6) and (7) in Proposition 3.3) with intervals F_k as bases, increasing heights h_k, measure $\mu(\mathcal{F}_k)$ uniformly bounded from below, and such that T^{h_k} is continuous on each floor of $\mathcal{F}_k$ and either contracts or expands at a uniform rate when restricted to F_k, implies $\dim_H(\mu) = 0$.

More precisely, we have the following.

PROPOSITION 3.3. *Let $T : [0, 1) \to [0, 1)$ be a piecewise C^2 bijection without periodic points, having positive derivative on each smoothness branch, and such that $\sup_{x \in [0,1)} |(T''(x))/(T'(x))| < \infty$. Let μ be a T-invariant ergodic probability measure. Suppose there exist a sequence of intervals $F_k \subseteq [0, 1)$ and an increasing sequence of natural numbers h_k such that:*

(a) *$\mathcal{F}_k = \bigsqcup_{j=0}^{h_k-1} T^j(F_k)$ is a Rohlin tower for any $k \geq 0$;*

(b) *$T^{h_k}|_{T^j(F_k)}$ is smooth for any $k \geq 0$ and any $0 \leq j < h_k$;*

(c) *$\inf_{k \geq 0} \mu(\mathcal{F}_k) > 0$;*

(d) *$\inf_{\substack{x \in F_k \\ k \geq 0}} |DT^{h_k}(x) - 1| > 0$;*

(e) *there exists a sequence of natural numbers M_k obeying*

$$\frac{M_k}{\log h_k} \to \infty, \tag{6}$$

such that

$$\bigcap_{m=0}^{M_k} T^{mh_k}(F_k) \neq \varnothing. \tag{7}$$

Then, $\dim_H(\mu) = 0$.

Proposition 3.3 will be proven in §4.

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To show that typical PL-homeomorphisms whose log-slope vector does not belong to the stable space fulfill the hypotheses of Proposition 3.3, we use renormalization techniques for interval exchange transformations (IETs) and treat PL-homeomorphisms as affine interval exchange transformations (AIETs) by parameterizing the circle $\mathbb{T}$ as

$$\begin{array}{cccc} \varphi : & [0,1) & \to & \mathbb{T} \\ & x & \mapsto & e^{2\pi i x}, \end{array}$$

and restricting ourselves to PL-homeomorphisms such that $\varphi(0)$ is a break point of f. Then, if f has $d \geq 1$ break points, the map $\varphi^{-1} \circ f \circ \varphi$ can be seen as a well-defined AIET on $d + 1$ intervals. See §2.3 for more details on this identification.

4. *Proof of the zero HD criterion*

The following lemma is a well-known fact.

LEMMA 4.1. *Let (T, X, μ) be an ergodic measure-preserving automorphism on a probability space. Then, for any $c > 0$ and any sequence of Rohlin towers,*

$$\mathcal{T}_k := \mathcal{T}(F_k, h_k) = \bigsqcup_{j=0}^{h_k - 1} T^j(F_k),$$

with $h_k \longrightarrow +\infty$ and $\mu(\mathcal{T}_k) > c$,

$$\mu_T\left(\bigcap_{n\geq 0}\bigcup_{k\geq n} \mathcal{T}_k\right) = 1.$$

Proof. Let

$$A = \bigcap_{n\geq 0}\bigcup_{k\geq n} \mathcal{T}_k.$$

Since A is the intersection of a decreasing sequence of sets of measure at least c, it follows that $\mu(A) \geq c$. Since μ is an ergodic T-invariant measure, it suffices to show that A is a T-invariant set. Notice that

$$A \Delta T^{-1}(A) \subseteq \bigcup_{k\geq n} T^{-1}(F_k) \cup T^{h_k - 1}(F_k)$$

for any $n \in \mathbb{N}$. Up to taking a subsequence and since $h_k \longrightarrow +\infty$, we may assume

$$\sum_{k\geq 0} \mu(F_k) < +\infty.$$

Therefore,

$$\mu(A\Delta T^{-1}(A)) \leq \lim_{n\to\infty} 2 \sum_{k\geq n} \mu(F_k) = 0.$$

Hence, A is a T-invariant set. By ergodicity, $\mu(A) = 1$. □

We are now in a position to prove Proposition 3.3.

Proof of Proposition 3.3. For any $k \geq 0$, let us denote the left and right endpoints of F_k by l_k and r_k, respectively. Notice that since T has no periodic points, equation (7) together

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with the continuity of $T^{h_k}|_{T^j(F_k)}$, for $0 \le j < h_k$, imply that either

$$T^{mh_k}(l_k) \in F_k \quad \text{for all } 0 < m \le M_k, \tag{8}$$

or

$$T^{mh_k}(r_k) \in F_k \quad \text{for all } 0 < m \le M_k.$$

Clearly, one of the two equations above must hold for infinitely many values of n. Hence, up to considering a subsequence, we may assume without loss of generality that one of the two equations holds for all $k \ge 0$. From now on, and for the sake of simplicity, let us assume that the first of the two equations holds for all $k \ge 0$, the other case being analogous. Moreover, by taking M_k bigger if necessary, we may assume that

$$T^{(M_k+1)h_k}(l_k) \notin F_k. \tag{9}$$

Similarly, by condition (d), we may assume without loss of generality that either $DT^{h_k}|_{F_k}$ is uniformly bigger than one for all $k \ge 0$ or it is uniformly smaller than one for all $k \ge 0$. For the sake of simplicity, let us assume that

$$\sigma = \sup_{\substack{x \in F_k \\ k \ge 0}} DT^{h_k}(x) < 1,$$

the other case being analogous.

Let $(L_k)_{k \ge 0}$ be a sequence of natural numbers such that

$$\frac{L_k}{\log h_k} \to \infty, \quad \frac{L_k}{M_k} \to 0.$$

Define

$$G_k = \bigsqcup_{j=L_k}^{M_k-1} T^{jh_k}((l_k, T^{h_k}(l_k))), \quad \mathcal{G}_k = \bigsqcup_{j=0}^{h_k-1} T^j(G_k), \quad \mathcal{X}_k = \bigcup_{n \ge k} \mathcal{G}_n$$

for any $k \ge 0$, and let

$$\mathcal{X} = \bigcap_{k \ge 0} \mathcal{X}_k.$$

We will show that $\mu(\mathcal{X}) = 1$ and $\dim_H(\mathcal{X}) = 0$.

Notice that $G_k \subseteq F_k$ and $\mathcal{G}_k \subseteq \mathcal{F}_k$. We shall see that although $\mathcal{G}_k$ has a very small Hausdorff content, its μ-measure is comparable to that of $\mathcal{F}_k$. More precisely, we can show the following.

CLAIM. *For any* $0 < s < 1$, *there exists* $C > 0$ *such that*

$$C_H^s(\mathcal{G}_k) \le Ce^{\log \sigma/2L_k}$$

for any $k \ge 0$, *where* C_H^s *denotes the s-dimensional Hausdorff content. Moreover,*

$$\inf_{k \ge 0} \mu(\mathcal{G}_k) > 0.$$

Before proving this claim, let us show how to conclude the proof of the proposition. By Lemma 4.1 and the previous claim, $\mu(\mathcal{X}) = 1$. Moreover, up to taking a subsequence, we

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may assume without loss of generality that

$$\sum_{k\geq 0} C_H^s(\mathcal{G}_k) < +\infty.$$

Thus,

$$C_H^s(\mathcal{X}) \leq \liminf_{k\to\infty} C_H^s(\mathcal{X}_k) \leq \liminf_{k\to\infty} \sum_{n\geq k} C_H^s(\mathcal{G}_n) = 0$$

for any $0 < s < 1$. Hence,

$$\dim_H(\mathcal{X}) = \inf\{s > 0 \mid C_H^s(\mathcal{X}) = 0\} = 0.$$

Therefore,

$$\dim_H(\mu) = \inf\{\dim_H(X) \mid \mu(X) = 1\} = 0.$$

Proof of the claim. Fix $k \geq 0$. Then

$$|G_k| = \sum_{j=L_k}^{M_k} |T^{jh_k}((l_k, T^{h_k}(l_k)))| \leq \sum_{j=L_k}^{M_k} \sigma^j |(l_k, T^{h_k}(l_k))| \leq C_\sigma \sigma^{L_k} |F_k|, \tag{10}$$

where $C_\sigma = (1-\sigma^{M_k - L_k})/(1-\sigma)$. A simple bounded distortion argument, together with equation (10), yields

$$|T^j(G_k)| \leq C_T C_\sigma \sigma^{L_k} |T^j(F_k)| \tag{11}$$

for any $0 \leq j < h_k$, where $C_T = \exp(\sup_{x\in[0,1)}(|T''(x)/T'(x)|))$. Indeed, since $G_k \subseteq F_k$ and $\sum_{j=0}^{h_k-1} |T^j(F_k)| \leq 1$, for any $0 < j < h_k$ and any $0 \leq i < j$, there exist, by the mean value theorem, $x_i \in G_k$, $y_i \in F_k$ and $z_{ij} \in T^i(F_k)$ such that

$$\begin{aligned}
\log \frac{|T^j(G_k)|}{|T^j(F_k)|} \frac{|F_k|}{|G_k|} &= \log \frac{(T^j)'(x_j)}{(T^j)'(y_j)} = \sum_{i=0}^{j-1} \log T'(T^i(x_j)) - \log T'(T^i(y_j)) \\
&\leq \sum_{i=0}^{j-1} \left|\frac{T''(z_{ij})}{T'(z_{ij})}\right| |T^i(x_j) - T^i(y_j)| \leq \sum_{i=0}^{j-1} \log(C_T)|T^i(F_k)| \\
&\leq \log(C_T).
\end{aligned}$$

The previous inequality, together with equation (10), implies equation (11).

Hence, for any $0 < s < 1$, it follows from equation (11) that

$$\begin{aligned}
C_H^s(\mathcal{G}_k) &\leq \sum_{j=0}^{h_k-1} |T^j(G_k)|^s \\
&\leq C_T C_\sigma \sigma^{L_k} \sum_{j=0}^{h_k-1} |T^j(F_k)|^s \\
&\leq C_T C_\sigma \sigma^{L_k} h_k^{1-s} \\
&= C_T C_\sigma \exp(L_k \log \sigma + (1-s)\log h_k) \\
&\leq C_T C_\sigma C_{h,s} \exp\left(\frac{\log \sigma}{2} L_k\right)
\end{aligned}$$

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for some positive constant $C_{h,s}$ independent of n, where from the second to third lines, we used Jensen's inequality on the concave function $x \mapsto x^s$ together with $\sum_{j=0}^{h_k-1} |T^j(F_k)| \leq 1$.

By equations (8) and (9),

$$M_k \mu((l_k, T^{h_k}(l_k))) \leq \mu(F_k) \leq (M_k + 1)\mu((l_k, T^{h_k}(l_k))).$$

Hence,

$$\frac{M_k - L_k}{M_k + 1} \leq \frac{\mu(G_k)}{\mu(F_k)} \leq \frac{M_k - L_k}{M_k},$$

which implies

$$\frac{\mu(\mathcal{G}_k)}{\mu(\mathcal{F}_k)} \longrightarrow 1. \qquad \square$$

This finishes the proof of the proposition. $\square$

5. *Proof of Theorem 3.2*

Theorem 3.2 will hold for irrational AIETs f of rotation type whose renormalizations $\{\mathcal{Z}^n(f)\}_{n\in\mathbb{N}}$ display certain prescribed behavior along some increasing subsequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ (see Lemma 5.2). This control on renormalizations will allow us to construct an appropriate sequence of Rohlin towers and to apply Proposition 3.3. We can prescribe a behavior on the subsequent renormalizations of f by imposing conditions on the map's combinatorial rotation number $\gamma(f) = (\lambda, \pi)$. Moreover, since f and the IET T associated with (λ, π) are conjugated (as well as their subsequent renormalizations), dynamical properties of T obtained using the renormalized maps can be easily translated to properties of f. Using strong ergodic properties of the map $\mathcal{Z}$ (see Theorem 5.7), we will show that the prescribed behavior on the renormalizations described in Lemma 5.2 is exhibited by a full-measure set of IETs.

For the sake of simplicity, let us start by introducing some notation that will be used in the remainder of this work. For any $d \geq 2$, we denote by $\pi^* = (\pi_0^*, \pi_1^*) \in \mathfrak{G}_d^0$ a fixed combinatorial datum satisfying

$$\pi_1 \circ {\pi_0}^{-1}(1) = d, \quad \pi_1 \circ {\pi_0}^{-1}(k) = k - 1. \tag{12}$$

Notice that although a permutation $\pi^* \in \mathfrak{G}_d^0$ verifying equation (12) always exists, π^* is not necessarily unique. For example, for $d = 4$, the permutations

$$\begin{pmatrix} A & D & B & C \\ D & B & C & A \end{pmatrix}, \quad \begin{pmatrix} A & C & B & D \\ C & B & D & A \end{pmatrix},$$

verify equation (12). In the following, we denote the last letters in the top and bottom rows of π^* by

$$\alpha^* = \pi_0^{*-1}(d), \quad \beta^* = \pi_1^{*-1}(d). \tag{13}$$

In Lemma 5.2, we will consider a subset of the IETs of rotation type whose iterates by $\mathcal{Z}$ belong to Δ_{π^*} infinitely many times. Our main interest in doing this is the properties in the lemma below, which we illustrate in Figure 1. In the following, we say that two intervals are adjacent if they share exactly one endpoint.

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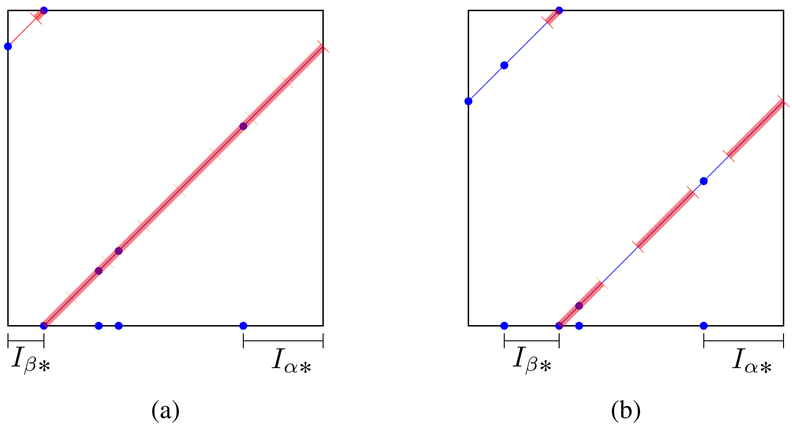

FIGURE 1. (a) An IET of rotation type with combinatorial data π satisfying equation (12). (b) An IET of rotation type with combinatorial data π not satisfying equation (12). In both cases, we denote $\alpha^* = \pi_0^{-1}(d)$ and $\beta^* = \pi_1^{-1}(d)$. The red (shaded) intervals represent the iterates $\{T^2(I_{\beta^*}), \ldots, T^{m+1}(I_{\beta^*})\}$, where m is the smallest positive natural number for which $T^m(I_\beta^*) \cap I_{\beta^*} \neq \varnothing$.

LEMMA 5.1. *Let $T = (\lambda, \pi^*) \in \Delta_{\pi^*}$, satisfying Keane's condition. Then,*

$$\{T(I_{\beta^*}), T^2(I_{\beta^*}), \ldots, T^{m+1}(I_{\beta^*})\} \quad \text{where } m = \left\lfloor \frac{1-\lambda_{\beta^*}}{\lambda_{\beta^*}} \right\rfloor,$$

are disjoint, adjacent intervals satisfying

$$\bigsqcup_{j=1}^{m} T^j(I_{\beta^*}) \subsetneq \bigsqcup_{\alpha \neq \beta^*} I_\alpha \subsetneq \bigsqcup_{j=1}^{m+1} T^j(I_{\beta^*}).$$

If $\alpha \neq \beta^*$ and $m_\alpha = \lfloor (\lambda_\alpha)/(\lambda_{\beta^*}) \rfloor > 2$, then at least $m_\alpha - 2$ of these iterates of I_{β^*} are contained in I_α. More precisely, there exist $1 \leq \ell_\alpha < L_\alpha \leq m+1$ with $L_\alpha - l_\alpha \geq m_\alpha - 1$ such that

$$\bigsqcup_{j=\ell_\alpha+1}^{L_\alpha-1} T^j(I_{\beta^*}) \subsetneq I_\alpha \subsetneq \bigsqcup_{j=\ell_\alpha}^{L_\alpha} T^j(I_{\beta^*}).$$

Furthermore, $\bigcap_{j=0}^{m_\alpha-2} T^j(I_\alpha) \neq \varnothing$.

Proof. Notice that $I_{\beta^*} = [0, \lambda_{\beta^*})$ and

$$T(x) = \begin{cases} x + (1-\lambda_{\beta^*}) & \text{if } x \in I_{\beta^*}, \\ x - \lambda_{\beta^*} & \text{otherwise.} \end{cases}$$

Hence, $T^j(I_{\beta^*}) = [1 - j\lambda_{\beta^*}, 1-(j-1)\lambda_{\beta^*})$ for any $1 \leq j \leq m+1$. In particular, the intervals $\{T(I_{\beta^*}), \ldots, T^{m+1}(I_{\beta^*})\}$ are disjoint and adjacent. Moreover, we have

$$\bigsqcup_{j=1}^{m} T^j(I_{\beta^*}) = [1 - m\lambda_{\beta^*}, 1) \subsetneq [\lambda_{\beta^*}, 1) = \bigsqcup_{\alpha \neq \beta^*} I_\alpha \subsetneq \bigsqcup_{j=1}^{m+1} T^j(I_{\beta^*}) = [1-(m+1)\lambda_{\beta^*}, 1).$$

If $\alpha \neq \beta^*$ is such that $m_\alpha = \lfloor \lambda_\alpha / \lambda_{\beta^*} \rfloor > 2$, since $\bigsqcup_{\alpha \neq \beta^*} I_\alpha \subseteq \bigsqcup_{j=1}^{m+1} T^j(I_{\beta^*})$, it follows that I_α intersects at least m_α intervals of $\{T(I_{\beta^*}), \ldots, T^{m+1}(I_{\beta^*})\}$. Hence, as these intervals are adjacent, there exists $1 \leq \ell_\alpha < L_\alpha \leq m+1$ such that $m_\alpha - 1 \leq L_\alpha - l$ and

$$\bigsqcup_{j=\ell_\alpha+1}^{L_\alpha-1} T^j(I_{\beta^*}) \subseteq I_\alpha \subseteq \bigsqcup_{j=\ell_\alpha}^{L_\alpha} T^j(I_{\beta^*}).$$

Notice that the inclusions in the equations above are strict since T would not verify Keane's condition otherwise. Moreover, it follows from the previous equation that

$$T^{L_\alpha-1}(I_{\beta^*}) \subseteq \bigcap_{i=0}^{m_\alpha-2} T^i\bigg(\bigsqcup_{j=\ell_\alpha+1}^{L_\alpha-1} T^j(I_{\beta^*})\bigg) \subseteq \bigcap_{i=0}^{m_\alpha-2} T^i(I_\alpha).$$

$\square$

Using the notation introduced above, we can explicitly state the generic condition in Theorem 3.2.

LEMMA 5.2. *For any $0 < c_0 < 1/10d$ sufficiently small and for any increasing sequence $\{C(n)\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ verifying*

$$\sum_{n\geq 1} \frac{1}{nC(n)} = +\infty, \tag{14}$$

the following holds. For a.e. $(\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ of rotation type, there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that:

(1) $\pi^{(n_k)} = \pi^*$;

(2) $\lambda_\alpha^{(n_k)} > n_k C(n_k) \lambda_{\beta^*}^{(n_k)}$ *for all* $\alpha \neq \beta^*$;

(3) $\lambda_\alpha^{(n_k)} > c_0$ *for all* $\alpha \neq \beta^*$;

(4) $h_\alpha^{(n_k)}/h_\beta^{(n_k)} > c_0$ *for all* $\alpha, \beta \in \mathcal{A}$;

(5) $(\log |h^{n_k}|)/n_k \leq c_0^{-1}$.

We postpone the proof of the lemma above to the end of this section.

Let us explain how the conditions in Lemma 5.2 will appear in the proof of (the second assertion of) Theorem 3.2 for log-slope vectors not belonging to the stable space. As mentioned before, this will be a consequence of Proposition 3.3 for which an appropriate sequence of Rohlin towers is required. To build Rohlin towers for a given AIET f with $\gamma(f) = (\lambda, \pi)$, we start by building towers for the IET T defined by (λ, π) to which f is conjugated.

From Lemmas 5.1 and 5.2, we immediately conclude the following.

COROLLARY 5.3. *Let $T = (\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ and $\{C(n)\}_{n\in\mathbb{N}}$, $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ as in Lemma 5.2. Then, for any $k \in \mathbb{N}$, there exist natural numbers $1 = l_0^k < l_1^k < \cdots < l_{d-1}^k$ such that $l_{i+1}^k - l_i^k \geq n_k C(n_k) - 1$ and*

$$\bigsqcup_{j=l_i^k+1}^{l_{i+1}^k-1} T^{(n_k)^j}\Big(I_{\beta^*}^{(n_k)}\Big) \subsetneq I^{(n_k)}_{\pi_0^{*-1}(d-i)} \subsetneq \bigsqcup_{j=l_i^k}^{l_{i+1}^k} T^{(n_k)^j}(I_{\beta^*}^{(n_k)})$$

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for $i = 1, \ldots, d-1$, where the unions in the previous equation consist of adjacent intervals. Moreover,

$$\bigcap_{j=0}^{n_k C(n_k)-2} T^{(n_k)^j}\left(I^{(n_k)}_{\pi_0^{*-1}(d-i)}\right) \neq \varnothing \tag{15}$$

for $i = 1, \ldots, d-1$.

Using Corollary 5.3, we can easily check that, along the subsequence given by Lemma 5.2, the Rauzy–Veech towers associated with T already satisfy many of the conditions in Proposition 3.3.

LEMMA 5.4. *Let $T = (\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ and $\{C(n)\}_{n\in\mathbb{N}}$, $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ as in Lemma 5.2. Fix $\alpha \in \mathcal{A}$ with $\alpha \neq \beta^*$. Then, the sequences*

$$\mathcal{F}_k := \bigsqcup_{j=0}^{h_k-1} T^{(n_k)^j}(F_k), \quad F_k := I^{(n_k)}_\alpha, \quad h_k := h^{(n_k)}_\alpha, \quad M_k = n_k C(n_k) - 2,$$

define Rohlin towers satisfying conditions (a)–(c) and (e) of Proposition 3.3.

Proof. Let $k \geq 0$. It follows directly from the definitions and the renormalization procedure that $\mathcal{F}_k$ is a well-defined Rohlin tower and that T is a translation when restricted to $T^j(F_k)$ for any $0 \leq j < h_k$. This proves conditions (a) and (b).

Since T is a translation on each floor of $\mathcal{F}_k$, it follows that

$$|\mathcal{F}_k| = \sum_{j=0}^{h_k-1} |T^j(F_k)| = h^{(n_k)}_\alpha |I^{(n_k)}_\alpha|. \tag{16}$$

By condition (3),

$$\max_{\beta\in\mathcal{A}} \frac{|I^{(n_k)}_\beta|}{|I^{(n_k)}_\alpha|} = \max_{\beta\in\mathcal{A}} \frac{\lambda^{(n_k)}_\beta}{\lambda^{(n_k)}_\alpha} \leq c_0^{-1}.$$

Hence, recalling that $[0, 1) = \bigsqcup_{\beta\in\mathcal{A}} \bigsqcup_{j=0}^{h^{(n_k)}_\beta - 1} T^j(I^{(n_k)}_\beta)$, by condition (4) and the previous inequality, we have

$$1 = \sum_{\beta\in\mathcal{A}} h^{(n_k)}_\beta |I^{(n_k)}_\beta| \leq d c_0^{-2} h^{(n_k)}_\alpha |I^{(n_k)}_\alpha|,$$

which, together with equation (16), implies condition (c).

By condition (5), it follows that

$$\frac{M_k}{\log h_k} = \frac{n_k C(n_k) - 2}{\log h^{(n_k)}_\alpha} \geq c_0 C(n_k) - \frac{2c_0}{n_k} \xrightarrow[k\to\infty]{} +\infty.$$

Since $\alpha \neq \beta^*$, there exists $1 \leq i < d$ such that $\alpha = \pi_0^{*-1}(d-i)$. Recalling that $T^{(n_k)}|_{I^{(n_k)}_\alpha} = T^{h^{(n_k)}_\alpha}$, it follows from equation (15) that

$$\bigcap_{m=0}^{M_k} T^{m h_k}(F_k) \neq \varnothing,$$

thus proving condition (e). □

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Therefore, if f is an AIET with $\gamma(f) = (\lambda, \pi)$ verifying Lemma 5.2, the Rauzy–Veech towers associated to f,

$$\bigsqcup_{j=0}^{h_\alpha^{(n_k)}-1} f^j(I_\alpha^{(n_k)}(f)) \quad \text{for } \alpha \neq \beta^*, \tag{17}$$

satisfy conditions (a)–(c) and (e) of Proposition 3.3. Indeed, for $\alpha \neq \beta^*$ fixed, since f and T are conjugated by some homeomorphism $h \in \mathrm{Hom}([0, 1))$, verifying $f \circ h = h \circ T$, then

$$\bigsqcup_{j=0}^{h_\alpha^{(n_k)}-1} f^j(I_\alpha^{(n_k)}(f)) = h\Bigg(\bigsqcup_{j=0}^{h_\alpha^{(n_k)}-1} T^j(I_\alpha^{(n_k)}(T))\Bigg)$$

and

$$\mu_f\Bigg(\bigsqcup_{j=0}^{h_\alpha^{(n_k)}-1} f^j(I_\alpha^{(n_k)}(f))\Bigg) = \Bigg|\bigsqcup_{j=0}^{h_\alpha^{(n_k)}-1} T^j(I_\alpha^{(n_k)}(T))\Bigg|,$$

where μ_f denotes the unique invariant probability measure of f.

Since, by Lemma 5.4, the towers $\bigsqcup_{j=0}^{h_\alpha^{(n_k)}-1} T^j(I_\alpha^{(n_k)}(T))$ verify conditions (a)–(c) and (e) in Proposition 3.3, it is readily seen that the towers in equation (17) also verify these conditions.

Notice that condition (d) for the towers in equation (17) is equivalent to

$$\inf_{k\geq 1} |\omega_\alpha^{(n_k)}| > 0, \tag{18}$$

where $\omega_\alpha^{(n_k)}$ is the log-slope vector of $f^{(n_k)}$. Recall that, by Proposition 2.3, if f has log-slope vector ω, then $\omega^{(n_k)} = B^{n_k}(\lambda, \pi)^T \omega$.

As we shall see in Corollary 5.6, if the log-slope vector of f does not belong to the stable space $E^s(\lambda, \pi)$ then, up to considering a subsequence, equation (18) is satisfied for at least one $\alpha \neq \beta^*$, and thus the second assertion of Theorem 3.2 would follow by Proposition 3.3 when applied to the towers given by equation (17) for this particular α.

To see that such $\alpha \in \mathcal{A} \setminus \{\beta^*\}$ indeed exists, we will use the following properties of the lengths cocycle (see §2.2.4 for the definition).

LEMMA 5.5. *Let $(\lambda, \pi) \in \Delta_d \times \mathfrak{S}_d^0$ infinitely renormalizable with π of rotation type. Then, for any $\omega \in \mathbb{R}^{\mathcal{A}}$ and for any $n \in \mathbb{N}$ such that $\pi^{(n)} = \pi$,*

$$\pi_{\mathrm{Ker}(\Omega_\pi)}(\omega^{(n)}) = \pi_{\mathrm{Ker}(\Omega_\pi)}(\omega),$$

where $\pi_{\mathrm{Ker}(\Omega_\pi)} : \mathbb{R}^{\mathcal{A}} \to \mathrm{Ker}(\Omega_\pi)$ denotes the orthogonal projection to $\mathrm{Ker}(\Omega_\pi)$ and $\omega^{(n)} = A^n(\lambda, \pi)^{-1}\omega$. Moreover,

$$\pi_{\mathrm{Ker}(\Omega_\pi)}(\omega) \neq 0$$

for any $\omega \in E^{cs}(\lambda, \pi) \setminus E^s(\lambda, \pi)$.

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Proof. Fix $(\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ infinitely renormalizable with π of rotation type and let $\omega \in \mathbb{R}^{\mathcal{A}}$. Then, for any $n \in \mathbb{N}$ such that $\pi^{(n)} = \pi$, and for any $v \in \operatorname{Ker}(\Omega_\pi)$,

$$\begin{aligned} |\langle \omega, v\rangle| &= |\langle ({B^{(n)}}^T)^{-1}\omega^{(n)}, v\rangle| \\ &= |\langle \omega^{(n)}, (B^{(n)})^{-1}v\rangle| \\ &= |\langle \omega^{(n)}, v\rangle|, \end{aligned}$$

where the last equality follows from Proposition 2.2. Hence,

$$\pi_{\operatorname{Ker}(\Omega_\pi)}(\omega^{(n)}) = \pi_{\operatorname{Ker}(\Omega_\pi)}(\omega)$$

for all $n \in \mathbb{N}$.

Recall that (see §2.2.9) for π of rotation type, we have

$$\begin{aligned} &E^{cs}(\lambda, \pi) = \lambda^\perp, && \dim\big(E^{cs}(\lambda, \pi)\big) = d - 1, \\ &E^{s}(\lambda, \pi) = \operatorname{Ker}(\Omega_\pi)^\perp \cap \lambda^\perp, && \dim\big(E^{s}(\lambda, \pi)\big) = 1. \end{aligned}$$

Hence,

$$(E^{cs}(\lambda, \pi) \setminus E^{s}(\lambda, \pi)) \cap \operatorname{Ker}(\Omega_\pi)^\perp = \{0\},$$

since otherwise $1 = \dim(\operatorname{Ker}(\Omega_\pi)^\perp \cap \lambda^\perp) = \dim(E^s(\lambda, \pi)) > 1$. Thus,

$$\pi_{\operatorname{Ker}(\Omega_\pi)}(\omega) \neq 0$$

for any $\omega \in E^{cs}(\lambda, \pi) \setminus E^s(\lambda, \pi)$. □

As a simple consequence of Lemmas 5.2 and 5.5, we have the following.

COROLLARY 5.6. *Let $(\lambda, \pi) \in \Delta_d \times \mathfrak{G}_d^0$ as in Lemma 5.2. Then, for any $\omega \in E^{cs}(\lambda, \pi) \setminus E^s(\lambda, \pi)$, there exists a constant $c_2 > 0$, depending only on ω, such that*

$$\inf_{k \geq 1} \max_{\alpha \neq \beta^*} |\omega_\alpha^{(n_k)}| > c_2,$$

where $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ is the sequence in Lemma 5.2.

Proof. Fix $k \geq 1$. By definition of π^*, we have

$$\operatorname{Ker}(\Omega_{\pi^*}) = \left\{ v \in \mathbb{R}^{\mathcal{A}} \;\middle|\; v_{\beta^*} = 0;\; v_{\alpha^*} = -\sum_{\delta \neq \alpha^*} v_\delta \right\}.$$

Then,

$$\pi_{\operatorname{Ker}(\Omega_{\pi^*})}(\pi_{e_{\beta^*}^\perp}(\omega^{(n_k)})) = \pi_{\operatorname{Ker}(\Omega_{\pi^*})}(\omega^{(n_k)}),$$

where $e_{\beta^*}^\perp = \{v \in \mathbb{R}^{\mathcal{A}} \mid v_{\beta^*} = 0\}$ and $\pi_{e_{\beta^*}^\perp} : \mathbb{R}^{\mathcal{A}} \to e_{\beta^*}^\perp$ denotes the orthogonal projection to $e_{\beta^*}^\perp$. By Lemma 5.5,

$$\pi_{\operatorname{Ker}(\Omega_{\pi^*})}(\pi_{v_{\beta^*}^\perp}(\omega^{(n)})) = \pi_{\operatorname{Ker}(\Omega_{\pi^*})}(\omega) \neq 0.$$

Hence, there exists $c > 0$, depending only on π^* and ω, such that

$$|\pi_{v_{\beta^*}^\perp}(\omega^{(n)})| > c.$$

□

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We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $T = (\lambda, \pi)$, f, and ω as in the statement of the theorem. Recall that in this case (see §2.2.9), $\omega \in E^{cs}(\lambda, \pi)$ and

$$E^{cs}(\lambda, \pi) = \lambda^{\perp}, \quad E^{s}(\lambda, \pi) = \lambda^{\perp} \cap \mathrm{Ker}(\Omega_{\pi})^{\perp}. \tag{19}$$

We start by showing that $\omega \in E^{s}(\lambda, \pi)$ if and only if f can be seen as a 2-IET. For the sake of simplicity, we will assume that $\pi = \pi^*$, with π^* as in equation (12), but an analogous argument will yield the same conclusion for any rotation-type combinatorial datum. By definition of π^*, we have

$$\mathrm{Ker}(\Omega_{\pi^*})^{\perp} = \mathrm{Vect}\bigg(e_{\beta^*}, \sum_{\delta \neq \beta^*} e_{\delta}\bigg), \tag{20}$$

where β^* is given by equation (13). In particular, the coordinates of a vector $\omega \in \mathrm{Ker}(\Omega_{\pi^*})^{\perp}$ have only two possible values. It follows from equations (12), (19), and (20) that f can be seen as a 2-IET if and only if $\omega \in E^{s}(\lambda, \pi^*)$.

Moreover, if $\omega \in E^{s}(\lambda, \pi)$, the existence of a smooth conjugacy between f and T, under generic assumptions on T, is a direct consequence of [**5**, Theorem 1].

For the rest of the proof, we will assume without loss of generality that $\pi = \pi^*$. In fact, by the ergodicity of the Zorich map $\mathcal{Z}$ when restricted to the $\mathcal{Z}$-invariant set $\bigcup_{\pi \in \mathfrak{R}} \Delta_{\pi}$, where $\mathfrak{R}$ denote the Rauzy class in $\mathfrak{S}_d^0$ that contains rotation-type permutations (see Proposition 2.1), it is sufficient to prove the theorem in this case.

Taking into account the observations above, the theorem will be proved if we show that, for a.e. $\lambda \in \Delta_d$, and for any AIET f with $\gamma(f) = (\lambda, \pi^*)$ and log-slope $\omega \in E^{cs}(\lambda, \pi^*) \setminus E^{s}(\lambda, \pi^*)$, where π^* satisfies equation (12), the Hausdorff dimension of its unique invariant measure μ_f is equal to zero.

Let $\lambda \in \Delta_d$ such that (λ, π^*) verifies the conclusions of Lemma 5.2 (and in particular of Corollary 5.3) and Corollary 5.6. By Corollary 5.6 and up to taking a subsequence, we can assume without loss of generality that there exists $\alpha \in \mathcal{A} \setminus \{\beta^*\}$ such that equation (18) holds, where β^* is given by equation (13). Notice that the set of such λ defines a full-measure set in Δ_d.

Let f be an AIET with combinatorial rotation number $\gamma(f) = (\lambda, \pi^*)$, log-slope $\omega \in E^{cs}(\lambda, \pi^*) \setminus E^{s}(\lambda, \pi^*)$, and unique invariant measure μ_f. Then, $\dim_H(\mu_f) = 0$ by applying Proposition 3.3 to the towers given by equation (17), which are defined using Corollary 5.3. Notice that these towers verify the hypotheses in Proposition 3.3 by Corollaries 5.3 and 5.6. □

The remainder of this work concerns the proof Lemma 5.2.

5.1. *Proof of Lemma 5.2.* Lemma 5.2 will be an application of the following 'Borel–Cantelli lemma' for the Zorich map, due to Aimino, Nicol, and Todd [**1**, Theorem 2.18].

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Before stating (a simplified version of) this result, let us introduce some notation. Let $\mathfrak{R}$ be a fixed Rauzy class. Let

$$\mathcal{P}_{\mathfrak{R}} = \{\Delta_{\pi,\epsilon} \mid \pi \in \mathfrak{R},\ \epsilon \in \{0, 1\}\},$$

and define

$$\mathcal{P}^n_{\mathfrak{R}} = \bigvee_{i=0}^{n-1} \mathcal{R}^{-i}(\mathcal{P}_{\mathfrak{R}})$$

for any $n \geq 1$. Notice that any $B \in \mathcal{P}^n_{\mathfrak{R}}$ is contained in a unique simplex of the form $\Delta_{\pi,\epsilon}$ for some $\pi \in \mathfrak{R}$ and $\epsilon \in \{0, 1\}$. For any $\epsilon \in \{0, 1\}$, we denote

$$\Delta_{\mathfrak{R},\epsilon} = \bigcup_{\pi\in\mathfrak{R}} \Delta_{\pi,\epsilon}.$$

Given $A \subseteq \Delta_{\pi,\epsilon}$, we denote by ∂A its boundary with respect to the usual Euclidean distance in $\Delta_{\pi,\epsilon}$. Similarly, for any $\delta > 0$, we denote by $B_\delta(A)$ the δ-neighborhood of A with respect to the usual Euclidean distance in $\Delta_{\pi,\epsilon}$.

THEOREM 5.7. (Borel–Cantelli lemma for Zorich acceleration [1]) *Let $\mathfrak{R}$ be a fixed Rauzy class, $n \geq 1$, and $\epsilon \in \{0, 1\}$. Suppose $B \in \mathcal{P}^n_{\mathfrak{R}}$ verifies $B \subseteq \Delta_{\mathfrak{R},\epsilon}$ and $\overline{B} \subseteq \Delta_d \times \mathfrak{R}$.*

Then, for any sequence $\{A_n\}_{n\geq1}$ of subsets of B such that

$$\sum_{n\geq1} \mu(A_n) = +\infty, \quad \sup_{n\geq1} \varlimsup_{\delta\searrow0} \frac{|B_\delta(\partial A_n)|}{\delta^\alpha} < +\infty$$

for some $0 < \alpha < 1$, we have

$$\frac{1}{E_n} \sum_{i=1}^{n} \chi_{A_i} \circ \mathcal{Z}^{2i}(x) \to 1$$

for a.e. $x \in \Delta_{\mathfrak{R},\epsilon}$, where $E_n = \sum_{i=1}^n \mu(A_i)$ for any $n \geq 1$.

In the following, we denote by $\mathfrak{R}$ the Rauzy class containing the permutations of rotation type. Recall that $\mathfrak{R}$ is endowed with an oriented graph structure associated with the Rauzy–Veech induction and that for any permutation in $\pi \in \mathfrak{R}$ and any $\epsilon \in \{0, 1\}$, there exists exactly one permutation $\pi^\epsilon \in \mathfrak{R}$ such that π can be obtained from π^ϵ after a Rauzy–Veech renormalization of type ϵ (to which we sometimes refer as a movement of type ϵ in the graph). Recall that we can denote this as $\pi^\epsilon(\epsilon) = \pi$ (we refer the reader to §§2.2.1, 2.2.2 for details on definitions and notation).

Note that associated to any admissible finite path γ in the oriented graph $\mathfrak{R}$, there exists a non-empty simplex Δ_γ in $\Delta_d \times \mathfrak{R}$ defined by the $(\lambda, \pi) \in \Delta_d \times \mathfrak{R}$ following the path γ under Rauzy–Veech induction. Indeed, denoting by $|\gamma|$ the length of the path γ, by $\gamma_i \in \mathcal{R}$ the ith permutation in γ, and by $\epsilon_i \in \{0, 1\}$ the type of movement (that is, top or bottom) used to go from γ_i to γ_{i+1}, then $\Delta_\gamma \subseteq \Delta_{\gamma_0,\epsilon_0}$ and we have

$$\Delta_\gamma = \bigcap_{i=0}^{|\gamma|-1} \mathcal{R}^{-i}(\Delta_{\gamma_i,\epsilon_i}) \in \mathcal{P}^{|\gamma|}_{\mathfrak{R}}. \tag{21}$$

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Moreover, since the matrices in the Rauzy–Veech cocycle depend only on combinatorial data and type (see §2.2.4), for any $(\lambda, \gamma_0) \in \Delta_\gamma$, we have $A^{|\gamma|-1}(\lambda, \gamma_0) = A_\gamma$, where

$$A_\gamma := A_{\gamma_0, \epsilon_0} \cdots A_{\gamma_{|\gamma|-1}, \epsilon_{|\gamma|-1}},$$

and A_{γ_i, ϵ_i} are as in equation (5). Notice that since the diagonal entries of Rauzy–Veech matrices are positive and all other entries are non-negative, the same is true for A_γ. Using the matrix A_γ, we can also write Δ_γ as

$$\Delta_\gamma = \left\{ \left(\frac{A_\gamma \lambda}{|A_\gamma \lambda|_1}, \gamma_0 \right) \,\middle|\, \lambda \in \Delta_d \right\}. \tag{22}$$

Given two finite admissible paths γ and γ' in $\mathcal{R}$ such that the last permutation in γ coincides with the first permutation in γ', the matrix associated with the concatenation of these paths, which we denote by $\gamma * \gamma'$, verifies $A_{\gamma * \gamma'} = A_\gamma A_{\gamma'}$.

It will be helpful to consider admissible finite paths γ in $\mathfrak{R}$ for which the associated matrix A_γ is *positive*, that is, such that all the entries of the matrix are positive. In this case, we say that the finite path γ is *positive*. These paths will play a crucial role in the proof of Lemma 5.2 (e.g. in showing condition (4), see Claim 2 and equation (27)). Notice that if a path is positive, its concatenation (either from left or right) with any other path is also positive.

Our main interest in considering positive matrices is the following property. If $A = (A_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ is a positive matrix, then

$$\max_{\alpha,\beta\in\mathcal{A}} \frac{(Av)_\alpha}{(Av)_\beta} \le d \max_{\alpha,\beta\in\mathcal{A}} A_{\alpha\beta} \quad \text{for any } v \in \mathbb{R}_+^{\mathcal{A}}. \tag{23}$$

Indeed, for any $v \in \mathbb{R}_+^{\mathcal{A}}$ and any $\alpha, \beta \in \mathcal{A}$,

$$\frac{(Av)_\alpha}{(Av)_\beta} = \frac{\sum_{\delta\in\mathcal{A}} A_{\alpha\delta} v_\delta}{\sum_{\delta\in\mathcal{A}} A_{\beta\delta} v_\delta} \le \sum_{\delta\in\mathcal{A}} \frac{A_{\alpha\delta}}{A_{\beta\delta}} \le d \max_{\sigma\delta\in\mathcal{A}} A_{\sigma\delta}.$$

In particular, if γ is a positive path in $\mathfrak{R}$, the associated simplex Δ_γ is compactly contained in $\Delta_d \times \mathfrak{R}$. Indeed, since in this case the associated matrix A_γ is positive, it follows from equations (22) and (23) that

$$\overline{\Delta_\gamma} \subseteq \left\{ v \in \Delta_d \,\middle|\, \max_{\alpha,\beta\in\mathcal{A}} \frac{v_\alpha}{v_\beta} \le d \max_{\alpha,\beta\in\mathcal{A}} (A_\gamma)_{\alpha\beta} \right\} \times \mathfrak{R} \subsetneq \Delta_d \times \mathfrak{R}.$$

Let us point out that positive paths within a Rauzy class starting at any given permutation always exist (see, e.g. [**22**, Lemma 1.2.4]). Moreover, since any two permutations in a Rauzy class can be connected by a finite path, there always exist positive finite paths starting/ending at any given permutation within a Rauzy class.

We are now in a position to prove Lemma 5.2.

Proof Lemma 5.2. To prove the lemma, we will define an appropriate admissible finite path γ in $\mathfrak{R}$ and an appropriate sequence of subsets $\{A_n\}_{n\ge1}$ in Δ_γ to which we will apply Theorem 5.7.

Let us start by defining the path γ. To avoid the use of double superscripts, let us denote by $\overline{\pi} = (\overline{\pi}_0, \overline{\pi}_1) \in \mathfrak{R}$ the pre-image of $\pi^* = (\pi_0^*, \pi_1^*)$ by a bottom movement, that is, $\overline{\pi}$

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verifies $\overline{\pi}(1) = \pi^*$. Let γ be a finite admissible positive path whose first permutation is π^*, its second permutation is $\pi^*(0)$, and its last permutation is $\overline{\pi}$. That is, γ is a positive path starting at π^* with a top movement and ending at $\overline{\pi}$. In particular, the associated set Δ_γ verifies $\Delta_\gamma \subseteq \Delta_{\pi^*,0} \subseteq \Delta_{\mathfrak{R},0}$.

To define the sequence of subsets $\{A_n\}_{n\geq 1}$ in Δ_γ, we will first define an auxiliar sequence of subsets $\{\widehat{A}_n\}_{n\geq 1}$ in Δ_d as follows. Denote $\delta^* = \overline{\pi}_0^{-1}(d)$ and let $0 < c_0 < (1)/(10d)$. Define

$$\widehat{A}_n := \left\{\lambda \in \Delta_d \;\middle|\; \min\left\{c_0^2, \frac{c_0}{nC(n)}\right\} > \lambda_{\beta^*} - \lambda_{\delta^*} > 0; \quad \min_{\alpha\in\mathcal{A}} \lambda_\alpha > c_0\right\}$$

for any $n \geq 1$. Clearly

$$|\widehat{A}_n| \geq \frac{c_1}{nC(n)} \tag{24}$$

for some $c_1 > 0$, depending only on c_0 and d.

CLAIM 1. *Let $n \geq 1$. For any $\lambda \in \widehat{A}_n$, we have $(\lambda, \overline{\pi}) \in \Delta_{\overline{\pi},1}$. Moreover, $\mathcal{R}(\lambda, \overline{\pi}) \in \Delta_{\pi^*,0}$ and its length vector λ' satisfies*

$$\min_{\alpha\neq\beta^*} \lambda'_\alpha > \max\{c_0, nC(n)\lambda'_{\beta^*}\}. \tag{25}$$

Proof. Let $\lambda \in \widehat{A}_n$. Notice that $\overline{\pi}(1) = \pi^*$ implies $\pi_1^* = \overline{\pi}_1$ and, in particular, $\pi_1^*(\beta^*) = d = \overline{\pi}_1(\beta^*)$. Since

$$\lambda_{\overline{\pi}_1^{-1}(d)} = \lambda_{\beta^*} > \lambda_{\delta^*} = \lambda_{\overline{\pi}_0^{-1}(d)},$$

the IET $(\lambda, \overline{\pi})$ is of bottom type. Hence, its Rauzy–Veech renormalization $\mathcal{R}(\lambda, \overline{\pi})$ has combinatorial datum $\overline{\pi}(1) = \pi^*$, and its length vector λ' is given by

$$\lambda'_\alpha = \begin{cases} \dfrac{\lambda_\alpha}{1-\lambda_{\beta^*}} & \text{if } \alpha \neq \beta^*, \\ \dfrac{\lambda_{\beta^*} - \lambda_{\delta^*}}{1-\lambda_{\beta^*}} & \text{if } \alpha = \beta^*. \end{cases} \tag{26}$$

In particular, it follows from the definition of $\widehat{A}_n$ that

$$nC(n)\lambda'_{\beta^*} < \frac{nC(n)}{1-\lambda_{\beta^*}} \min\left\{c_0^2, \frac{c_0}{nC(n)}\right\} \leq \frac{c_0}{1-\lambda_{\beta^*}}.$$

Since $\min_{\alpha\in\mathcal{A}} \lambda_\alpha > c_0$ by definition of $\widehat{A}_n$, the previous equation together with equation (26) implies equation (25). Moreover, $\mathcal{R}(\lambda, \overline{\pi}) = (\lambda', \pi^*)$ is of top type since

$$\lambda'_{\pi_0^{*-1}(d)} \geq \min_{\alpha\neq\beta^*} \lambda'_\alpha > \lambda'_{\beta^*} = \lambda'_{\pi_1^{*-1}(d)}. \qquad \square$$

Define, for any $n \geq 1$,

$$A_n := \left\{(\lambda, \pi) \in \Delta_\gamma \;\middle|\; \frac{A_\gamma^{-1}\lambda}{|A_\gamma^{-1}\lambda|_1} \in \widehat{A}_n\right\}.$$

CLAIM 2. *Let $n \geq 1$. There exists $N > 0$, depending only on γ, such that for any $(\lambda, \pi) \in A_n$, $\lambda^{(N)}$ satisfies equation (25), $\pi^{(N)} = \pi^*$, and $B^N(\lambda, \pi) = A_\gamma A_{\overline{\pi},1}$.*

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Proof of the claim. Fix $n \geq 1$. By definition of A_n and γ, any $(\lambda, \pi) \in A_n$ follows the path γ under Rauzy–Veech induction and $\mathcal{R}^{|\gamma|-1}(\lambda, \pi) \in \widehat{A}_n \times \{\overline{\pi}\}$. By Claim 1, the IET $\mathcal{R}^{|\gamma|-1}(\lambda, \pi)$ is of bottom type, $\mathcal{R}^{|\gamma|}(\lambda, \pi)$ is of top type with combinatorial datum π^*, and the length vector of $\mathcal{R}^{|\gamma|}(\lambda, \pi)$ satisfies equation (25).

Since the types of $\mathcal{R}^{|\gamma|-1}(\lambda, \pi)$ and $\mathcal{R}^{|\gamma|}(\lambda, \pi)$ are different, there exists $1 < N \leq |\gamma|$, depending only on γ, such that

$$\mathcal{Z}^N(\lambda, \pi) = (\lambda^{(N)}, \pi^{(N)}) = \mathcal{R}^{|\gamma|}(\lambda, \pi).$$

In particular, $\pi^{(N)} = \pi^*$ and $\lambda^{(N)}$ satisfies equation (25). Finally, we have

$$B^N(\lambda, \pi) = A^{|\gamma|}(\lambda, \pi) = A^{|\gamma|-1}(\lambda, \pi)A(\mathcal{R}^{|\lambda|-1}(\lambda, \pi)) = A_\gamma A_{\overline{\pi},1}. \qquad \square$$

As γ is positive, Δ_γ is compactly contained in $\Delta_d \times \mathfrak{R}$. In particular, since the sets A_n are contained in Δ_γ, they are also compactly contained in $\Delta_d \times \mathfrak{R}$. Therefore, since $\mu_{\mathcal{Z}}$ is equivalent to the Lebesgue measure in Δ_d and its density is uniformly bounded away from the boundary of Δ_d (see §2.2.7), there exists $c_2 > 0$, depending only on γ, such that

$$\mu(A_n) \geq c_2|A_n|.$$

Moreover, by definition of A_n and equation (24),

$$|A_n| \geq c_3|\widehat{A}_n| \geq \frac{c_1 c_3}{nC(n)}$$

for some $c_3 > 0$ depending only on γ. Hence,

$$\sum_{n \geq 1} \mu(A_n) = +\infty = \sum_{n \geq 1} \mu(A_{2n+N}),$$

since $C(n)$ is an increasing sequence verifying equation (14).

Recalling that $\Delta_\gamma \in \mathcal{P}_{\mathfrak{R}}^{|\gamma|}$ (see equation (21)), it follows by Theorem 5.7 that the set

$$\bigcap_{m \geq 1} \bigcup_{n \geq m} \mathcal{Z}^{-2n}(A_{2n+N})$$

has full measure in $\Delta_{\mathfrak{R},0}$. Thus, for a.e. $(\lambda, \pi) \in \Delta_{\mathfrak{R},0}$, there exists an increasing sequence $\{n_k\}_{k \geq 1} \subseteq \mathbb{N}$ such that

$$(\lambda^{(n_k-N)}, \pi^{(n_k-N)}) \in A_{n_k}$$

for all $k \geq 1$, where N is as in Claim 2. In particular, it follows from Claim 2 that $\pi^{(n_k)}$ and $\lambda^{(n_k)}$ verify conditions (1), (2), and (3).

By Claim 2, for any $k \geq 1$,

$$h^{(n_k)} = B^N(\lambda^{(n_k-N)}, \pi^{(n_k-N)})^T h^{(n_k-N)} = (A_\gamma A_{\overline{\pi},1})^T h^{(n_k-N)}.$$

Since $(A_\gamma A_{\overline{\pi},1})^T = A_{\overline{\pi},1}^T A_\gamma^T$ is positive, by equation (23) and the previous equation, there exists $c_\gamma > 0$, depending only on γ and d, such that

$$\frac{h_\alpha^{(n_k)}}{h_\beta^{(n_k)}} \geq c_\gamma \tag{27}$$

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for any $k \geq 1$ and any $\alpha, \beta \in \mathcal{A}$. By taking c_0 smaller if necessary, we may assume without loss of generality that $c_0 < c_\gamma$. This proves condition (4).

Since a.e. IET verifying Keane's condition admits an Oseledet's filtration associated to the heights cocycle B^T, we may assume that, for (λ, π) as above,

$$\lim_{n\to\infty} \frac{\log \|B^n(\lambda, \pi)^T\|}{n} = \theta_1,$$

where θ_1 is the biggest Lyapunov exponent associated with (λ, π) and B^T (see §2.2.9). Hence, as $h^{(n)} = B^n(\lambda, \pi)^T \overline{1}$ for all $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} \frac{\log |h^{(n)}|}{n} = \theta_1.$$

Notice that since $\mathcal{Z}$ is ergodic on $\Delta_d \times \mathfrak{R}$, the biggest Lyapunov exponent is the same for a.e. IET as above. Thus, assuming without loss of generality that $c_0 < \theta_1^{-1}$ and up to considering a subsequence of $\{n_k\}_{k\in\mathbb{N}}$, we may assume that condition (5) holds for all $k \geq 1$.

By considering the set

$$\mathcal{Z}^{-1}\bigg(\bigcap_{m\geq 1} \bigcup_{n\geq m} \mathcal{Z}^{-2n}(A_{2n+N+1})\bigg)$$

and recalling that $\mathcal{Z}^{-1}(\Delta_{\mathfrak{R},0}) = \Delta_{\mathfrak{R},1}$ (up to a zero measure set), it is easy to show that the same conclusions hold for a.e. $(\lambda, \pi) \in \Delta_{\mathfrak{R},1}$. This finishes the proof. □

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