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Separating H-sets by Open Sets

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Abstract. In an H-closed, Urysohn space, disjoint H-sets can be separated by disjoint open sets. This is not true for an arbitrary H-closed space even if one of the H-sets is a point. In this paper, we provide a systematic study of those spaces in which disjoint H-sets can be separated by disjoint open sets.

1 Introduction

Let X be a Hausdorff topological space and denote the topology by $\tau(X)$; $\mathrm{cl}_X(A)$ (respectively, $\mathrm{int}_X(A)$) will denote the closure (respectively, the interior) of A in X. The family $\{\mathrm{int}_X \mathrm{cl}_X(U) : U \in \tau(X)\}$ is an open base for a coarser Hausdorff topology. The space with the topology generated by the open base is denoted by X_s (note that $\tau(X_s) \subseteq \tau(X)$). The space X_s is called the *semiregularization* of X, and X is called *semiregular* if and only if $X = X_s$. Two nonempty disjoint subsets A and B can be separated by disjoint open sets in X if and only if they can be separated in X_s. A subset A of X is called an H-set [11] (respectively, an N-set) in X if, given any open cover $\mathcal{U}$ of A in X, there exists a finite subfamily $\{U_i : i = 1, 2, 3, \dots, n\} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^n \mathrm{cl}_X(U_i)$ (respectively, $A \subseteq \bigcup_{i=1}^n \mathrm{int}_X \mathrm{cl}_X(U_i)$.) Obviously, every N-set is an H-set, and N-sets and H-sets in a Hausdorff space are closed. Further, if A is an N-set in X, then A is compact in X_s. Now in a Hausdorff space, any two nonempty disjoint compact sets can be separated by disjoint open sets. Hence, any two nonempty disjoint N-sets can be separated by disjoint open sets in a Hausdorff space. If X is also regular, then N-sets and H-sets are all compact. Hence, in a T_3 (*i.e.,* regular and Hausdorff) space, any two nonempty disjoint H-sets can be separated by disjoint open sets. A space X is called *Urysohn* if given any two distinct points x, y in X, there exist open sets U and V in X such that $x \in U, y \in V$, and $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \varnothing$. A space X is called H-*closed* [1] if X is closed in every Hausdorff space in which X is embedded. A space X is called *completely*, *i.e., functionally*, Hausdorff if for any two distinct points x and y in X, there exists a continuous map $f\colon X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. H-sets in Hausdorff spaces are interesting but not completely understood. Here are some basic properties of H-sets.

Proposition 1.1 (i) *An H-set in a Hausdorff space is a closed set.*
(ii) *If X is a subspace of Y and A is an H-set in X, then A is an H-set in Y.*
(iii) *A set A in a space X is an H-set in X if and only if A is an H-set in X_s.*
(iv) *An H-closed subspace in a Hausdorff space is also an H-set.*
(v) *A space X is H-closed and Urysohn if and only if X_s is compact and Hausdorff.*

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(vi) *A subset A of a Urysohn H-closed space X is an H-set if and only if A is compact in X_s.*

(vii) *Two disjoint H-sets in a Urysohn, H-closed space can be separated by open sets.*

(viii) *Let A be an H-set in a space X and $U \in \tau(X)$ such that $A \subseteq U \subseteq X$. Then A is an H-set in U.*

Proof The results (i)–(vi) are contained in [9, 12] and (vii) is an immediate consequence of (vi). To show (viii), let $\mathcal{U}$ be $\tau(U)$-open cover of A. Now, $\mathcal{U}$ is a $\tau(X)$-open cover of A. There is a finite subfamily $\mathcal{S}$ such that $A \subseteq \mathrm{cl}_X(\cup\mathcal{S})$. Thus, $A \subseteq U \cap \mathrm{cl}_X(\cup\mathcal{S}) = \mathrm{cl}_U(\cup\mathcal{S})$. ■

Remark In general, the converse of Proposition 1.1(ii) is false. The usual example is to use an H-set that is discrete in a space. For example, the discrete subspace $\kappa\omega\backslash\omega$ is an H-set in $\kappa\omega$ as noted in [9] (this follows from the fact that $(\kappa\omega)_s = \beta\omega$ and thus $\kappa\omega\backslash\omega$ is an H-set by Proposition 1.1(iii)). However, as noted in Proposition 1.1(v), the converse is true when the intermediate set is open.

Theorem 1.2 *Suppose A is an H-set in a Hausdorff space X and let $x \in X\backslash A$. Then there exists open sets V and W in X such that $A \subseteq \mathrm{cl}_X V$, $x \in W$, and $W \cap \mathrm{cl}_X V = \varnothing$.*

Proof For each $a \in A$, there exist open sets O_a and O_x such that $a \in O_a$, $x \in O_x$, and $O_a \cap O_x = \varnothing$. The family $\mathcal{U} = \{O_a : a \in A\}$ is an open cover of the H-set A in X. Hence, there exists a finite subset $B \subseteq A$ such that $A \subseteq \mathrm{cl}_X \bigcup\{O_a : a \in B\} = \bigcup\{\mathrm{cl}_X O_a : a \in B\}$. Take $V = \bigcup\{O_a : a \in B\}$ and $W = \bigcap\{W_a : a \in B\}$ to complete the proof. ■

Theorem 1.2 emphasizes the differences between separating a point from a compact set and separating a point from an H-set.

Corollary 1.3 *Suppose A and B are two disjoint, nonempty H-sets in X. Then there exist open sets U and V in X such that $A \subseteq \mathrm{cl}_X U$, $B \cap U = \varnothing$, $B \subseteq \mathrm{cl}_X V$, and $A \cap V = \varnothing$.*

Theorem 1.4 (i) *Suppose $X = \bigoplus\{X_i : 1 \leq i \leq n\}$ is a topological sum of spaces $\{X_i : 1 \leq i \leq n\}$ and A_i is an H-set in X_i for each i. Then the subspace $\bigcup\{A_i : 1 \leq i \leq n\}$ is an H-set in X.*

(ii) *The product of two H-sets is an H-set.*

Proof The proof of (i) is straightforward. To prove (ii), let A be an H-set in X and B an H-set in Y. We want to show that $A\times B$ is an H-set in $X\times Y$. Let $\mathcal{C}$ be an open cover of $A\times B$ using open sets in $X\times Y$. We can assume that $\mathcal{C} = \{U_{ab}\times V_{ab} : (a, b) \in A\times B\}$ where U_{ab} is an open set in X containing a and V_{ab} is an open set in Y containing b. Fix $a \in A$. Then $\{V_{ab} : b \in B\}$ is an open cover of B, and there is a finite subset $F_a \subseteq B$ such that $B \subseteq \mathrm{cl}_Y(\bigcup\{V_{ab} : b \in F_a\})$. Let $U_a = \bigcap\{U_{ab} : b \in F_a\}$. Note that $a \in U_a$.

Then

$$U_a \times B \subseteq U_a \times \mathrm{cl}_Y(\bigcup\{V_{ab} : b \in F_a\}) \subseteq \mathrm{cl}_X U_a \times \mathrm{cl}_Y(\bigcup\{V_{ab} : b \in F_a\})$$
$$= \mathrm{cl}_{X\times Y}(U_a \times (\bigcup\{V_{ab} : b \in F_a\}) = \mathrm{cl}_{X\times Y}(\bigcup\{U_a \times V_{ab} : b \in F_a\}).$$

Note that $\{U_a : a \in A\}$ is an open cover of A and there is a finite subset $G \subseteq A$ such that $A \subseteq \mathrm{cl}_X(\bigcup\{U_a : a \in G\})$. Let $F = \bigcup\{F_a : a \in G\}$. Now

$$\begin{aligned} A \times B &\subseteq \mathrm{cl}_X(\textstyle\bigcup\{U_a : a \in G\}) \times B \\ &\subseteq \mathrm{cl}_{X\times Y}(\textstyle\bigcup\{U_a : a \in G\}) \times B) \\ &= \mathrm{cl}_{X\times Y}(\textstyle\bigcup\{U_a \times B : a \in G\}) \\ &\subseteq \mathrm{cl}_{X\times Y}(\textstyle\bigcup\{\mathrm{cl}_{X\times Y}(\bigcup\{U_a \times V_{ab} : b \in F_a\}) : a \in G\}) \\ &= \mathrm{cl}_{X\times Y}(\mathrm{cl}_{X\times Y}(\textstyle\bigcup\{\bigcup\{U_a \times V_{ab} : b \in F_a\} : a \in G\})) \\ &\subseteq \mathrm{cl}_{X\times Y}(\textstyle\bigcup\{\bigcup\{U_{ab} \times V_{ab} : b \in F_a\} : a \in G\}) \\ &\subseteq \mathrm{cl}_{X\times Y}(\textstyle\bigcup\{U_{ab} \times V_{ab} : (a,b) \in G \times F\}). \end{aligned}$$

This shows that $A \times B$ is an H-set in $X \times Y$. ■

Corollary 1.5 *The finite product of H-sets is an H-set.*

Remark We do not know whether the infinite product of H-sets is also an H-set. Vermeer [12, (4.6)] has conjectured that if A is an H-set in X, there is a compact Hausdorff space Y and a θ-continuous function $f : Y \to X$ such that $f[Y] = A$. If this conjecture is correct, then the infinite product of H-sets is an H-set.

Example 1.6 ([7]) Let $X = \mathbb{R} \cup \{p, q\}$ where $\mathbb{R}$ is the space of reals with the usual topology and p, q are elements not in $\mathbb{R}$. A set U is defined to be open if $U \cap \mathbb{R}$ is open in $\mathbb{R}$ and $p \in U$ (respectively, $q \in U$) implies that for some $m \in \mathbb{N}, \bigcup\{(2n, 2n+1) \cup (-2n-1, -2n) : n \geqq m\} \subseteq U$ (respectively, $\bigcup\{(2n-1, 2n) \cup (-2n, -2n+1) : n \geqq m\} \subseteq U$). The space X is an H-closed space but the H-sets $\{q\}$ and $\{p\} \cup \mathbb{N}$ in X cannot be separated by open sets in X. Also, the intersection of two H-sets $\{p\} \cup \mathbb{N}$ and $\{q\} \cup \mathbb{N}$ is $\mathbb{N}$, which is not an H-set in X.

Example 1.7 ([10]) Let

$$X = \Big\{ \Big(\frac{1}{n}, \frac{1}{m}\Big) : n \in \mathbb{N}, |m| \in \mathbb{N} \Big\} \cup \Big\{ \Big(\frac{1}{n}, 0\Big) : n \in \mathbb{N} \Big\} \cup \{(0,1), (0,-1)\}.$$

Let $\mathcal{V}$ be an ultrafilter on $\mathbb{N}$. Topologize X as follows: a set $U \subseteq X$ is open in X if and only if $U \cap (X \backslash \{(0,1), (0,-1)\})$ is open in the topology induced by the usual topology of the plane $\mathbb{R}^2$ and if $(0,1) \in U$ (respectively, $(0,-1) \in U$), then there is a set $K \in \mathcal{V}$ such that $\{(\frac{1}{n}, \frac{1}{m}) : n \in K, m \in \mathbb{N}\} \subseteq U$ (respectively, $\{(\frac{1}{n}, -\frac{1}{m}) : n \in K, m \in \mathbb{N}\} \subseteq U$). Then X is a non-Urysohn (hence, non regular), non-H-closed Hausdorff space such that every H-set in X is compact, whence any two disjoint nonempty H-sets in X can be separated by disjoint open sets in X.

Theorem 1.8 *In a completely Hausdorff space X, disjoint H-sets can be separated by a real-valued continuous function.*

Proof The first step is to show that a point $a \in X$ and an H-set B such that $a \notin B$ can be separated by a real-valued continuous function. For each point $b \in B$, there is a real-valued continuous function f_b such that $f_b(b) = 0$ and $f_b(a) = 1$. Define $g_b = \max\{f_b, \frac{1}{4}\} - \frac{1}{4}$. Note that $g_b^{\leftarrow}(0) = f_b^{\leftarrow}[(-\infty, \frac{1}{4}]]$ is a neighborhood of b. Let $z(g_b) = g_b^{\leftarrow}(0)$. Then the neighborhood cover $\{z(g_b) : b \in B\}$ of B has a finite subfamily $\{z(g_b) : b \in A\}$ for some finite $A \subseteq B$ such that $B \subseteq \bigcup \mathrm{cl}_X(\{z(g_b) : b \in A\})$. Let $g = \min\{g_b : b \in A\}$. Now $z(g_b) \subseteq z(g)$ for all $b \in A$ and $g(a) = \frac{3}{4}$. Thus, $B \subseteq \bigcup \mathrm{cl}_X(\{z(g_b) : b \in A\}) \subseteq \mathrm{cl}_X(z(g)) = z(g)$. Thus, a and B are separated by g. The final step of separating two disjoint H-sets B and C by a real-valued continuous function is similar to the first step. ■

By Proposition 1.1(vii), two disjoint nonempty H-sets of a Urysohn H-closed space can be separated by open sets in X. Actually, more is true, using Theorem 1.8. As X_s is compact Hausdorff, both X_s and X are completely Hausdorff. By Theorem 1.8, disjoint H-sets in X can be separated by a real-valued continuous function.

A function $f\colon X \to Y$ between two spaces X and Y is *θ-continuous* [4] at $x \in X$ if for each open neighborhood V of $f(x)$ in Y there is an open neighborhood U of x in X such that $f[\mathrm{cl}_X] \subseteq \mathrm{cl}_Y V$. If f is θ-continuous at each x, then f is said to be θ-continuous on X. A *θ-homeomorphism* is a bijection $f\colon X \to Y$ such that both f and its inverse $f^{\leftarrow}$ are θ-continuous. A function $f\colon X \to Y$ is called *irreducible* if f is onto and for each proper subset A of X, $f[A] \neq Y$. A function $f\colon X \to Y$ is called *compact* if for each $y \in Y$, $f^{\leftarrow}(y) = \{x \in X : f(x) = y\}$ is compact in X; f is called a *perfect* map if f is both closed and compact. For any Hausdorff space X there exists an extremally disconnected zero-dimensional space EX, called the *absolute* of X and a perfect, irreducible θ-continuous surjection $k_X\colon \mathrm{EX} \to X$ [6].

Remark By Theorem 1.8, for a Hausdorff space X, disjoint H-sets in EX can be separated by a real-valued continuous function.

2 Separation Properties

In this section, we examine closely those spaces in which disjoint H-sets can be separated by disjoint open sets or disjoint open sets whose closures are also disjoint. We will use the following symbols to classify spaces with separation properties for H-sets.

A space X has property λ_1 (respectively, $c\lambda_1$) if, whenever B is any nonempty H-set in X and $x \in X\backslash B$, there are open sets U and V in X such that $x \in U, B \subseteq V$, and $U \cap V = \varnothing$ (respectively, $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \varnothing$).

A space X has property λ_2 (respectively, $c\lambda_2$) if, whenever A and B are disjoint, nonempty H-sets in X, there are open sets U and V in X such that $A \subseteq U, B \subseteq V$, and $U \cap V = \varnothing$ (respectively, $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \varnothing$).

Recall [8] that a space X is S(3) (respectively S(4)) if for every pair of distinct points p, q, there are open sets $\{U_0, U_1, V_0, V_1\}$ such that $p \in U_0 \subseteq \mathrm{cl}\, U_0 \subseteq U_1$, $q \in V_0 \subseteq \mathrm{cl}\, V_0 \subseteq V_1$, and $U_1 \cap V_1 = \varnothing$ (respectively, $\mathrm{cl}\, U_1 \cap \mathrm{cl}\, V_1 = \varnothing$). Also, recall that S(2) is the same as Urysohn and S(1) is the same as Hausdorff.

Theorem 2.1 (i) *An* S(4) *space X has property $c\lambda_2$.*
(ii) *An* S(3) *space X has property $c\lambda_1$.*

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(iii) *A Urysohn space X has property λ_1.*
(iv) *A space X with property $c\lambda_1$ is Urysohn and has property λ_2.*
(v) *A space X with property λ_1 is Hausdorff, and a compact set disjoint from an H-set can be separated by disjoint open sets.*
(vi) *An H-closed space X is Urysohn if and only if X is λ_1*

Proof The proofs of (iv) and (v) are clear. The proofs of (i)–(iii) are similar. We provide the proof for (ii). Let A be an H-set in an S(3) space X and $p \notin A$. For each $a \in A$, there are open sets U_a^0, U_a^1 and V_a^0, V_a^1 such that $a \in U_a^0 \subseteq \operatorname{cl} U_a^0 \subseteq U_a^1$, $p \in V_a^0 \subseteq \operatorname{cl} V_a^0 \subseteq V_a^1$, and $U_a^1 \cap V_a^1 = \varnothing$. There is a finite subset $F \subseteq A$ such that $A \subseteq \bigcup\{\operatorname{cl} U_a^0 : a \in F\}$. Let $V_0 = \bigcap\{V_a^0 : a \in F\}$ and $V_1 = \bigcap\{V_a^1 : a \in F\}$. Note that $A \subseteq \bigcup\{U_a^1 : a \in F\}$, $p \in V_0 \subseteq \operatorname{cl} V_0 \subseteq V_1$ and $V_1 \cap (\bigcup\{U_a^1 : a \in F\}) = \varnothing$. Thus, $\operatorname{cl} V_0 \cap \operatorname{cl}(\bigcup\{U_a^1 : a \in F\}) = \varnothing$. This shows that X is $c\lambda_1$. To prove (vi), one direction follows from (iii). To show the other direction, recall [9] that in an H-closed space, the closure of an open set is H-closed and by Proposition 1.1(iv), an H-closed subspace is an H-set. Thus, it follows that for an H-closed space X with property λ_1, the semiregularization X_s is regular, and hence compact. So X is Urysohn. ■

Corollary 2.2 (i) *A space that is T_3 or completely Hausdorff has property $c\lambda_2$.*
(ii) *A semiregular, H-closed space with property λ_1 is compact.*

We now have this implication diagram:

$$\begin{array}{ccccccccc}
S(4) & \longrightarrow & S(3) & \longrightarrow & c\lambda_1 & \longrightarrow & \text{Urysohn} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
c\lambda_2 & \longrightarrow & c\lambda_1 & \longrightarrow & \lambda_2 & \longrightarrow & \lambda_1 & \longrightarrow & \text{Hausdorff}
\end{array}$$

Remark Recall [9] that for an open set U in a space X, $\operatorname{cl}_X U = \operatorname{cl}_{X_s}(\operatorname{int}_X \operatorname{cl}_X U)$. Now $\operatorname{int}_X \operatorname{cl}_X U \in \tau(X_s)$. Thus, X has property λ_1 (respectively, $\lambda_2, c\lambda_1, c\lambda_2$) if and only if X_s has property λ_1 (respectively, $\lambda_2, c\lambda_1, c\lambda_2$).

Example 2.3 **(A space with property λ_1 but not λ_2.)** Consider the Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1) \backslash \{(\omega_1, \omega)\}$ The space T is a zero-dimensional dense subspace of the compact Hausdorff product space $(\omega_1 + 1) \times (\omega + 1)$. A slight modification of the technique listed in [9, 2R] shows that

(i) If U is an open set in T, C is a cofinal subset of ω and $\{\omega_1\} \times C \subseteq U$, there is some $\alpha \in \omega_1$ such that $[\alpha, \omega_1) \times C \subseteq U$ and $[\alpha, \omega_1) \times \{\omega\} \subseteq \operatorname{cl}_T U$.
(ii) If U is an open set in T and $[\alpha, \omega_1) \times \{\omega\} \subseteq U$ for some $\alpha \in \omega_1$, then $\{\omega_1\} \times [n, \omega) \subseteq \operatorname{cl}_T U$ for some $n \in \omega$.

Now let $Z = T \times \omega$. For $\alpha \in \omega_1$ identify (α, ω, n) and $(\alpha, \omega, n+1)$ if n is odd. For $m \in \omega$ identify (ω_1, m, n) and $(\omega_1, m, n+1)$ if n is even. Call the resulting space Y. For $n \geq 2$, let Y_n be the image of $T \times \{0, 1, \ldots, n\}$, and let $T_i = T \times \{i\}$, $i = 0, 1, 2, \ldots$.

Let $S_2 = \{a, b\} \cup Y_2$, where $\{a, b\} \cap Y_2 = \varnothing$. A set $U \subseteq S_2$ is open in S_2 if $U \cap Y_2$ is open in Y_2 and $a \in U$ (respectively, $b \in U$) implies for some $\alpha \in \omega_1$, $((\alpha, \omega_1) \times \omega \times \{0\}) \subseteq U$ (respectively, $((\alpha, \omega_1) \times \omega \times \{2\}) \subseteq U$).

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The subspace Y_2 of S_2 is a zero-dimensional and locally compact space. To show that S_2 is Urysohn, we need to show only that the points a and b can be S_2-separated. Consider the open neighborhoods $U = \{a\} \cup (\omega_1 \times \omega \times \{0\})$ of a and $V = \{b\} \cup (\omega_1 \times \omega \times \{2\})$ of b. Note that $\mathrm{cl}_{S_2} U = \{a\} \cup T_0$, $\mathrm{cl}_{S_2} V = \{b\} \cup T_2$, and $\mathrm{cl}_{S_2} U \cap \mathrm{cl}_{S_2} V = \varnothing$.

So S_2 is λ_1 by Theorem 2.1(iii). The sets $A = \{a\} \cup (\{\omega_1\} \times \omega \times \{0\})$ and $B = \{b\} \cup (\omega_1 \times \{\omega\} \times \{2\})$ are disjoint H-sets. By (i), for any open set W of S_2 that contains A, $\mathrm{cl}_{S_2} W$ contains $[\alpha, \omega_1) \times \{\omega\} \times \{2\}$ for some $\alpha \in \omega_1$. So $(\alpha, \omega, 2) \in \mathrm{cl}_{S_2} W \cap B$. Thus, A and B cannot be separated by disjoint open sets and S_2 is not λ_2. Since S_2 is not λ_2, it cannot be $c\lambda_1$.

***Example 2.4* (A space with property $c\lambda_1$ but not $c\lambda_2$)** We will use the notation of Example 2.3. Let $S_3 = \{a, b\} \cup Y_3$, where $\{a, b\} \cap Y_3 = \varnothing$. A set $U \subseteq S_3$ is open in S_3 if $U \cap Y_3$ is open in Y_3 and $a \in U$ (respectively, $b \in U$) implies for some $\alpha \in \omega_1, ((\alpha, \omega_1) \times \omega \times \{0\}) \subseteq U$ (respectively, $((\alpha, \omega_1) \times \omega \times \{3\}) \subseteq U$).

The subspace Y_3 is a zero-dimensional and locally compact space. To show that space S_3 is S(3), we need to show only that the points a and b can be S(3)-separated. Consider the open neighborhoods $U_0 = \{a\} \cup (\omega_1 \times \omega \times \{0\})$ and $U_1 = \{a\} \cup T_0 \cup (\omega_1 \times \omega \times \{1\})$ of a and $V_0 = \{b\} \cup (\omega_1 \times \omega \times \{3\})$ and $V_1 = \{b\} \cup T_3 \cup (\omega_1 \times \omega \times \{2\})$ of b. Note that $U_1 \cap V_1 = \varnothing$, $\mathrm{cl}_{S_3} U_0 = \{a\} \cup T_0 \subseteq U_1$, and $\mathrm{cl}_{S_3} V_0 = \{b\} \cup T_3 \subseteq V_1$. This shows that the points a and b can be S(3)-separated.

So S_3 is $c\lambda_1$ by Theorem 2.1(ii). The subspaces $\{a\} \cup T_0$ and $\{b\} \cup T_3$ are H-closed and disjoint. The sets $A = \{a\} \cup (\{\omega_1\} \times \omega \times \{0\})$ and $B = \{b\} \cup (\{\omega_1\} \times \omega \times \{3\})$ are H-sets and disjoint. By Example 2.3(i) for any open set U of S_3 that contains A, $\mathrm{cl}_{S_3} U$ contains $[\alpha, \omega_1) \times \{\omega\} \times \{2\}$ for some $\alpha \in \omega_1$. By Example 2.3(ii) for any open set V of S_3 that contains B, $\mathrm{cl}_{S_3} V$ contains $[\beta, \omega_1) \times \{\omega\} \times \{2\}$ for some $\beta \in \omega_1$. So $(\max\{\alpha, \beta\}, \omega, 2) \in \mathrm{cl}_{S_3} U \cap \mathrm{cl}_{S_3} V$. Thus, A and B cannot be separated by disjoint open sets whose closures are disjoint and this shows that S_3 is not $c\lambda_2$.

***Example 2.5* (A space with property λ_2 but not $c\lambda_1$)** This space is a modification of the space described in Example 2.3. Let $Y_2' = Y_2 \backslash (\omega_1 \times \{\omega\} \times \{0\})$, $S_2' = \{a, b\} \cup Y_2'$, where $\{a, b\} \cap Y_2' = \varnothing$. Let $\mathcal{U}$ be a free ultrafilter on ω. A set $U \subseteq S_2'$ is open in S_2' if $U \cap Y_2'$ is open in Y_2', $a \in U$ implies for some $\alpha \in \omega_1$ and $V \in \mathcal{U}$, $((\alpha, \omega_1) \times V \times \{0\}) \subseteq U$, and $b \in U$ implies for some $\alpha \in \omega_1, (\alpha, \omega_1] \times \omega \times \{2\} \subseteq U$. The space S_2' is Urysohn (and hence λ_1 by Theorem 2.1(iii)), $\{b\} \cup T_2$ is H-closed, and Y_2 is a zero-dimensional, locally compact space.

The set $B = \{b\} \cup (\omega_1 \times \{\omega\} \times \{2\})$ is an H-set but the set $A = \{a\} \cup (\{\omega_1\} \times \omega \times \{0\})$ is not. Clearly a and B can be separated by disjoint open sets. Let V be an open set in S_2' that contains B. By Example 2.3(ii), $\mathrm{cl}_{S_2'} V$ meets a tail of A. However, for any open set U of a, $\mathrm{cl}_{S_2'} U$ contains a cofinal subset of A. Thus, a and B cannot be separated by open sets whose closures are disjoint. This shows that S_2' is not $c\lambda_1$.

To show that S_2' is λ_2, let C and D be disjoint H-sets. If C or D is compact, we are done by Theorem 2.1(v). If $C \cap \{a, b\} = \varnothing$, then C is an H-set in the T_3 subspace $S_2' \backslash \{a, b\}$ by Proposition 1.1(v). Thus, C is a compact subspace. We can assume that $a \in C$ and $b \in D$.

Let $S = \{n \in \omega : (\omega_1 + 1) \times \{n\} \times \{0\} \cap C \neq \varnothing\}$. Assume that $S \in \mathcal{U}$. Then there is some $R \in \mathcal{U}$ such that $R \subset S$ and $S \backslash R$ is infinite. Let $U = \{a\} \cup (\omega \times R \times \{0\})$; for $n \in$

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$S\, V_n = (\omega_1+1)\times\{n\}\times\{0,1\}$ (note that V_n is clopen in S_2'); $V = \bigcup\{V_n : n \in S\}$; and for $p \in C\backslash(\{a\}\cup V)$, there is a clopen set W_p in S_2' such that $p \in W_p \subseteq T_1\cup T_2\cup\{b\}$. So, $U\cup\{V_n : n \in S\}\cup\{W_p : p \in C\backslash(\{a\}\cup V)$ is an open cover of the H-set C. There are finite subsets $F \subset S$ and $G \subseteq C\backslash(\{a\}\cup V)$ such that $C \subseteq \mathrm{cl}_{S_2'} U \cup \bigcup\{V_n : n \in F\}\cup\bigcup\{W_p : p \in G\}$. Note that $\mathrm{cl}_{S_2'} U \subseteq \{a\}\cup(\omega_1+1)\times R\times\{0\}$. For $m \in S\backslash(R\cup F)$, $C \cap ((\omega_1+1)\times\{m\}\times\{0\}) \neq \varnothing$. But $((\omega_1+1)\times\{m\}\times\{0\}) \cap [\mathrm{cl}_{S_2'} U \cup \bigcup\{V_n : n \in F\}\cup\bigcup\{W_p : p \in G\}] = \varnothing$. This contradiction implies that $S \notin \mathcal{U}$. Hence there is some $P \in \mathcal{U}$ such that $((\omega+1)\times P\times\{0\}) \cap C = \varnothing$. This shows that $C\backslash\{a\}$ is an H-set in a T_3 space and hence compact. It follows that C is compact.

Added in Proof R. Hodel [5] has shown that an infinite product of H-sets is an H-set. This answers the question contained in the remark following Corollary 1.5.

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