# Remark on Zero Sets of Holomorphic Functions in Convex Domains of Finite Type

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*Abstract.* We prove that if the $(1, 1)$-current of integration on an analytic subvariety $V \subset D$ satisfies the uniform Blaschke condition, then $V$ is the zero set of a holomorphic function $f$ such that $\log |f|$ is a function of bounded mean oscillation in $bD$. The domain $D$ is assumed to be smoothly bounded and of finite d'Angelo type. The proof amounts to non-isotropic estimates for a solution to the $\bar{\partial}$-equation for Carleson measures.

## 1 Statement of result

J. Bruna, P. Charpentier, and Y. Dupain [3] solved the Henkin–Skoda problem for smoothly bounded convex domains of strict finite type [11]. Recall that the problem amounts to proving that the Blaschke condition characterizes zero varieties of functions belonging to the Nevanlinna class. K. Diederich and J. E. Fornaess provided a construction of support functions for each smoothly bounded convex domain of finite type [7]. K. Diederich and E. Mazzilli solved the Henkin–Skoda problem completely for convex domains of finite type, *i.e.,* without the assumption that a domain is additionally of strict type [8]. This fact was also proved by A. Cumenge [6]. Interestingly, her method relies on estimates of the Bergman kernel obtained in [13].

The problem of characterizing zero varieties of several variable functions is much more complicated than the one variable analog. For instance, it is not known how to characterize zero varieties of bounded holomorphic functions. However, it was N. Varopoulos ([15, 16], see also [1]) who formulated a condition on a subvariety $V$ of a strictly pseudoconvex domain that guarantees that there exists a function $f$ such that $\log |f| \in \text{BMO}(bD)$ and $V$ is the zero variety of $f$. We will recall this condition, known as the uniform Blaschke condition, later on (Definition 2.5). J. Bruna and S. Grellier [4] proved that if $D$ is of finite strict type, then the uniform Blaschke condition implies that $V$ is the zero set of $f \in \mathcal{H}^p$ for some $p > 0$.

The aim of this paper is to complete the picture by proving the following result.

**Theorem 1.1** *Assume that $D \subset \mathbb{C}^n$, $n > 1$ is a smoothly bounded convex domain of finite (not necessarily strict) type. Assume that $V$ is a subvariety of $D$ satisfying the uniform Blaschke condition, then there exists a holomorphic function $u$ with $\log |u| \in$ BMO$(bD)$ whose zero set is equal to $V$.*

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In view of results obtained in [3,4] the proof follows from the regularity result for the $\overline{\partial}$-equation.

**Theorem 1.2** *Assume that $D \subset \mathbb{C}^n$, $n > 1$ is a smoothly bounded convex domain of finite type. There exists an operator $K$ such that for a $\overline{\partial}$-closed smooth $(0,1)$-form $f$*

(i)  $\overline{\partial} K f = f$,
(ii)  $K f \in \mathrm{BMO}(bD)$,

*provided $|f|_\kappa dV$ is a Carleson measure in $D$.*

The symbol $|\cdot|_\kappa$ stands for a suitably defined non-isotropic norm of a $(0,1)$-form $f$ and $\mathrm{BMO}(bD)$ denotes the space of functions of bounded mean oscillation. Both definitions will be formulated in the next section.

We will briefly sketch the standard argument, which reduces the proof of Theorem 1.1 to Theorem 1.2. Each divisor in $D$ defines a $(1,1)$-current in $D$. By the Poincaré–Lelong theorem, if $\Theta$ is a divisor of a meromorphic function $f$, then in the sense of currents

$$\Theta = \frac{\sqrt{-1}}{\pi} \partial \overline{\partial} \log |f|.$$

This reduces the question whether a subvariety $V$ is the zero set of a holomorphic function belonging to some function space to the regularity problem for the $\partial \overline{\partial}$-equation. More specifically, in order to prove Theorem 1.2 we need to find a real-valued solution to the equation

$$(1.1) \qquad\qquad \frac{\sqrt{-1}}{\pi} \partial \overline{\partial} u = \Theta,$$

for a positive closed $(1,1)$-current $\Theta$. Indeed, a standard cohomological argument shows that if $D$ is convex and $u$ solves (1.1), then it holds $u = \log |f|$ with a holomorphic function $f$.

Equation (1.1) is solved in the following two steps. First, one solves $\sqrt{-1}dW = \theta$ in such a way that $W = -\overline{W}$. Next, if $v$ is a solution to $\overline{\partial} v = W_{0,1}$, then the function $u = 2\Re v$ satisfies the condition $\sqrt{-1}\partial\overline{\partial} u = \Theta$. Importantly, Bruna–Grellier proved the following result [4].

**Theorem 1.3** *Let $D$ be a smoothly bounded domain of finite type in $\mathbb{C}^n$, $n > 1$. Assume that $\Theta$ is a closed positive $(1,1)$-current satisfying the uniform Blaschke condition. Then there exists a real-valued solution to $dw = \Theta$ such that $|w|_\kappa$ is a Carleson measure.*

Observe that this fact together with Theorem 1.2 suffices to complete the proof of Theorem 1.1. Indeed, a standard regularization and approximation argument allows us to assume that $f$ in Theorem 1.2 is smooth. As for the method of the proof of Theorem 1.2, we are guided by results and methods contained in the above-mentioned papers: [3,4,7,8]. We will also make use of estimates obtained in [9,10].

## 2   Preliminaries

Let $D \subset \mathbb{C}^n$, $n > 1$ be a smoothly bounded convex domain of finite type $M$ defined by a smooth function $r$, which is non-degenerate on $bD$. We assume that the reader

is familiar with the definition and properties of domains of finite type (see [12, 13]). This is why we will recall only some basic concepts here. Namely, for $\varepsilon > 0$ and a vector $v \neq 0$ one defines a complex directional boundary distance

$$(2.1) \qquad \tau(z, v, \varepsilon) := \sup\{\delta > 0 : |r(z + \lambda v) - r(z)| \leq \varepsilon, |\lambda| \leq \delta\}.$$

We keep to a rather standard notation and write $\tau_k(z, \varepsilon)$, $1 \leq k \leq n$ in order to denote $\tau(z, v_k, \varepsilon)$ when $v_1, \ldots, v_n$ is an $\varepsilon$-extremal basis at $z$ [12, 13].

Next, following [3], we introduce a non-isotropic norm on the space of covectors at $z \in D$. If $\theta$ is a smooth 1-form, then one defines $|\theta(z)|_\kappa$ as

$$|\theta(z)|_\kappa = \sup\left\{ |\theta(z)(v)| \frac{\tau(z, v, \varrho/2)}{\varrho} : v \neq 0 \right\},$$

where $\varrho = |r(z)|$. Also, if $\Theta$ is a $(1, 1)$-form, one sets

$$|\Theta(z)|_\kappa = \sup\left\{ |\theta(z)(v_1, v_2)| \frac{\tau(z, v_1, \varrho/2)\tau(z, v_2, \varrho/2)}{\varrho^2} : v_1, v_2 \neq 0 \right\}.$$

Definition of the boundary distances (2.1) is the first step in turning some neighbourhood of $\bar{D}$ (and consequently $bD$) into a space of homogeneous type (see [5] for the definition and properties). In this case the structure consists of a pseudometric $d$ and the Lebesgue measure. The pseudometric $d$ is defined as

$$d(z, \zeta) := \inf\{\varepsilon > 0 : \zeta \in P_\varepsilon(z), z \in P_\varepsilon(\zeta)\},$$

where $P_\varepsilon(z)$ is a polydisk defined by means of the $\varepsilon$-extremal basis.

We must again refer the reader to [12, 13] for definitions and properties of these objects. However, for the sake of completeness we will gather, mostly following the exposition in [9], some of their basic properties below.

**Proposition 2.1** (i) *For each $c > 0$ there exists $b(c) > 0$ such that*

$$P_{c\varepsilon}(z) \subset bP_\varepsilon(z), cP_\varepsilon(z) \subset P_{b\varepsilon}(z).$$

(ii) *We have $\tau_1(z, \varepsilon) \sim \varepsilon$ and $\varepsilon^{1/2} \lesssim \tau_k(z, \varepsilon) \lesssim \varepsilon^{1/M}$, $k = 2, \ldots, n$.*
(iii) *For $\zeta \in P_\varepsilon(z)$ we have $|z - \zeta| \lesssim \varepsilon^{1/M}$ and $\zeta \notin P_\varepsilon(z)$ implies $|z - \zeta| \gtrsim \varepsilon$.*
(iv) *Let $w$ be any orthonormal coordinate system centred at $z$ and $v_j$ be the unit vector in the $w_j$-direction. Then*

$$\left| \frac{\partial^{|\alpha+\beta|} r(z)}{\partial w^\alpha \partial \bar{w}^\beta} \right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, v_j, \varepsilon)^{\alpha_j + \beta_j}}.$$

We will need the following elementary fact.

**Lemma 2.2** *Let $p$ be an arbitrary point in $bD$ and assume that $\varepsilon > 0$ is sufficiently small. For any $\zeta, \eta \in P_\varepsilon(p) \cap bD$ there exists a smooth curve $\gamma_{\zeta,\eta} \colon [0, 1] \to bD \cap P_\varepsilon(p)$ such that $\gamma_{\zeta,\eta}(0) = \zeta$ and $\gamma_{\zeta,\eta}(1) = \eta$ satisfying the condition*

$$|\Re(\gamma'_{\zeta,\eta})_i| + |\Im(\gamma'_{\zeta,\eta})_i| \lesssim \tau_i(p, \varepsilon),$$

$i = 1, \ldots, n$. *The involved constants are uniform with respect to $p, \zeta$ and $\eta$ and $\varepsilon > 0$.*

**Proof** Fix $p$ and let $\Phi$ be a projection onto the tangent space at $p$. Choose an $\varepsilon$-extremal basis $v_1, \ldots, v_n$ at $p$. We may assume that the equation of the tangent space at $p$ is $\Re w_1 = 0$, where $w_1$ stands for the coordinate with respect to $v_1$. Consequently, the set $\Phi(bD \cap P_\varepsilon(p))$ is now precisely equal to the set of all these points in $P_\varepsilon(p)$ which satisfy the condition $\Re w_1 = 0$. Choose a line segment $\gamma \colon [0,1] \to \Phi(bD \cap P_\varepsilon(p))$ with $\gamma(0) = \zeta$ and $\gamma(1) = \eta$.

Now it is enough to take $\gamma_{\zeta,\eta} := \Phi^{-1} \circ \gamma$. Since $\Phi$ is a projection, its Jacobian matrix in $\varepsilon$-extremal coordinates at $p$ is a block matrix of the form $(I|\operatorname{grad} r)^T$, where $bD$ is locally near $p$ a graph of $r$ over the tangent plane at $p$. Therefore, we have trivially uniform estimates by $\tau_j(p, \varepsilon)$, $j = 2, \ldots, n$ for both the real and the imaginary part of $(\gamma_{\zeta,\eta})'_j$. The same holds true for the imaginary part of $(\gamma_{\zeta,\eta})'_1$. As for the real part of $(\gamma_{\zeta,\eta})'_1$, observe that the implicit function theorem gives the estimate

$$
|\Re(\gamma'_{\zeta,\eta})_1(t)| \lesssim \sum_{i=1}^n \frac{\varepsilon}{\tau_i(p,\varepsilon)} \left( \frac{\varepsilon}{\tau_1(p,\varepsilon)} \right)^{-1} \tau_i(p,\varepsilon) \sim \tau_1(p,\varepsilon),
$$

by Proposition 2.1. ■

Next we recall the definition of the space of functions of bounded mean oscillation on $bD$ and the concept of a Carleson measure.

**Definition 2.3** The space $\mathrm{BMO}(bD)$ consists of all functions $f$, which are locally integrable in $bD$ with respect to the surface measure $\sigma$ and satisfy the condition

$$
\|f\|_{\mathrm{BMO}} := \sup_{\substack{\varepsilon > 0 \\ p \in bD}} \frac{1}{\sigma(bD \cap P_\varepsilon(p))} \int_{P_\varepsilon(p) \cap bD} |f - f_{P_\varepsilon(p) \cap bD}| \, d\sigma < \infty,
$$

where $f_{P_\varepsilon(p) \cap bD}$ stands for the mean value of $f$ over $P_\varepsilon(p) \cap bD$.

**Definition 2.4** A positive Borel measure $\mu$ is called a Carleson measure if

$$
\mu(P_\varepsilon(p) \cap D) \le C\sigma(P_\varepsilon(p) \cap bD)
$$

for each $p \in bD$ and $\varepsilon > 0$ with a constant $C$ independent of $p$ and $\varepsilon > 0$.

Observe that $\sigma(P_\varepsilon(p) \cap bD) \sim \varepsilon^{-1}|P_\varepsilon(p)|$ and the involved constant is independent of $p$ and $\varepsilon$.

**Definition 2.5** A positive $(1,1)$-current $\Theta$ satisfies a uniform Blaschke condition in a smoothly bounded convex domain of finite type $D$, if the measure

$$
\operatorname{dist}(\,\cdot\,, bD)|\Theta|_\kappa
$$

is a Carleson measure in $D$.

Throughout the paper the symbol $S$ stands for the support function for the domain $D$, the existence of which was proved in [7]. Also, we have

$$S(z, \zeta) = \sum_{i=1}^{n} Q_i(z, \zeta)(z_i - \zeta_i),$$

with functions $Q_1, \ldots, Q_n$ holomorphic in $z$ and smooth in $\zeta$.

We must emphasize at this moment the fact that Diedrich–Fornaess [7] proved the corresponding estimates for $S$ on $bD$ only. However, as indicated in [9], one can easily obtain similar results in the whole domain $D$ (see also [8]). Obviously, this requires $S$ to be extended in a suitable way inside of $D$ (see [8, 9]). Most importantly, one has the following estimate.

***Lemma 2.6*** **([7, 8])** *For $z, \zeta \in U$ sufficiently close to $bD$ and $\varepsilon > 0$ sufficiently small $|S(z, \zeta)| \gtrsim \varepsilon$, if $\zeta \in P_\varepsilon(z) \setminus P_{1/2\varepsilon}(z)$ or $z \in P_\varepsilon(\zeta) \setminus P_{1/2\varepsilon}(\zeta)$. Furthermore, if $z \in bD$, then $|r(\zeta) + S(z, \zeta)| \gtrsim \varepsilon$, if $\zeta \in (P_\varepsilon(z) \setminus P_{\varepsilon/2}(z)) \cap D$.*

The symbols $Q, P$ stand for the corresponding Leray forms

$$Q(z, \zeta) = \sum_{i=1}^{n} Q_i(z, \zeta) d\zeta_i, \quad P(z, \zeta) = \sum_{i=1}^{n} Q_i(\zeta, z) d\zeta_i,$$

which may be assumed to be defined in $\bar{D} \times \bar{D}$. Fix a point $z_0$ sufficiently close to $bD$ and choose small $\varepsilon > 0$. Denote by $w^* = (w_1^*, \ldots, w_n^*)$ coordinates with respect to an $\varepsilon$-extremal basis at $z_0$. Let $\Phi^*$ be a unitary transformation such that $w^* = \Phi^*(\zeta - z_0)$ and define the pullback $Q^*(w^*) = \bar{\Phi}^*\big(z_0, z_0 + (\bar{\Phi}^*)^T w^*\big)$. Lemma 3.3 in [9] now says that

$$(2.2) \qquad |Q_k^*(w^*)| \lesssim \frac{\varepsilon}{\tau_k(z_0, \varepsilon)}, \qquad \left|\frac{\partial Q_k^*(w^*)}{\partial z_j}\right| \lesssim \frac{\varepsilon}{\tau_k(z_0, \varepsilon)},$$
$$\left|\frac{\partial Q_k^*(w^*)}{\partial \bar{w}_j^*}\right| \lesssim \frac{\varepsilon}{\tau_j(z_0, \varepsilon)\tau_k(z_0, \varepsilon)},$$

$j, k = 1, \ldots, n$, provided $|w_j^*| \leq \tau_j(z_0, \varepsilon)$. The involved constants are uniform with respect to $z_0$ and $\varepsilon$.

The same estimates hold with respect to the second variable ([9, Lemma 3.4]). Namely, if $\zeta_0$ is fixed and $w^* = \Phi^*(\zeta - \zeta_0)$, $w_* = \Phi^*(z - \zeta_0)$ with an appropriate choice of a unitary transformation $\Phi^*$, then for $|w_{*j}| < \tau_j(\zeta_0, \varepsilon)$

$$(2.3) \qquad |Q_k^*(w_*)| \lesssim \tau_k(\zeta_0, \varepsilon), \qquad \left|\frac{\partial Q_k^*(w_*)}{\partial z_j}\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)},$$
$$\left|\frac{\partial Q_k^*(w_*)}{\partial \bar{w}_j}\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_j(\zeta_0, \varepsilon)}.$$

The symbol $Q^*(w_*)$ stands for $\bar{\Phi}^* Q\big(\zeta_0 + (\bar{\Phi}^*)^T w_*, \zeta_0\big)$.

## 3   Proof of Theorem 1.2

As a $\overline{\partial}$-solution operator we make use of a certain Berndtsson–Andersson type operator [2]. Importantly, it was shown in [8] that we may assume that the operator

$$Kf(z) = \int_D K(\zeta, z) \wedge f(\zeta)$$

is defined by the following kernel

$$(3.1) \quad K(\zeta, z) = K_1(\zeta, z) + K_2(\zeta, z) = \frac{r(\zeta)P(z, \zeta) \wedge (\overline{\partial}_\zeta Q(z, \zeta))^{n-1}}{S(\zeta, z)[r(\zeta) + S(z, \zeta)]^n}$$

$$+ \frac{P(z, \zeta) \wedge \overline{\partial} r(\zeta) \wedge Q(z, \zeta) \wedge (\overline{\partial}_\zeta Q(z, \zeta))^{n-2}}{S(\zeta, z)[r(\zeta) + S(z, \zeta)]^n},$$

provided $z \in bD$. Actually $K$ is of this form only sufficiently close to the boundary of $D$. We may ignore this obstacle, since the kernel is bounded on compact subsets of $D$. Also, arguments provided in [14], and in the case of finite type domains in [8], show that the operator $K$ is well defined and solves the $\overline{\partial}$-equation.

As for the estimates proving Theorem 1.2, observe that

$$\|Kf\|_{\mathrm{BMO}}$$

$$\leq \sup_{\substack{p \in bD \\ \varepsilon > 0}} \frac{1}{\sigma(bD \cap P_\varepsilon(p))^2} \int_{(bD \cap P_\varepsilon(p)) \times (bD \cap P_\varepsilon(p))} |Kf(z) - Kf(\eta)| d\sigma(z) d\sigma(\eta).$$

In the following, $\varrho$ always stands for $|r(\zeta)|$. Let $v_1, \dots, v_n$ be any orthonormal basis of $\mathbb{C}^n$ (with respect to the standard Hermitian structure in $\mathbb{C}^n$). Then

$$\int_D K(\zeta, z) \wedge f(\zeta) = \sum_{i=1}^n (-1)^{i+1} \int_D K(\zeta, z)(\widehat{v}_i) f(\zeta)(\bar{v}_i) \, dV(\zeta).$$

Naturally, for a fixed $1 \leq i \leq n$, we may write

$$(3.2) \quad \int_{bD \cap P_\varepsilon(p)} \left| \int_{\varrho \leq \varepsilon} K(\zeta, z)(\widehat{v}_i) f(z)(\bar{v}_i) dV(\zeta) \right| d\sigma(z)$$

$$\leq \int_{CP_\varepsilon(p)} d\mu_i(\zeta) \int_{bD \cap P_\varepsilon(p)} |K(\zeta, z)(\widehat{v}_i)| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\sigma(z)$$

$$+ \sum_{k=1}^\infty \int_{bD \cap P_\varepsilon(p)} d\sigma \int_{\substack{P_\varepsilon^k(p) \\ \varrho \leq \varepsilon}} |K(\zeta, z)(\widehat{v}_i)| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\mu_i(\zeta),$$

where $C > 1$ and

$$P_\varepsilon^k(p) := CP_{2^{k+1}\varepsilon}(p) \setminus \bigcup_{j=1}^k CP_{2^j \varepsilon}(p).$$

Since

$$d\mu[v_1, \ldots, v_n]_i(\zeta) = d\mu_i(\zeta) := \frac{\tau(\zeta, v_i, \varrho/2)}{\varrho}|f(\zeta)(\bar{v}_i)|dV(\zeta) \leq |f(\zeta)|_\kappa dV(\zeta),$$

the measure $\mu_i$ is a Carleson measure in $D$ for any $1 \leq i \leq n$ and any choice of an orthonormal basis $v_1, \ldots, v_n$.

To prove that (3.2) can be estimated uniformly by $\sigma(bD \cap P_\varepsilon(p))$, it is enough to show that

$$\int_{bD \cap CP_\varepsilon(p)} |K(\zeta, z)(\widehat{v}_i)| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\sigma(z) \lesssim 1,$$

$$\int_{\substack{P_\varepsilon^k(p) \\ \varrho \leq \varepsilon}} |K(\zeta, z)(\widehat{v}_i)| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\mu_i(\zeta) \lesssim C_k,$$

where the sequence $(C_k)$ is summable. We will start with the first integral, which can be estimated by

$$(3.3) \quad \left\{ \int_{bD \cap CP_\varrho(\zeta)} + \sum_{k=0}^\infty \int_{bD \cap (CP_{2^{k+1}\varrho}(\zeta) \setminus CP_{2^k\varrho}(\zeta))} \right\} |K(\zeta, z)(\widehat{v}_i)| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\sigma(z),$$

with possibly a different constant $C > 1$.

***Remark 3.1*** The estimate of the integral which appears above is essentially [3, Lemma 4.4]. However, the authors implicitly assume that $v_i$ is tangential, since only under this assumption Lemma 2.3 in [3] holds true.

We will provide estimates only for $K_1$ in (3.1). The second term of the kernel can be dealt with in a similar manner: apart from estimates (2.2) and (2.3) one must also make use of Proposition 2.1. Now fix $\zeta$ and for each $k \in \mathbb{N}$ choose a $2^k\varepsilon$-extremal basis at $\zeta$, which will be denoted by the same symbols $v_1, \ldots, v_n$. The first term of the nominator of the kernel can be estimated by the expression of the form

$$\sum \varrho|Q_l^*(w^*)| \prod_{s=1}^{n-1} \left| \frac{\partial Q_{\alpha_s}^*(w_*)}{\partial \bar{w}_{\beta_s}} \right| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)},$$

where the sum is taken over all $1 \leq l, i \leq n$, multiindices $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}$ such that $\{\alpha_1 < \cdots < \alpha_{n-1}\}$ does not contain $l$ and $\{\beta_1, \ldots, \beta_{n-1}\}$ does not contain $i$ and is of cardinality $n - 1$. This, in view of estimates (2.3) and Lemma 2.6, shows that the first term of the kernel can be estimated in $CP_{2^{k+1}\varrho}(\zeta) \setminus CP_{2^k\varrho}(\zeta), k \in \mathbb{N}$ by the sum of expressions of the form

$$(3.4) \quad \varrho \frac{1}{(2^k\varrho)^{n+1}} \frac{\varrho}{\tau_i(\zeta, \varrho/2)} \frac{(2^k\varrho)^n}{\tau_l(\zeta, 2^k\varrho) \prod_{s=1}^{n-1} \tau_{\alpha_s}(\zeta, 2^k\varrho)\tau_{\beta_s}(\zeta, 2^k\varrho)},$$

with the same convention concerning the indices. Now if $i = 1$, we estimate (3.4) by $2^{-k}\sigma(bD \cap P_{2^k\varrho}(z))^{-1}$, since $\varrho \sim \tau_1(\zeta, \varrho/2)$ and

$$\tau_l(\zeta, 2^k\varrho) \prod_{s=1}^{n-1} \tau_{\alpha_s}^2(\zeta, 2^k\varrho) \sim (2^k\varrho)^{-1} |P_{2^k\varrho}(\zeta)| \sim \sigma(bD \cap P_{2^k\varrho}(\zeta)).$$

If $i > 1$, we have

$$\frac{\varrho}{\tau_i(\zeta, \varrho/2)} = \frac{1}{2^k} \frac{2^k\varrho}{\tau_i(\zeta, 2^k\varrho)} \frac{\tau_i(\zeta, 2^k\varrho)}{\tau_i(\zeta, \varrho/2)} \lesssim 2^{-k/2} \frac{2^k\varrho}{\tau_i(\zeta, 2^k\varrho)},$$

since then

$$\tau(\zeta, \nu_i, \varepsilon_1) \lesssim \tau(\zeta, \nu_i, \varepsilon_2) \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{1/2},$$

if $\varepsilon_2 \leq \varepsilon_1$. To sum this up we estimate (3.4) by

$$2^{-k/2}\varrho \frac{1}{|P_{2^k\varrho}(\zeta)|} \sim 2^{-k/2} \frac{\varrho}{2^k\varrho\sigma(bD \cap P_{2^k\varrho}(\zeta))}.$$

Similar estimates also hold true for $z \in CP_\varrho(\zeta) \cap bD$, *i.e.*, for the first term in (3.3). Indeed, observe that the term $S(z, \zeta)$ in the denominator of the kernel does not pose any problem, since the fact that $z \in bD$ implies that $z \notin P_{c_1\varrho/2}(\zeta)$, with a uniform constant $c > 0$. Consequently, Lemma 2.6 can be applied.

Altogether we obtain an estimate of (3.3), which is independent of $\zeta$

$$\int_{bD \cap P_\varepsilon(p)} |K(\zeta, z)(\widehat{\nu_i})| \frac{\varrho}{\tau(z, \nu_i, \varrho/2)} d\sigma(z) \lesssim \sum_{k=1}^{\infty} C_k \frac{1}{\sigma(bD \cap P_{2^k\varrho}(\zeta))} \int_{bD \cap P_{2^k\varrho}(\zeta)} d\sigma(z)$$

$$\lesssim 1.$$

Now we estimate

$$(3.5) \qquad \int_{\substack{CP_{2^{k+1}\varepsilon}(p) \setminus CP_{2^k\varepsilon}(p) \\ \varrho \leq \varepsilon}} |K(\zeta, z)(\widehat{\nu_i})| \frac{\varrho}{\tau(\zeta, \nu_i, \varrho/2)} d\mu_i(\zeta),$$

for a fixed $z \in bD$ and $k \in \mathbb{N}$, where $C > 1$ is fixed. We postpone the precise choice of $\nu_1, \ldots, \nu_n$ for a moment. Using (2.2) this time, we are led to consider

$$(3.6) \quad \frac{\varrho}{(2^k\varepsilon)^{n+1}} \frac{\varrho}{\tau(\zeta, \nu_i, \varrho/2)} \frac{(2^k\varepsilon)^n}{\tau_l(z, 2^k\varepsilon) \prod_{s=1}^{n-1} \tau_{\alpha_s}(z, 2^k\varepsilon)\tau_{\beta_s}(z, 2^k\varepsilon)}$$

$$\lesssim \frac{1}{(2^k\varepsilon)^{n+1}} \frac{\varepsilon}{\tau_i(z, 2^k\varepsilon)} \frac{(2^k\varepsilon)^{n+1}}{\tau_l(z, 2^k\varepsilon) \prod_{s=1}^{n-1} \tau_{\alpha_s}(z, 2^k\varepsilon)\tau_{\beta_s}(z, 2^k\varepsilon)}.$$

Notice that we made use of the assumption that $\varrho \leq \varepsilon$. Indeed, this implies that

$$\frac{\varrho}{\tau(\zeta, \nu_i, \varrho/2)} \lesssim \frac{\varepsilon}{\tau(\zeta, \nu_i, \varepsilon)} \lesssim \frac{2^k\varepsilon}{\tau(\zeta, \nu_i, 2^k\varepsilon)}.$$

Furthermore, if $v_1, \ldots, v_n$ in (3.5) is chosen to be a $2^k \varepsilon$-basis at $z$, then by [3, (10)] we also have

$$\frac{2^k \varepsilon}{\tau(\zeta, v_i, 2^k \varepsilon)} \sim \frac{2^k \varepsilon}{\tau_i(z, 2^k \varepsilon)},$$

since $z \in \tilde{C} P_{2^k \varepsilon}(\zeta) \subset P_{b(\bar{C}) 2^k \varepsilon}(\zeta)$ for some uniform $\tilde{C} > 1$, where $b(\bar{C})$ is defined in Proposition 2.1. Recall that $z$ here is fixed and belongs to $P_\varepsilon(p)$.

Also, $P_\varepsilon(p) \Subset CP_\varepsilon(p)$, if $C > 1$. This allows us to apply Lemma 2.6, since the fact $z \in P_\varepsilon(p)$ implies that there exists a uniform constant $c > 0$ such that $\zeta \notin P_{c\varepsilon}(z)$, if for some $k \in \mathbb{N}$, $\zeta \in CP_{2^{k+1}\varepsilon}(p) \setminus \bigcup_{j=1}^k CP_{2^j \varepsilon}(p)$. Recall that $\mu_i(P_{2^k\varepsilon}(p)) \lesssim \sigma(bD \cap P_{2^k\varepsilon}(p))$ and, as a result, we obtain an estimate of (3.5), which is independent of $\zeta$ and summable with respect to $k$. Observe that $v_1, \ldots, v_n$ is this time the $2^k \varepsilon$-extremal basis at $z$ but, obviously, not necessarily at $\zeta$. This is why we have written $\tau(\zeta, v_i, \varrho)$ in the denominator of (3.6).

Lastly, we deal with the integral over the set $\varrho \geq \varepsilon$. We have

$$\int_{\varrho \gtrsim \varepsilon} |K(\zeta, z)(\widehat{v}_i) - K(\zeta, \eta)(\widehat{v}_i)| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\mu_i(\zeta)$$

$$\leq \int_0^1 \int_{\varrho \gtrsim \varepsilon} \left| \frac{d}{dt} K(\zeta, \gamma_{z,\eta}(t))(\widehat{v}_i) \right| \frac{\varrho}{\tau(\zeta, v_i, \varrho/2)} d\mu_i(\zeta) dt,$$

where $\gamma_{z,\eta}$ is a curve from Lemma 2.2. For a fixed $t \in [0,1]$ denote $q = \gamma_{z,\eta}(t)$. As before, we will show necessary estimates only for a typical term of the kernel. In the worst case we have to deal with

$$(3.7) \quad \left| (\gamma_{\zeta,\eta})'_j(t) \right| \frac{|\partial_j S(q, \zeta)| + |\partial_j (r + S(q, \zeta))|}{|r + S(q, \zeta)|^{n+1} |S(\zeta, q)|}$$

$$\times \left| r(\zeta) P(q, \zeta) \wedge (\overline{\partial}_\zeta Q(q, \zeta))^{n-1} (\widehat{v}_i) \right| \frac{\varrho}{\tau_i(\zeta, \varrho/2)},$$

where $\partial_j$ stands either for the real or the imaginary part of the derivative with respect to $w_j$.

For a given $\varepsilon > 0$ choose $\varepsilon$-extremal coordinates at $p$. Let $\Phi^*$ be the corresponding unitary transformation such that $w^* = \Phi^*(z - p)$. Although both $z$ and $\zeta$ vary in formula (3.7), we still want to have estimates with the boundary directional derivatives at point $p$. Therefore, we write

$$Q^*(w^*, \eta^*) := \bar{\Phi}^* Q(p + (\bar{\Phi}^*)^T w^*, p + (\bar{\Phi}^*)^T \eta^*),$$

where $\eta^* = \Phi^*(\zeta - p)$. Under the assumption that $|w_1^*| \leq C$, $|w_j^*| \lesssim \tau_j(p, \varepsilon)$, $j = 2, \ldots, n$ and $|\eta^*| \lesssim \tau_j(p, \varepsilon)$, $j = 1, \ldots, n$,

$$|Q_k^*(w^*, \eta^*)| \lesssim \frac{\varepsilon}{\tau_k(p, \varepsilon)}, \quad \left| \frac{Q_k^*(w^*, \eta^*)}{\partial \bar{\eta}_j^*} \right| \lesssim \frac{\varepsilon}{\tau_k(p, \varepsilon) \tau_j(p, \varepsilon)}.$$

This is proved in [10, Lemma 3.1].

Therefore choosing a $2^k\varepsilon$-extremal basis at $p$ (denoted again by $v_1, \ldots, v_n$) we obtain the following estimate for terms of the kernel in $P_{2^{k+1}\varepsilon}(p) \setminus P_{2^k\varepsilon}(p)$ with a summable sequence $(c_k)$:

$$\tau_j(p, \varepsilon) \frac{1}{(2^k\varepsilon)^{n+2}} \frac{2^k\varepsilon}{\tau_j(p, 2^k\varepsilon)} \frac{\varrho^2}{\tau(\zeta, v_i, \varrho/2)} \frac{(2^k\varepsilon)^n}{\tau_l(p, 2^k\varepsilon) \prod_{s=1}^{n-1} \tau_{\alpha_s}(p, 2^k\varepsilon) \tau_{\beta_s}(p, 2^k\varepsilon)}$$

$$\lesssim \frac{\tau_j(p, \varepsilon)}{\tau_j(p, 2^k\varepsilon)} \frac{2^k\varepsilon}{|P_{2^k\varepsilon}(p)|} \lesssim c_k \frac{1}{\sigma(bD \cap P_{2^k\varepsilon}(p))}.$$

We used Lemma 2.2, Lemma 2.6, Proposition 2.1, the fact that $\epsilon/\tau_i(\zeta, \epsilon)$ is non-decreasing with respect to $\epsilon > 0$, and $\tau(\zeta, v, \varepsilon) \sim \tau(p, v, \varepsilon)$ if $\zeta \in P_\varepsilon(p)$. This completes the proof. ∎

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