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BIG COHEN–MACAULAY TEST IDEALS IN EQUAL CHARACTERISTIC ZERO VIA ULTRAPRODUCTS

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Abstract. Utilizing ultraproducts, Schoutens constructed a big Cohen–Macaulay (BCM) algebra $\mathcal{B}(R)$ over a local domain R essentially of finite type over $\mathbb{C}$. We show that if R is normal and Δ is an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, then the BCM test ideal $\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta})$ of $(\widehat{R}, \widehat{\Delta})$ with respect to $\widehat{\mathcal{B}(R)}$ coincides with the multiplier ideal $\mathcal{J}(\widehat{R}, \widehat{\Delta})$ of $(\widehat{R}, \widehat{\Delta})$, where $\widehat{R}$ and $\widehat{\mathcal{B}(R)}$ are the $\mathfrak{m}$-adic completions of R and $\mathcal{B}(R)$, respectively, and $\widehat{\Delta}$ is the flat pullback of Δ by the canonical morphism $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$. As an application, we obtain a result on the behavior of multiplier ideals under pure ring extensions.

§1. Introduction

A (balanced) big Cohen–Macaulay (BCM) algebra over a Noetherian local ring $(R, \mathfrak{m})$ is an R-algebra B such that every system of parameters is a regular sequence on B. Its existence implies many fundamental homological conjectures including the direct summand conjecture (now a theorem). Hochster and Huneke [14], [15] proved the existence of a BCM algebra in equal characteristic, and André [1] settled the mixed characteristic case. Recently, using BCM algebras, Ma and Schwede [18], [19] introduced the notion of BCM test ideals as an analog of test ideals in tight closure theory.

The test ideal $\tau(R)$ of a Noetherian local ring R of positive characteristic was originally defined as the annihilator ideal of all tight closure relations of R. Since it turned out that $\tau(R)$ was related to multiplier ideals via reduction to characteristic p, the definition of $\tau(R)$ was generalized in [11], [29] to involve effective $\mathbb{Q}$-Weil divisors Δ on $\operatorname{Spec} R$ and ideals $\mathfrak{a} \subseteq R$ with real exponent $t > 0$. In these papers, it was shown that multiplier ideals coincide, after reduction to characteristic $p \gg 0$, with such generalized test ideals $\tau(R, \Delta, \mathfrak{a}^t)$. In positive characteristic, Ma-Schwede's BCM test ideals are the same as the generalized test ideals. In this paper, we study BCM test ideals in equal characteristic zero.

Using ultraproducts, Schoutens [24] gave a characterization of log-terminal singularities, an important class of singularities in the minimal model program. He also gave an explicit construction of a BCM algebra $\mathcal{B}(R)$ in equal characteristic zero: $\mathcal{B}(R)$ is described as the ultraproduct of the absolute integral closures of Noetherian local domains of positive characteristic. He defined a closure operation associated with $\mathcal{B}(R)$ to introduce the notions of $\mathcal{B}$-rationality and $\mathcal{B}$-regularity, which are closely related to BCM rationality and BCM regularity defined in [19], and proved that $\mathcal{B}$-rationality is equivalent to being rational singularities. The aim of this paper is to give a geometric characterization of BCM test ideals associated with $\mathcal{B}(R)$. Our main result is stated as follows:

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THEOREM 1.1 (Theorem 6.4). *Let R be a normal local domain essentially of finite type over $\mathbb{C}$. Let Δ be an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R+\Delta$ is $\mathbb{Q}$-Cartier, where K_R is a canonical divisor on $\operatorname{Spec} R$. Suppose that $\widehat{R}$ and $\widehat{\mathcal{B}(R)}$ are the $\mathfrak{m}$-adic completions of R and $\mathcal{B}(R)$, and $\widehat{\Delta}$ is the flat pullback of Δ by the canonical morphism $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$. Then we have*

$$\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta}) = \mathcal{J}(\widehat{R}, \widehat{\Delta}),$$

where $\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta})$ is the BCM test ideal of $(\widehat{R}, \widehat{\Delta})$ with respect to $\widehat{\mathcal{B}(R)}$ and $\mathcal{J}(\widehat{R}, \widehat{\Delta})$ is the multiplier ideal of $(\widehat{R}, \widehat{\Delta})$.

The inclusion $\mathcal{J}(\widehat{R}, \widehat{\Delta}) \subseteq \tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta})$ is obtained by comparing reductions of the multiplier ideal modulo $p \gg 0$ to its approximations. We prove the opposite inclusion by combining an argument similar to that in [25] with the description of multiplier ideals as the kernel of a map between local cohomology modules in [29]. As an application of Theorem 1.1, we show the next result about a behavior of multiplier ideals under pure ring extensions, which is a generalization of [31, Cor. 5.30].

THEOREM 1.2 (Corollary 7.11). *Let $R \hookrightarrow S$ be a pure local homomorphism of normal local domains essentially of finite type over $\mathbb{C}$. Suppose that R is $\mathbb{Q}$-Gorenstein. Let Δ_S be an effective $\mathbb{Q}$-Weil divisor such that $K_S+\Delta_S$ is $\mathbb{Q}$-Cartier, where K_S is a canonical divisor on $\operatorname{Spec} S$. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal, and let $t>0$ be a positive rational number. Then we have*

$$\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \mathfrak{a}^t).$$

In [31], we defined ultra-test ideals, a variant of test ideals in equal characteristic zero, to generalize the notion of ultra-F-regularity introduced by Schoutens [24]. Theorem 1.2 was proved by using ultra-test ideals under the assumption that $\mathfrak{a}$ is a principal ideal. The description of multiplier ideals as BCM test ideals associated with $\mathcal{B}(R)$ (Theorem 1.1) and a generalization of module closures in [20] enables us to show Theorem 1.2 without any assumptions.

As another application of Theorem 1.1, we give an affirmative answer to one of the conjectures proposed by Schoutens [24, Rem. 3.10], which says that $\mathcal{B}$-regularity is equivalent to being log-terminal singularities (see Theorem 8.2).

This paper is organized as follows: in the preliminary section, we give definitions of multiplier ideals, test ideals, and BCM test ideals. In §3, we quickly review the theory of ultraproducts in commutative algebra including non-standard and relative hulls. In §4, we prove some fundamental results on BCM algebras constructed via ultraproducts following [23]. In §5, we review the relationship between approximations and reductions modulo $p \gg 0$ and consider approximations of multiplier ideals. In §6, we show Theorem 1.1, the main theorem of this paper. In §7, using a generalized module closure, we show Theorem 1.2 as an application of Theorem 1.1. In §8, we show that $\mathcal{B}$-regularity is equivalent to log-terminal singularities. Finally in §9, we discuss a question, a variant of [7, Quest. 2.7], to handle BCM algebras that cannot be constructed via ultraproducts, and consider the equivalence of BCM-rationality and being rational singularities.

§2. Preliminaries

Throughout this paper, all rings will be commutative with unity.

2.1 Multiplier ideals

Here, we briefly review the definition of multiplier ideals and refer the reader to [16], [21] for more details. Throughout this subsection, we assume that X is a normal integral scheme essentially of finite type over a field of characteristic zero or $X = \operatorname{Spec} \widehat{R}$, where $(R, \mathfrak{m})$ is a normal local domain essentially of finite type over a field of characteristic zero and $\widehat{R}$ is its $\mathfrak{m}$-adic completion.

Definition 2.1. A proper birational morphism $f : Y \to X$ between integral schemes is said to be a *resolution of singularities* of X if Y is regular. When Δ is a $\mathbb{Q}$-Weil divisor on X and $\mathfrak{a} \subseteq \mathcal{O}_X$ is a nonzero coherent ideal sheaf, a resolution $f : Y \to X$ is said to be a *log resolution* of $(X, \Delta, \mathfrak{a})$ if $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is invertible and if the union of the exceptional locus $\operatorname{Exc}(f)$ of f and the support F and the strict transform $f_*^{-1}\Delta$ of Δ is a simple normal crossing divisor.

If $f : Y \to X$ is a proper birational morphism with Y a normal integral scheme and Δ is a $\mathbb{Q}$-Weil divisor, then we can choose K_Y such that $f^*(K_X + \Delta) - K_Y$ is a divisor supported on the exceptional locus of f. With this convention:

Definition 2.2. Let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a nonzero coherent ideal sheaf, and let $t > 0$ be a positive real number. Then the *multiplier ideal sheaf* $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ associated with $(X, \Delta, \mathfrak{a}^t)$ is defined by

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = f_*\mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta) + tF \rfloor).$$

where $f : Y \to X$ is a log resolution of $(X, \Delta, \mathfrak{a})$. Note that this definition is independent of the choice of log resolution.

Definition 2.3. Let X be a normal integral scheme essentially of finite type over a field of characteristic zero. We say that X *has rational singularities* if X is Cohen–Macaulay at x and if for any projective birational morphism $f : Y \to \operatorname{Spec} \mathcal{O}_{X,x}$ with Y a normal integral scheme, the natural morphism $f_*\omega_Y \to \omega_{X,x}$ is an isomorphism.

2.2 Tight closure and test ideals

In this subsection, we quickly review the basic notion of tight closure and test ideals. We refer the reader to [4], [11], [13], [29].

Definition 2.4. Let R be a normal domain of characteristic $p > 0$, let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor, let $\mathfrak{a} \subseteq R$ be a nonzero ideal, and let $t > 0$ be a real number. Let $E = \bigoplus E(R/\mathfrak{m})$ be the direct sum, taken over all maximal ideals $\mathfrak{m}$ of R, of the injective hulls $E_R(R/\mathfrak{m})$ of the residue fields $R/\mathfrak{m}$.

(1) Let I be an ideal of R. The $(\Delta, \mathfrak{a}^t)$*-tight closure*$I^{*\Delta, \mathfrak{a}^t}$ *of* I is defined as follows: $x \in I^{*\Delta, \mathfrak{a}^t}$ if and only if there exists a nonzero element $c \in R^\circ$ such that

$$c\mathfrak{a}^{\lceil t(q-1) \rceil} x^q \subseteq I^{[q]} R(\lceil (q-1)\Delta \rceil)$$

for all large $q = p^e$, where $I^{[q]} = \{f^q | f \in I\}$ and $R^\circ = R \setminus \{0\}$.

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(2) If M is an R-module, then the $(\Delta, \mathfrak{a}^t)$-*tight closure* $0_M^{*\Delta,\mathfrak{a}^t}$ is defined as follows: $z \in 0_M^{*\Delta,\mathfrak{a}^t}$ if and only if there exists a nonzero element $c \in R^\circ$ such that

$$(c\mathfrak{a}^{\lceil t(q-1)\rceil})^{1/q} \otimes z = 0 \quad \text{in} \quad R(\lceil (q-1)\Delta\rceil)^{1/q} \otimes_R M$$

for all large $q = p^e$.

(3) The *(big) test ideal* $\tau(R, \Delta, \mathfrak{a}^t)$ *associated with*$(R, \Delta, \mathfrak{a}^t)$ is defined by

$$\tau(R, \Delta, \mathfrak{a}^t) = \mathrm{Ann}_R(0_E^{*\Delta,\mathfrak{a}^t}).$$

When $\mathfrak{a} = R$, then we simply denote the ideal $\tau(R, \Delta)$. We call the triple $(R, \Delta, \mathfrak{a}^t)$ is *strongly F-regular* if $\tau(R, \Delta, \mathfrak{a}^t) = R$.

Definition 2.5 [8]. Let R be an F-finite Noetherian local domain of characteristic $p > 0$ of dimension d. We say that R is *F-rational* if any ideal $I = (x_1, \ldots, x_d)$ generated by a system of parameters satisfies $I = I^*$.

2.3 Big Cohen–Macaulay algebras

In this subsection, we will briefly review the theory of BCM algebras. Throughout this subsection, we assume that local rings $(R, \mathfrak{m})$ are Noetherian.

Definition 2.6. Let $(R, \mathfrak{m})$ be a local ring, and let $\mathbf{x} = x_1, \ldots, x_n$ be a system of parameters. R-algebra B is said to be *BCM with respect to* $\mathbf{x}$ if $\mathbf{x}$ is a regular sequence on B. B is called a *(balanced) BCM algebra* if it is BCM with respect to $\mathbf{x}$ for every system of parameters $\mathbf{x}$.

Remark 2.7 [5, Cor. 8.5.3]. If B is BCM with respect to $\mathbf{x}$, then the $\mathfrak{m}$-adic completion $\widehat{B}$ is (balanced) BCM.

About the existence of BCM algebras of residue characteristic $p > 0$, the following are proved in [3], [14].

Theorem 2.8. *If $(R, \mathfrak{m})$ is an excellent local domain of residue characteristic $p > 0$, then the p-adic completion of absolute integral closure R^+ is a (balanced) BCM R-algebra.*

Using BCM algebras, we can define a class of singularities.

Definition 2.9. If R is an excellent local ring of dimension d, and let B be a BCM R-algebra. We say that R is *BCM-rational with respect to B* (or simply BCM_B-rational) if R is Cohen–Macaulay and if $H_\mathfrak{m}^d(R) \to H_\mathfrak{m}^d(B)$ is injective. We say that R is *BCM-rational* if R is BCM_B-rational for any BCM algebra B.

We explain BCM test ideals introduced in [19].

Setting 2.10. Let $(R, \mathfrak{m})$ be a normal local domain of dimension d.

(i) $\Delta \geqslant 0$ is a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier.
(ii) Fixing Δ, we also fix an embedding $R \subseteq \omega_R \subseteq \operatorname{Frac} R$, where ω_R is the canonical module.
(iii) Since $K_R + \Delta$ is effective and $\mathbb{Q}$-Cartier, there exist an integer $n > 0$ and $f \in R$ such that $n(K_R + \Delta) = \mathrm{div}(f)$.

Definition 2.11. With notation as in Setting 2.10, if B is a BCM $R[f^{1/n}]$-algebra, then we define $0_{H_\mathfrak{m}^d(\omega_R)}^{B, K_R+\Delta}$ to be $\operatorname{Ker}\psi$, where ψ is the homomorphism determined by the

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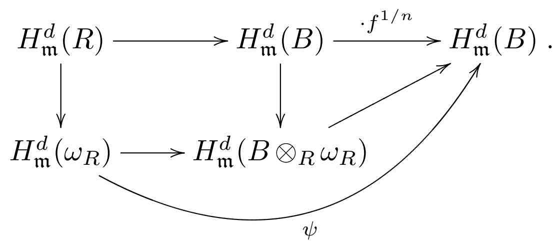
below commutative diagram:



If R is $\mathfrak{m}$-adically complete, then we define

$$\tau_B(R,\Delta) = \operatorname{Ann}_R 0^{B,K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)}.$$

We call $\tau_B(R,\Delta)$ *the BCM test ideal of* (R,Δ) *with respect to* B. We say that (R,Δ) *is BCM regular with respect to* B (or simply BCM_B regular) if $\tau_B(R,\Delta) = R$.

PROPOSITION 2.12 [19]. *Let* $(R,\mathfrak{m})$ *be a complete normal local domain of characteristic* $p > 0$*, let* $\Delta \geqslant 0$ *be an effective* $\mathbb{Q}$*-Weil divisor on* $\operatorname{Spec} R$*, and let* B *be a BCM* R^+*-algebra. Fix an effective canonical divisor* $K_R \geqslant 0$*. Suppose that* $K_R + \Delta$ *is* $\mathbb{Q}$*-Cartier. Then*

$$\tau_B(R,\Delta) = \tau(R,\Delta).$$

§3. Ultraproducts

3.1 Basic notions

In this subsection, we quickly review basic notions from the theory of ultraproduct. The reader is referred to [22], [26] for details. We fix an infinite set W. We use $\mathcal{P}(W)$ to denote the power set of W.

DEFINITION 3.1. A nonempty subset $\mathcal{F} \subseteq \mathcal{P}(W)$ is called a *filter* if the following two conditions hold.

(i) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(ii) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq W$, then $B \in \mathcal{F}$.

DEFINITION 3.2. Let $\mathcal{F}$ be a filter on W.

(1) $\mathcal{F}$ is called an *ultrafilter* if for all $A \in \mathcal{P}(W)$, we have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, where A^c is the complement of A.
(2) $\mathcal{F}$ is called *principal* if there exists a finite subset $A \subseteq W$ such that $A \in \mathcal{F}$.

REMARK 3.3. By Zorn's lemma, non-principal ultrafilters always exist.

REMARK 3.4. Ultrafilters are an equivalent notion to two-valued finitely additive measures. If we have an ultrafilter $\mathcal{F}$ on W, then

$$m(A) := \begin{cases} 1 & (A \in \mathcal{F}) \\ 0 & (A \notin \mathcal{F}) \end{cases}$$

is a two-valued finitely additive measure. Conversely, if $m : \mathcal{P}(W) \to \{0,1\}$ is a nonzero finitely additive measure, then $\mathcal{F} := \{A \subseteq W | m(A) = 1\}$ is an ultrafilter. Here, $\mathcal{F}$ is principal if and only if there exists an element w_0 of W such that $m(\{w_0\}) = 1$. Hence, $\mathcal{F}$ is not principal if and only if $m(A) = 0$ for any finite subset A of W.

DEFINITION 3.5. Let A_w be a family of sets indexed by W and $\mathcal{F}$ be an ultrafilter on W. Suppose that $a_w \in A_w$ for all $w \in W$ and φ is a predicate. We say $\varphi(a_w)$ holds *for almost all* w if $\{w \in W | \varphi(a_w) \text{ holds}\} \in \mathcal{F}$.

REMARK 3.6. This is an analog of "almost everywhere" or "almost surely" in analysis. The difference is that m is not countably but finitely additive. We can consider elements in $\mathcal{F}$ as "large" sets and elements in the complement $\mathcal{F}^c$ as "small" sets. If $\mathcal{F}$ is not principal, all finite subsets of W are "small."

DEFINITION 3.7. Let A_w be a family of sets indexed by W and $\mathcal{F}$ be a non-principal ultrafilter on W. The *ultraproduct of* A_w is defined by

$$\operatorname*{ulim}_w A_w = A_\infty := \prod_w A_w / \sim,$$

where $(a_w) \sim (b_w)$ if and only if $\{w \in W | a_w = b_w\} \in \mathcal{F}$. We denote the equivalence class of (a_w) by $\operatorname{ulim}_w a_w$.

REMARK 3.8 [17, Sec. 3]. If A_w are local rings, then the ultraproduct is equivalent to the localization of $\prod A_w$ at a maximal ideal.

EXAMPLE 3.9. We use $^*\mathbb{N}$ and $^*\mathbb{R}$ to denote the ultraproduct of $|W|$ copies of $\mathbb{N}$ and $\mathbb{R}$, respectively. $^*\mathbb{N}$ is a semiring and $^*\mathbb{R}$ is a field (see Definition-Proposition 3.10 and Theorem 3.20). $^*\mathbb{N}$ is a non-standard model of Peano arithmetic. $^*\mathbb{R}$ is a system of hyperreal numbers used in non-standard analysis.

DEFINITION-PROPOSITION 3.10. Let $A_{1w}, \ldots, A_{nw}$, B_w be families of sets indexed by W and $\mathcal{F}$ be a non-principal ultrafilter. Suppose that $f_w : A_{1w} \times \cdots \times A_{nw} \to B_w$ is a family of maps. Then we define the *ultraproduct* $f_\infty = \operatorname{ulim}_w f_w : A_{1\infty} \times \cdots \times A_{n\infty} \to B_\infty$ *of* f_w by

$$f_\infty(\operatorname*{ulim}_w a_{1w}, \ldots, \operatorname*{ulim}_w a_{nw}) := \operatorname*{ulim}_w f_w(a_{1w}, \ldots, a_{nw}).$$

This is well-defined.

COROLLARY 3.11. *Let A_w be a family of rings. Suppose that B_w is an A_w-algebra and M_w is an A_w-module for almost all w. Then the following hold:*

(1) *A_∞ is a ring.*
(2) *B_∞ is an A_∞-algebra.*
(3) *M_∞ is an A_∞-module.*

Proof. Let $0 := \operatorname{ulim}_w 0$, $1 := \operatorname{ulim}_w 1$ in A_∞, B_∞ and $0 := \operatorname{ulim}_w 0$ in M_∞. By the above Definition–Proposition, A_∞, B_∞ have natural additions, subtractions, and multiplications and we have a natural ring homomorphism $A_\infty \to B_\infty$. Similarly, M_∞ has a natural addition and a scalar multiplication between elements of M_∞ and A_∞. ∎

PROPOSITION 3.12. *Suppose that, for almost all w, we have an exact sequence*

$$0 \to L_w \to M_w \to N_w \to 0$$

of abelian groups. Then

$$0 \to \operatorname*{ulim}_w L_w \to \operatorname*{ulim}_w M_w \to \operatorname*{ulim}_w N_w \to 0$$

is an exact sequence of abelian groups. In particular, $\operatorname{ulim}_w : \prod_w \mathrm{Ab} \to \mathrm{Ab}$ *is an exact functor.*

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Proof. Let $f_w : L_w \to M_w$ and $g_w : M_w \to N_w$ be the morphisms in the given exact sequence. Here, we only prove the injectivity of $\mathrm{ulim}_w f_w$ and the surjectivity of $\mathrm{ulim}_w g_w$. Suppose that $\mathrm{ulim}_w f_w(a_w) = 0$ for $\mathrm{ulim}_w a_w \in \mathrm{ulim}_w L_w$. Then $f_w(a_w) = 0$ for almost all w. Since f_w is injective for almost all w, we have $a_w = 0$ for almost all w. Therefore, $\mathrm{ulim}_w a_w = 0$ in $\mathrm{ulim}_w L_w$. Hence, $\mathrm{ulim}_w f_w$ is injective. Next, let $\mathrm{ulim}_w c_w$ be any element in $\mathrm{ulim}_w N_w$. Since g_w is surjective for almost all w, there exists $b_w \in M_w$ such that $g_w(b_w) = c_w$ for almost all w. Let $b = \mathrm{ulim}_w b_w$. Then we have $(\mathrm{ulim}_w g_w)(b) = \mathrm{ulim}_w g_w(b_w) = \mathrm{ulim}_w c_w$. Hence, $\mathrm{ulim}_w g_w$ is surjective. The rest of the proof is similar. □

Łoś's theorem is a fundamental theorem in the theory of ultraproducts. We will prepare some notions needed to state the theorem.

Definition 3.13. The *language $\mathcal{L}$ of rings* is the set defined by

$$\mathcal{L} := \{0, 1, +, -, \cdot\}.$$

Definition 3.14. *Terms of $\mathcal{L}$* are defined as follows:

(i) 0, 1 are terms.
(ii) Variables are terms.
(iii) If s, t are terms, then $-(s), (s)+(t), (s)\cdot(t)$ are terms.
(iv) A string of symbols is a term only if it can be shown to be a term by finitely many applications of the above three rules.

We omit parentheses and "·" if there is no ambiguity.

Example 3.15. $1+1$, $x_1(x_2+1)$,$-(-x)$ are terms.

Definition 3.16. *Formulas of $\mathcal{L}$* are defined as follows:

(i) If s, t are terms, then $(s = t)$ is a formula.
(ii) If φ, ψ are formulas, then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi), (\neg\varphi)$ are formulas.
(iii) If φ is a formula and x is a variable, then $\forall x\varphi, \exists x\varphi$ are formulas.
(iv) A string of symbols is a formula only if it can be shown to be a formula by finitely many applications of the above three rules.

We omit parentheses if there is no ambiguity and use $\neq$, $\nexists$ in the usual way.

Remark 3.17. $\varphi \wedge \psi$ means "φ and ψ," $\varphi \vee \psi$ means "φ or ψ," $\varphi \to \psi$ means "φ implies ψ," and $\neg\varphi$ means "φ does not hold."

Example 3.18. 0=1, $x = 0 \wedge y \neq 1$, $\forall x \forall y(xy = yx)$ are formulas.

Remark 3.19. Variables in a formula φ which is not bounded by $\forall$ or $\exists$ are called free variables of φ. If $x_1, \ldots, x_n$ are free variables of φ, we denote $\varphi(x_1, \ldots, x_n)$ and we can substitute elements of a ring for $x_1, \ldots, x_n$.

Theorem 3.20 (Łoś's theorem in the case of rings). *Suppose that $\varphi(x_1, \ldots, x_n)$ is a formula of $\mathcal{L}$ and A_w is a family of rings indexed by a set W endowed with a non-principal ultrafilter. Let $a_{iw} \in A_w$. Then $\varphi(\mathrm{ulim}_w a_{1w}, \ldots, \mathrm{ulim}_w a_{nw})$ holds in A_∞ if and only if $\varphi(a_{1w}, \ldots, a_{nw})$ holds in A_w for almost all w.*

Remark 3.21. Even if A_w are not rings, replacing $\mathcal{L}$ properly, we can get the same theorem as above. We use one in the case of modules.

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EXAMPLE 3.22. Let A be a ring. If a property of rings is written by some formula, we can apply Łoś's theorem.

(1) A is a field if and only if $\forall x(x=0 \vee \exists y(xy=1))$ holds.
(2) A is a domain if and only if $\forall x \forall y(xy=0 \to (x=0 \vee y=0))$ holds.
(3) A is a local ring if and only if

$$\forall x \forall y(\nexists z(xz=1) \wedge \nexists w(yw=1) \to \nexists u((x+y)u=1))$$

holds.
(4) The condition that A is an algebraically closed field is written by countably many formulas, that is, the formula in (1) and for all $n \in \mathbb{N}$,

$$\forall a_0 \dots a_{n-1} \exists x(x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0).$$

(5) The condition that A is Noetherian cannot be written by formulas. Indeed, if $W = \mathbb{N}$ with some non-principal ultrafilter and $A_w = \mathbb{C}[\![x]\!]$, then $\mathrm{ulim}_n x^n \neq 0$ is in $\cap_n \mathfrak{m}_\infty^n$, where $\mathfrak{m}_\infty$ is the maximal ideal of A_∞. Hence, A_∞ is not Noetherian.

PROPOSITION 3.23 ([22, 2.8.2]; see Example 3.22). *If almost all K_w are algebraically closed field, then K_∞ is an algebraically closed field.*

THEOREM 3.24 (Lefschetz principle [22, Th. 2.4]). *Let W be the set of prime numbers endowed with some non-principal ultrafilter. Then*

$$\operatorname*{ulim}_{p \in \mathcal{W}} \overline{\mathbb{F}_p} \cong \mathbb{C}.$$

Proof. Let $C = \mathrm{ulim}_p \overline{\mathbb{F}_p}$. By the above theorem, C is an algebraically closed field. For any prime number q, we have $q \neq 0$ in $\overline{\mathbb{F}_p}$ for almost all p. Hence, $q \neq 0$ in C, that is, C is of characteristic zero. We can check that C has the same cardinality as $\mathbb{C}$. If two algebraically closed uncountable field of characteristic zero have the equal cardinality, then they are isomorphic. Hence, $C \cong \mathbb{C}$. (Note that this isomorphism is not canonical.) ∎

3.2 Non-standard hulls

In this subsection, we will introduce the notion of non-standard hulls along [22], [26]. Throughout this subsection, let $\mathcal{P}$ be the set of prime numbers and we fix a non-principal ultrafilter on $\mathcal{P}$ and an isomorphism $\mathrm{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$.

Let $\mathbb{C}[X_1, \dots, X_n]_\infty := \mathrm{ulim}_p \overline{\mathbb{F}_p}[X_1, \dots, X_n]$. Then we have the following proposition.

PROPOSITION 3.25 [22, Th. 2.6]. *We have a natural map $\mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}[X_1, \dots, X_n]_\infty$, which is faithfully flat.*

DEFINITION 3.26. The ring $\mathbb{C}[X_1, \dots, X_n]_\infty$ is said to be the *non-standard hull of* $\mathbb{C}[X_1, \dots, X_n]$.

REMARK 3.27. If $n \geqslant 1$, then $\mathbb{C}[X_1, \dots, X_n]_\infty$ is not Noetherian. Let $y = \mathrm{ulim}_p X_1^p$. Then, for any integer $l \geqslant 1$, $X_1^p \in (X_1, \dots, X_n)^l$ for almost all p. Hence, $y \in (X_1, \dots, X_n)^l$ for any l by Łoś's theorem. Therefore, $\cap_l (X_1, \dots, X_n)^l \neq 0$. By Krull's intersection theorem, $\mathbb{C}[X_1, \dots, X_n]_\infty$ is not Noetherian.

DEFINITION 3.28. Suppose that R is a finitely generated $\mathbb{C}$-algebra. Let

$$R \cong \mathbb{C}[X_1, \dots, X_n]/I$$

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be a presentation of R. The *non-standard hull* R_∞ *of* R is defined by

$$R_\infty := \mathbb{C}[X_1,\dots,X_n]_\infty / I\mathbb{C}[X_1,\dots,X_n]_\infty.$$

REMARK 3.29. The non-standard hull is independent of a representation of R. If $R \cong \mathbb{C}[X_1,\dots,X_n]/I \cong \mathbb{C}[Y_1,\dots,Y_m]/J$, then $\overline{\mathbb{F}_p}[X_1,\dots,X_n]/I_p \cong \overline{\mathbb{F}_p}[Y_1,\dots,Y_m]/J_p$ for almost all p (see Definitions 3.33 and 3.35).

REMARK 3.30. The natural map $R \to R_\infty$ is faithfully flat since this is a base change of the homomorphism $\mathbb{C}[X_1,\dots,X_n] \to \mathbb{C}[X_1,\dots,X_n]_\infty$. By faithfully flatness, we have $IR_\infty \cap R = R$ for any ideal $I \subseteq R$.

DEFINITION 3.31. Let $a \in \mathbb{C}$. Since $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$, we have a family $(a_p)_p$ of elements of $\overline{\mathbb{F}_p}$ such that $\operatorname{ulim} a_p = a$. Then we call $(a_p)_p$ an *approximation of* a.

PROPOSITION 3.32. *Let* $I = (f_1,\dots,f_s)$ *be an ideal of* $\mathbb{C}[X_1,\dots,X_n]$ *and* $f_i = \sum a_{i\nu} X^\nu$. *Let* $I_p = (f_{1p},\dots,f_{sp})\overline{\mathbb{F}_p}[X_1,\dots,X_n]$, *where* $f_{ip} = \sum a_{i\nu p} X^\nu$ *and each* $(a_{i\nu p})_p$ *is an approximation of* $a_{i\nu}$. *Then we have*

$$I\mathbb{C}[X_1,\dots,X_n]_\infty = \operatorname*{ulim}_p I_p$$

and

$$R_\infty \cong \operatorname*{ulim}_p (\overline{\mathbb{F}_p}[X_1,\dots,X_n]/I_p).$$

DEFINITION 3.33. Let R be a finitely generated $\mathbb{C}$-algebra.

(1) In the setting of Proposition 3.32, a family R_p is said to be an *approximation of* R if R_p is an $\overline{\mathbb{F}_p}$-algebra and $R_p \cong \overline{\mathbb{F}_p}[X_1,\dots,X_n]/I_p$ for almost all p. Then we have $R_\infty \cong \operatorname{ulim}_p R_p$.
(2) For an element $f \in R$, a family f_p is said to be an *approximation of* f if $f_p \in R_p$ for almost all p and $f = \operatorname{ulim}_p f_p$ in R_∞. For $f \in R_\infty$, we define an *approximation of* f in the same way.
(3) For an ideal $I = (f_1,\dots,f_s) \subseteq R$, a family I_p is said to be an *approximation of* I if I_p is an ideal of R_p and $I_p = (f_{1p},\dots,f_{sp})$ for almost all p. For finitely generated ideal $I \subseteq R_\infty$, we define *an approximation of* I in the same way.

REMARK 3.34. This is an abuse of notation since approximations should be denoted by $(R_p)_p$, $(f_p)_p$, $(I_p)_p$, and so forth.

DEFINITION 3.35. Let $\varphi : R \to S$ be a $\mathbb{C}$-algebra homomorphism between finitely generated $\mathbb{C}$-algebras. Suppose that $R \cong \mathbb{C}[X_1,\dots,X_n]/I$ and $S \cong \mathbb{C}[Y_1,\dots,Y_m]/J$. Let $f_i \in \mathbb{C}[Y_1,\dots,Y_m]$ be a lifting of the image of $X_i \bmod I$ under φ. Then we define an *approximation* $\varphi_p : R_p \to S_p$ *of* φ as the morphism induced by $X_i \longmapsto f_{ip}$. Let $\varphi_\infty := \operatorname{ulim}_p \varphi_p$, then the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R_\infty & \xrightarrow{\varphi_\infty} & S_\infty \end{array}$$

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Proposition 3.36 [22, Cor. 4.2], [26, Th. 4.3.4]. *Let R be a finitely generated $\mathbb{C}$-algebra. An ideal $I \subseteq R$ is prime if and only if I_p is prime for almost all p if and only if IR_∞ is prime.*

Definition 3.37. Let R be a local ring essentially of finite type over $\mathbb{C}$. Suppose that $R \cong S_\mathfrak{p}$, where S is a finitely generated $\mathbb{C}$-algebra and $\mathfrak{p}$ is a prime ideal of S. Then we define the *non-standard hull R_∞ of R* by

$$R_\infty := (S_\infty)_{\mathfrak{p}S_\infty}.$$

Remark 3.38. Since $S \to S_\infty$ is faithfully flat, $R \to R_\infty$ is faithfully flat.

Definition 3.39. Let S be a finitely generated $\mathbb{C}$-algebra, let $\mathfrak{p}$ be a prime ideal of S, and let $R \cong S_\mathfrak{p}$.

(1) A family R_p is said to be an *approximation of R* if R_p is an $\overline{\mathbb{F}_p}$-algebra and $R_p \cong (S_p)_{\mathfrak{p}_p}$ for almost all p. Then we have $R_\infty \cong \operatorname{ulim}_p R_p$.
(2) For an element $f \in R$, a family f_p is said to be an *approximation of f* if $f_p \in R_p$ for almost all p and $f = \operatorname{ulim}_p f_p$ in R_∞. For $f \in R_\infty$, we define an *approximation of f* in the same way.
(3) For an ideal $I = (f_1, \ldots, f_s) \subseteq R$, a family I_p is said to be an *approximation of I* if I_p is an ideal of R_p and $I_p = (f_{1p}, \ldots, f_{sp})$ for almost all p. For finitely generated ideal $I \subseteq R_\infty$, we define *an approximation of I* in the same way.

Definition 3.40. Let S_1, S_2 be finitely generated $\mathbb{C}$-algebras, and let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals of S_1, S_2, respectively. Suppose that $R_i \cong (S_i)_{\mathfrak{p}_i}$ and $\varphi : R_1 \to R_2$ is a local $\mathbb{C}$-algebra homomorphism. Let $S_1 \cong \mathbb{C}[X_1, \ldots, X_n]/I$ and f_j/g_j be the image of X_j under φ, where $f_j \in S_2$, $g_j \in S_2 \setminus \mathfrak{p}_2$. Then we say that a homomorphism $R_{1p} \to R_{2p}$ induced by $X_j \longmapsto f_{jp}/g_{jp}$ is an *approximation of φ*. Let $\varphi_\infty := \operatorname{ulim}_p \varphi_p$. Then the following commutative diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R_\infty & \xrightarrow{\varphi_\infty} & S_\infty \end{array}.$$

Definition 3.41. Let R be a finitely generated $\mathbb{C}$-algebra or a local ring essentially of finite type over $\mathbb{C}$, and let M be a finitely generated R-module. Write M as the cokernel of a matrix A, that is, given by an exact sequence

$$R^m \xrightarrow{A} R^n \to M \to 0,$$

where m, n are positive integers. Let A_p be an approximation of A defined by entrywise approximations. Then the cokernel M_p of the matrix A_p is called an *approximation of M* and the ultraproduct $M_\infty := \operatorname{ulim}_p M_p$ is called the *non-standard hull of M*. M_∞ is a finitely generated R_∞-module and independent of the choice of matrix A.

Remark 3.42. Tensoring the above exact sequence with R_∞, we have an exact sequence

$$R_\infty^m \xrightarrow{A} R_\infty^n \to M \otimes_R R_\infty \to 0.$$

Taking the ultraproduct of exact sequences

$$R_p^m \xrightarrow{A_p} R_p^n \to M_p \to 0,$$

we have an exact sequence

$$R_\infty^m \xrightarrow{A} R_\infty^n \to M_\infty \to 0.$$

Therefore, $M_\infty \cong M \otimes_R R_\infty$. Note that if m,n are not integers but infinite cardinals, then the naive definition of an approximation of A does not work and the ultraproduct of $R_p^{\oplus n}$ is not necessarily equal to $R_\infty^{\oplus n}$.

Here, we state basic properties about non-standard hulls and approximations.

Proposition 3.43 [22, 2.9.5, 2.9.7, Ths. 4.5 and 4.6], [26, §4.3]; cf. [2, 5.1]. *Let R be a local ring essentially of finite type over $\mathbb{C}$, then the following hold:*

(1) *R has dimension d if and only if R_p has dimension d for almost all p.*
(2) *$\mathbf{x} = x_1, \ldots, x_i$ is an R-regular sequence if and only if $\mathbf{x}_p = x_{1p}, \ldots, x_{ip}$ is an R_p-regular sequence for almost all p if and only if $\mathbf{x}$ is an R_∞-regular sequence.*
(3) *$\mathbf{x} = x_1, \ldots, x_d$ is a system of parameters of R if and only if $\mathbf{x}_p$ is a system of parameters of R_p for almost all p.*
(4) *R is regular if and only if R_p is regular for almost all p.*
(5) *R is Gorenstein if and only if R_p is Gorenstein for almost all p.*
(6) *R is Cohen–Macaulay if and only if R_p is Cohen–Macaulay for almost all p.*

Proposition 3.44 [31, Prop. 3.9]. *Let R be a local ring essentially of finite type over $\mathbb{C}$. The following conditions are equivalent to each other.*

(1) *R is normal.*
(2) *R_p is normal for almost all p.*
(3) *R_∞ is normal.*

Definition 3.45. Let R be a normal local domain essentially of finite type over $\mathbb{C}$, and let $\Delta = \sum_i a_i \Delta_i$ be a $\mathbb{Q}$-Weil divisor. Assume that Δ_i are prime divisors and $\mathfrak{p}_i$ is a prime ideal associated with Δ_i for each i. Suppose that $\mathfrak{p}_{ip}$ is an approximation of $\mathfrak{p}_i$ and Δ_{ip} is a divisor associated with $\mathfrak{p}_{ip}$. We say $\Delta_p := \sum_i a_i \Delta_{ip}$ is an *approximation of* Δ.

Remark 3.46. If Δ is an effective integral divisor, then this definition is compatible with Definition 3.33 by [22, Th. 4.4]. Hence, if Δ is $\mathbb{Q}$-Cartier, then Δ_p is $\mathbb{Q}$-Cartier for almost all p.

Lastly, we review some singularities introduced by Schoutens via ultraproducts.

Definition 3.47 [22, Def. 5.2], [25, Def. 3.1]. Suppose that R is a finitely generated $\mathbb{C}$-algebra or a local domain essentially of finite type over $\mathbb{C}$. Let $I \subseteq R$ be an ideal. The *generic tight closure $I^{*\mathrm{gen}}$ of I* is defined by

$$I^{*\mathrm{gen}} = (\operatorname*{ulim}_p I_p)^* \cap R.$$

Remark 3.48. The generic tight closure $I^{*\mathrm{gen}}$ of I does not depend on the choice of approximation of I since any two approximations are almost equal.

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DEFINITION 3.49 [25, Def. 4.1 and Rem. 4.7], [23, Def. 4.3]. Suppose that R is a finitely generated $\mathbb{C}$-algebra or a local ring essentially of finite type over $\mathbb{C}$.

(1) R is said to be *weakly generically F-regular* if $I^{*\mathrm{gen}} = I$ for any ideal $I \subseteq R$.
(2) R is said to be *generically F-regular* if $R_{\mathfrak{p}}$ is weakly generically F-regular for any prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.
(3) Let R be a local ring essentially of finite type over $\mathbb{C}$. R is said to be *generically F-rational* if $I^{*\mathrm{gen}} = I$ for some ideal I generated by a system of parameters.

PROPOSITION 3.50 [25, Th. 4.3]. *If R is generically F-rational, then $I^{*\mathrm{gen}} = I$ for any ideal I generated by part of a system of parameters.*

PROPOSITION 3.51 [25, Th. 6.2], [23, Prop. 4.5 and Th. 4.12]. *If R is generically F-rational if and only if R_p is F-rational for almost all p if and only if R has rational singularities.*

DEFINITION 3.52 [24, 3.2]. Let R be a local ring essentially of finite type over $\mathbb{C}$ and R_p be an approximation. Let $\varepsilon := \operatorname{ulim}_p e_p \in {}^*\mathbb{N}$. Then an *ultra-Frobenius* $F^{\varepsilon} : R \to R_\infty$ *associated with* ε is defined by $x \longmapsto \operatorname{ulim}_p(F_p^{e_p}(x_p))$, where F_p is a Frobenius morphism in characteristic p.

DEFINITION 3.53 [24, Def. 3.3]. Let R be a local domain essentially of finite type over $\mathbb{C}$. R is said to be *ultra-F-regular* if, for each $c \in R^{\circ}$, there exists $\varepsilon \in {}^*\mathbb{N}$ such that

$$R \xrightarrow{cF^{\varepsilon}} R_\infty$$

is pure.

PROPOSITION 3.54 [24, Th. A]. *Let R be a $\mathbb{Q}$-Gorenstein normal local domain essentially of finite type over $\mathbb{C}$. Then R is ultra-F-regular if and only if R has log-terminal singularities.*

3.3 Relative hulls

In this subsection, we introduce the concept of relative hulls and approximations of schemes, cohomologies, and so forth. We refer the reader to [22], [24], [25].

DEFINITION 3.55 (Cf. [25]). Let R be a local ring essentially of finite type over $\mathbb{C}$. Suppose that X is a finite tuple of indeterminates and $f \in R[X]$ is a polynomial such that $f = \sum_\nu a_\nu X^\nu$, where ν is a multi-index. If $a_{\nu p}$ is an approximation of a_ν for each ν, then the sequence of polynomials $f_p := \sum_\nu a_{\nu p} X^\nu$ is said to be an *R-approximation of f.* If $I := (f_1, \dots, f_s)$ is an ideal in $R[X]$, then we call $I_p := (f_{1p}, \dots, f_{sp}) R_p[X]$ an *R-approximation* of I, and if $S = R[X]/I$, then we call $S_p := R_p[X]/I_p$ an *R-approximation* of S.

REMARK 3.56. Any two R-approximations of a polynomial f are almost equal. Similarly, any two R-approximations of an ideal I are almost equal.

DEFINITION 3.57 (Cf. [25]). Let S be a finitely generated R-algebra, and let S_p be an R-approximation of S, then we call $S_\infty = \operatorname{ulim}_p S_p$ the *(relative) R-hull of S.*

DEFINITION 3.58 (Cf. [24]). If X is an affine scheme $\operatorname{Spec} S$ of finite type over $\operatorname{Spec} R$, then we call $X_p := \operatorname{Spec} S_p$ is an *R-approximation of X.*

DEFINITION 3.59 (Cf. [24]). Suppose that $f : Y \to X$ is a morphism of affine schemes of finite type over $\operatorname{Spec} R$. If $X = \operatorname{Spec} S, Y = \operatorname{Spec} T$ and $\varphi : S \to T$ is the morphism

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corresponding to f, then we call $f_p : Y_p \to X_p$ is an *R-approximation of f*, where f_p is a morphism of R_p-schemes induced by an R-approximation $\varphi_p : S_p \to T_p$.

DEFINITION 3.60 (Cf. [24]). Let S be a finitely generated R-algebra, and let M be a finitely generated S-module. Write M as the cokernel of a matrix A, that is, given by an exact sequence

$$S^m \xrightarrow{A} S^n \to M \to 0,$$

where m,n are positive integers. Let A_p be an R-approximation of A defined by entrywise R-approximations. Then the cokernel M_p of the matrix A_p is called an *R-approximation of M* and the ultraproduct $M_\infty := \operatorname{ulim}_p M_p$ is called the *R-hull of M*. M_∞ is independent of the choice of the matrix A and $M_\infty \cong M \otimes_S S_\infty$.

REMARK 3.61. If M is not finitely generated, then we cannot define an R-approximation of M in this way. It is crucial that any two R-approximations of A is equal for almost all p.

DEFINITION 3.62 [24]. Let X be a scheme of finite type over $\operatorname{Spec} R$. Let $\mathfrak{U} = \{U_i\}$ is a finite affine open covering of X and U_{ip} be an R-approximation of U_i. Gluing $\{U_{ip}\}$ together, we obtain a scheme X_p of finite type over $\operatorname{Spec} R_p$. We call X_p an *R-approximation of X*.

REMARK 3.63. Suppose that $\{U_{ijk}\}_k$ is a finite affine open covering of $U_i \cap U_j$ and $\varphi_{ijk} : \mathcal{O}_{U_i}|_{U_k} \cong \mathcal{O}_{U_j}|_{U_k}$ are isomorphisms. Then R-approximations $\varphi_p : \mathcal{O}_{U_{ip}}|_{U_{kp}} \to \mathcal{O}_{U_{jp}}|_{U_{kp}}$ are isomorphisms for almost all p (note that indices ijk are finitely many). Hence, we can glue these together. For any other choice of finite affine open covering $\mathfrak{U}'$ of X, the resulting R-approximation X'_p is isomorphic to X_p for almost all p.

DEFINITION 3.64 (Cf. [24]). Suppose that $f : Y \to X$ is a morphism between schemes of finite type over $\operatorname{Spec} R$. Let $\mathfrak{U}$, $\mathfrak{V}$ be finite affine open coverings of X and Y, respectively, such that for any $V \in \mathfrak{V}$, there exists some $U \in \mathfrak{U}$ such that $f(V) \subseteq U$. Let $\mathfrak{U}_p$, $\mathfrak{V}_p$ be R-approximations of $\mathfrak{U}$, $\mathfrak{V}$ and $(f|_V)_p$ an *R-approximation of $f|_V$*. We define an R-approximation f_p of f by the morphism determined by $(f|V)_p$.

REMARK 3.65. In the same way as the above Remark 3.63, $(f|_V)_p$ and $(f|_{V'})_p$ agree on $V \cap V'$ for any two opens $V, V' \in \mathfrak{V}$ for almost all p.

DEFINITION 3.66 (Cf. [24]). Let X be a scheme of finite type over $\operatorname{Spec} R$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathfrak{U}$ be a finite affine open covering of X. For any $U \in \mathfrak{U}$, we have an R-approximation M_{Up} of M_U such that M_U is a finitely generated $\mathcal{O}_U$-module and $\widetilde{M_U} \cong \mathcal{F}|_U$. We define an *$R$-approximation $\mathcal{F}_p$ of $\mathcal{F}$* by the coherent $\mathcal{O}_{X_p}$-module determined by $\widetilde{M_{Up}}$.

DEFINITION 3.67 (Cf. [24]). Let X be a separated scheme of finite type over $\operatorname{Spec} R$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the ultra-cohomology of $\mathcal{F}$ is defined by

$$H^i_\infty(X, \mathcal{F}) := \operatorname*{ulim}_p H^i(X_p, \mathcal{F}_p).$$

REMARK 3.68. In the above setting, let $\mathfrak{U} = \{U_i\}_{i=1,\dots,n}$ be a finite affine open covering of X, let

$$C^j(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_j} \mathcal{F}(U_{i_0 \dots i_j}),$$

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where $U_{i_0\ldots i_j} := U_{i_0} \cap \cdots \cap U_{i_j}$, and let

$$(C^j(\mathfrak{U},\mathcal{F}))_p := \prod_{i_0\ldots i_j} (\mathcal{F}(U_{i_0\ldots i_j}))_p,$$

where $\mathcal{F}(U_{i_0\ldots i_j})_p$ is an R-approximation considered as $\mathcal{O}(U_{i_0\ldots i_j})$-module. Then

$$(C^j(\mathfrak{U},\mathcal{F}))p$$

coincides with the jth term of the Čech complex of X_p, $\mathfrak{U}_p$, and $\mathcal{F}_p$. We have a commutative diagram

$$\begin{array}{ccccc} C^{j-1}(\mathfrak{U},\mathcal{F}) & \longrightarrow & C^j(\mathfrak{U},\mathcal{F}) & \longrightarrow & C^{j+1}(\mathfrak{U},\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{ulim}_p(C^{j-1}(\mathfrak{U},\mathcal{F}))_p & \longrightarrow & \mathrm{ulim}_p(C^j(\mathfrak{U},\mathcal{F}))_p & \longrightarrow & \mathrm{ulim}_p(C^{j+1}(\mathfrak{U},\mathcal{F}))_p. \end{array}$$

Since $\mathrm{ulim}_p(\text{-})$ is an exact functor, we have

$$\check{H}^j(\mathfrak{U},\mathcal{F}) \to \mathrm{ulim}_p \check{H}^j(\mathfrak{U}_p,\mathcal{F}_p).$$

If X is separated, then X_p is separated for almost all p. This can be checked by taking a finite affine open covering and observing that if the diagonal morphism $\Delta_{X/\operatorname{Spec} R}$ is a closed immersion, then $\Delta_{X_p/\operatorname{Spec} R_p}$ is also a closed immersion for almost all p. Hence, we have the map

$$H^j(\mathfrak{U},\mathcal{F}) \to \mathrm{ulim}_p H^j(\mathfrak{U}_p,\mathcal{F}_p).$$

Note that we do not know whether this map is injective or not.

PROPOSITION 3.69. *Let R be a local ring essentially of finite type over $\mathbb{C}$ of dimension d, $\mathbf{x} = x_1,\ldots,x_d$ a system of parameters and M a finitely generated R-module. Then we have a natural homomorphism $H^d_{\mathfrak{m}}(M) \to \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(M_p)$.*

Proof. Since $M_{x_1\cdots\hat{x_i}\cdots x_d}$ is a finitely generated $R_{x_1\cdots\hat{x_i}\cdots x_d}$-module and $M_{x_1\cdots x_d}$ is a finitely generated $R_{x_1\cdots x_d}$-module, we have an R-approximation $(M_{x_1\cdots\hat{x_1}\cdots x_d})_p \cong (M_p)_{x_{1p}\cdots\hat{x_{ip}}\cdots x_{dp}}$ and $(M_{x_1\cdots x_d})_p \cong (M_p)_{x_{1p}\cdots x_{dp}}$ for almost all p. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_i M_{x_1\cdots\hat{x_i}\cdots x_d} & \longrightarrow & M_{x_1\cdots x_d} \\ \downarrow & & \downarrow \\ \bigoplus_i \mathrm{ulim}_p(M_p)_{x_{1p}\cdots\hat{x_{ip}}\cdots x_{dp}} & \longrightarrow & \mathrm{ulim}_p(M_p)_{x_{1p}\cdots x_{dp}} \end{array}.$$

Taking the cokernel of rows, we have the desired map. ∎

REMARK 3.70. We do not know whether $H^d_{\mathfrak{m}}(M) \to \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(M_p)$ is injective or not.

PROPOSITION 3.71. *Let R be a local ring essentially of finite type over $\mathbb{C}$ of dimension d, $\mathbf{x} = x_,\ldots,x_d$ be a system of parameters and M_p be an R_p-module for almost all p. Then we have a natural homomorphism $H^d_{\mathfrak{m}}(\mathrm{ulim}_p M_p) \to \mathrm{ulim}_p H^d_{\mathfrak{m}}(M_p)$.*

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Proof. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_i (\mathrm{ulim}_p M_p)_{x_1\cdots \hat{x_i}\cdots x_d} & \longrightarrow & (\mathrm{ulim}_p M_p)_{x_1\cdots x_d} \\ \downarrow & & \downarrow \\ \bigoplus_i \mathrm{ulim}_p (M_p)_{x_{1p}\cdots \hat{x_{ip}}\cdots x_{dp}} & \longrightarrow & \mathrm{ulim}_p (M_p)_{x_{1p}\cdots x_{dp}} \end{array}.$$

∎

Taking the cokernel of rows, we have the desired map.

§4. Big Cohen–Macaulay algebras constructed via ultraproducts

In [23], Schoutens constructed the canonical BCM algebra in characteristic zero. Following the idea of [23], we will deal with BCM algebras constructed via ultraproducts in slightly general settings. In this section, suppose that $(R, \mathfrak{m})$ is a local domain essentially of finite type over $\mathbb{C}$ and R_p is an approximation of R.

DEFINITION 4.1 [23, §2]. Suppose that R is a local domain essentially of finite type over $\mathbb{C}$. Then we define the *canonical BCM algebra* $\mathcal{B}(R)$ *of* R by

$$\mathcal{B}(R) := \operatorname*{ulim}_p R_p^+.$$

SETTING 4.2. Let R be a local domain essentially of finite type over $\mathbb{C}$ of dimension d, and let B_p be a BCM ${R_p}^+$-algebra for almost all p. We use B to denote $\mathrm{ulim}_p B_p$.

REMARK 4.3. By Theorem 2.8, we can set $B_p = R_p^+$ and $B = \mathcal{B}(R)$ in Setting 4.2.

PROPOSITION 4.4. *$\mathcal{B}(R)$ is a domain over R^+-algebra.*

Proof. By Łoś's theorem, $\mathcal{B}(R)$ is a domain over $R_\infty = \mathrm{ulim}_p R_p$. Hence, $\mathcal{B}(R)$ is an R-algebra. Let $f = \sum a_n x^n \in \mathcal{B}(R)[x]$ be a monic polynomial in one variable over $\mathcal{B}(R)$ and let $f_p = \sum a_{np} x^n$ be an approximation of f. Since f_p is a monic polynomial for almost all p and R_p^+ is absolutely integrally closed, f_p has a root c_p in R_p^+ for almost all p. Hence, $c := \mathrm{ulim}_p c_p \in \mathcal{B}(R)$ is a root of f by Łoś's theorem. Hence, $\mathcal{B}(R)$ is absolutely integrally closed. In particular, $\mathcal{B}(R)$ contains an absolute integral closure R^+ of R. ∎

COROLLARY 4.5. *In Setting 4.2, B is an R^+-algebra.*

Proof. Since B_p is an R_p^+-algebra for almost all p, B is an R^+-algebra by the above proposition. ∎

PROPOSITION 4.6. *In Setting 4.2, B is a BCM R-algebra.*

Proof. Assume that B is not a BCM R-algebra. Since $B_p \neq \mathfrak{m}_p B_p$ for almost all p, we have $B \neq \mathfrak{m}B$. Hence, there exists part of system of parameters $x_1, \dots, x_i$ of R such that $(x_1, \dots, x_{i-1})B \subsetneq (x_1, \dots, x_{i-1})B :_B x_i$. Then there exists $y \in B$ such that $x_i y \in (x_1, \dots, x_{i-1})B$ and $y \notin (x_1, \dots, x_{i-1})B$. Taking approximations, we have $x_{ip} y_p \in (x_{1p}, \dots, x_{(i-1)p})B_p$ and $y_p \notin (x_{1p} \dots, x_{(i-1)p})B_p$ for almost all p. Since $x_{1p}, \dots, x_{ip}$ is part of a system of parameters of R_p and B_p is a BCMR_p-algebra for almost all p, $x_{1p}, \dots, x_{ip}$ is a regular sequence for almost all p. This is a contradiction. Therefore, B is a BCM R-algebra. ∎

LEMMA 4.7. *In Setting 4.2, the natural homomorphism $H^d_{\mathfrak{m}}(B) \to \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(B_p)$ is injective.*

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Proof. Let $x = x_1 \cdots x_d$ be the product of a system of parameters and $[\frac{z}{x^t}]$ be an element of $H^d_{\mathfrak{m}}(B)$ such that the image in $\mathrm{ulim}_p H^d_{\mathfrak{m}_p}(B_p)$ is zero. Then there exists $s_p \in \mathbb{N}$ such that $x^{s_p} z \in (x_{1p}^{s_p+t}, \ldots, x_{dp}^{s_p+t}) B_p$ for almost all p. Since B_p is a BCM R_p-algebra for almost all p, $z \in (x_{1p}^t, \ldots, x_{dp}^t) B_p$ for almost all p. Hence, $z \in (x_1^t, \ldots, x_d^t) B$ and $[\frac{z}{x^t}] = 0$ in $H^d_{\mathfrak{m}}(B)$. □

We generalize [23, Th. 4.2] to the cases other than the canonical BCM algebra.

Proposition 4.8 (Cf. [23, Th. 4.2], [19, Prop. 3.7]). *In Setting 4.2, R is BCM_B-rational if and only if R has rational singularities. In particular, R has rational singularities if R is BCM-rational.*

Proof. Let $x := x_1 \cdots x_d$ is the product of a system of parameters. Suppose that R has rational singularities. By [23, Prop. 4.11] and [9], R_p is F-rational for almost all p. Let $\eta := [\frac{z}{x^t}]$ be an element of $H^d_{\mathfrak{m}}(R)$ such that $\eta = 0$ in $H^d_{\mathfrak{m}}(B)$. Then we have a commutative diagram

$$\begin{array}{ccc} H^d_{\mathfrak{m}}(R) & \longrightarrow & \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(R_p) \\ \downarrow & & \downarrow \\ H^d_{\mathfrak{m}}(B) & \longrightarrow & \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(B_p). \end{array}$$

By [19, Prop. 3.5], $H^d_{\mathfrak{m}_p}(R_p) \to H^d_{\mathfrak{m}_p}(B_p)$ is injective for almost all p. Hence, $\mathrm{ulim}_p H^d_{\mathfrak{m}_p}(R_p) \to \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(B_p)$ is injective. Therefore, $[\frac{z_p}{x_p^t}] = 0$ in $H^d_{\mathfrak{m}_p}(R_p)$ for almost all p. Since R_p is Cohen–Macaulay for almost all p, we have $z_p \in (x_{1p}^t, \ldots, x_{dp}^t)$ for almost all p. Hence, $z \in (x_1^t, \ldots, x_d^t)$ by Łoś's theorem. Therefore, $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B)$ is injective. Conversely, suppose that R is BCM_B-rational. Let $I = (x_1, \ldots, x_d)$ be an ideal generated by the system of parameters. Let $z \in I^{*\mathrm{gen}}$. Since $I_p^* \subseteq I_p B_p \cap R_p$ by [27, Th. 5.1] for almost all p, we have $[\frac{z_p}{x_p}] = 0$ in $H^d_{\mathfrak{m}_p}(B_p)$ for almost all p. Since $H^d_{\mathfrak{m}}(B) \to \mathrm{ulim}_p H^d_{\mathfrak{m}_p}(B_p)$ and $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B)$ are injective, we have $[\frac{z}{x}] = 0$ in $H^d_{\mathfrak{m}}(R)$. Since R is Cohen–Macaulay, $z \in I$. Therefore, R is generically F-rational. By Proposition 3.51 (see [25, Th. 6.2]), R has rational singularities. □

§5. Approximations of multiplier ideals

In this section, we will explain the relationship between approximations and reductions modulo $p \gg 0$. Note that an isomorphism $\mathrm{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$ is fixed.

Definition 5.1. Let R be a finitely generated $\mathbb{C}$-algebra. A pair (A, R_A) is called a *model of R* if the following two conditions hold:

(i) $A \subseteq \mathbb{C}$ is a finitely generated $\mathbb{Z}$-subalgebra.
(ii) R_A is a finitely generated A-algebra such that $R_A \otimes_A \mathbb{C} \cong R$.

Proposition 5.2 [23, Lem. 4.10]. *Let A be a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$. There exists a family $(\gamma_p)_p$ which satisfies the following two conditions:*

(i) *$\gamma_p : A \to \overline{\mathbb{F}_p}$ is a ring homomorphism for almost all p.*
(ii) *For any $x \in A$, $x = \mathrm{ulim}_p \gamma_p(x)$.*

Proposition 5.3 (Cf. [23, Cor. 4.10]). *Let R be a finitely generated $\mathbb{C}$-algebra, and let $\mathbf{a} = a_1, \ldots, a_l$ be finitely many elements of R. Let R_p be an approximation of R. Then there exists a model (A, R_A) which satisfies the following conditions:*

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(i) *There exists a family* (γ_p) *as in Proposition 5.2.*
(ii) $\mathbf{a} \subseteq R_A$.
(iii) $R_A \otimes_A \overline{\mathbb{F}_p} \cong R_p$ *for almost all* p.
(iv) *For any* $x \in R_A$, *the ultraproduct of the image of* x *under* $\mathrm{id}_{R_A} \otimes_A \gamma_p$ *is* x.

Proof. Let $X = X_1, \ldots, X_n$ and $R \cong \mathbb{C}[X]/I$ for some ideal $I \subseteq \mathbb{C}[X]$. Take any model (A, R_A) which contains $\mathbf{a}$. Enlarging this model, we may assume that there exits an ideal $I_A \subseteq A[X]$ such that $R_A \cong A[X]/I_A$ and $I_A \otimes_A \mathbb{C} = I$ in $\mathbb{C}[X]$. Take (γ_p) as in Proposition 5.2. Let $I = (f_1, \ldots, f_m)$. For $f = \sum_\nu c_\nu X^\nu \in A[X] \subseteq \mathbb{C}[X]$, by the definition of approximations, $f_p := \sum_\nu \gamma_p(c_\nu) X^\nu \in \overline{\mathbb{F}_p}[X]$ is an approximation of f. Hence, by the definition of approximations of finitely generated $\mathbb{C}$-algebras, $R_A \otimes_A \overline{\mathbb{F}_p} \cong \overline{\mathbb{F}_p}[X]/(f_{1p}, \ldots, f_{mp})\overline{\mathbb{F}_p}[X]$ is an approximation of R. Since two approximations are isomorphic for almost all p, $R_A \otimes_A \overline{\mathbb{F}_p} \cong R_p$ for almost all p. The condition (iv) is clear by the above argument. ∎

REMARK 5.4. Let $\mathfrak{p} = (x_1, \ldots, x_n) \subseteq R$ be a prime ideal. Enlarging the model (A, R_A), we may assume that $x_1, \ldots, x_n \in R_A$. Let μ_p be the kernel of $\gamma_p : A \to \overline{\mathbb{F}_p}$. Then this is a maximal ideal of A and A/μ_p is a finite field. $\mathfrak{p}_{\mu_p} = (x_1, \ldots, x_n)R_A/\mu_p R_A$ is prime for almost all p since this is a reduction to $p \gg 0$. On the other hand, $\mathfrak{p}_p := (x_1, \ldots, x_n)R_A \otimes_A \overline{\mathbb{F}_p} \subseteq R_p$ is an approximation of $\mathfrak{p}$. Hence, $\mathfrak{p}_p$ is prime for almost all p. Here, $(R_p)_{\mathfrak{p}_p}$ is an approximation of $R_\mathfrak{p}$. Thus we have a flat local homomorphism $(R_A/\mu_p R_A)_{\mathfrak{p}_{\mu_p}} \to R_p$ with $\mathfrak{p}_{\mu_p} R_p = \mathfrak{p}_p$. Moreover, if $\mathfrak{p}$ is maximal, then $\mathfrak{p}_{\mu_p}, \mathfrak{p}_p$ are maximal for almost all p. Then, the map $R_A/\mathfrak{p}_{\mu_p} \to R_p/\mathfrak{p}_p \cong \overline{\mathbb{F}_p}$ is a separable field extension since $R_A/\mathfrak{p}_{\mu_p}$ is a finite field.

The next result is a generalization of [31, Th. 4.6] from ideal pairs to triples.

PROPOSITION 5.5. *Let R be a normal local domain essentially of finite type over $\mathbb{C}$, let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, let $\mathfrak{a}$ be a nonzero ideal, and let $t > 0$ be a real number. Suppose that R_p, Δ_p, $\mathfrak{a}_p$ are approximations. Then $\tau(R_p, \Delta_p, \mathfrak{a}_p^t)$ is an approximation of $\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$.*

Proof. Let $R = S_\mathfrak{p}$, where S is a normal domain of finite type over $\mathbb{C}$ and $\mathfrak{p}$ is a prime ideal. Let $\mathfrak{m}$ be a maximal ideal contains $\mathfrak{p}$. Then there exists a model (A, S_A) of S such that the properties in Proposition 5.3 hold and S_A containing a system of generators of $\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ and Δ_A, $\mathfrak{a}_A$ can be defined properly. Let μ_p be maximal ideals of S_A as in Remark 5.4, and let $\mathfrak{m}_{\mu_p}, \mathfrak{p}_{\mu_p}$ be reductions to $p \gg 0$. Since, for almost all p, $(S_A/\mu_p)_{\mathfrak{m}_{\mu_p}} \to (S_\mathfrak{m})_p$ is a flat local homomorphism such that $S_A/\mathfrak{m}_{\mu_p} \to (S/\mathfrak{m})_p \cong \overline{\mathbb{F}_p}$ is a separable field extension, we have

$$\tau((S_A/\mu_p)_{\mathfrak{m}_{\mu_p}}, \Delta_{(S_A/\mu_p)_{\mathfrak{m}_{\mu_p}}}, \mathfrak{a}^t_{(S_A/\mu_p)_{\mathfrak{m}_{\mu_p}}})(S_\mathfrak{m})_p = \tau((S_\mathfrak{m})_p, \Delta_{\mathfrak{m}_p}, \mathfrak{a}^t_{\mathfrak{m}_p}),$$

by a generalization of [28, Lem. 1.5]. Since the localization commutes with test ideals [10, Prop. 3.1], we have

$$\tau((S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}, \Delta_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}}, \mathfrak{a}^t_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}})R_p = \tau(R_p, \Delta_p, \mathfrak{a}_p^t)$$

for almost all p. Since the reduction of multiplier ideals modulo $p \gg 0$ is the test ideal [29, Th. 3.2], $\tau((S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}, \Delta_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}}, \mathfrak{a}^t_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}})$ is a reduction of

$$\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$$

to characteristic $p \gg 0$. Hence, $\tau(R_p, \Delta_p, \mathfrak{a}_p^t)$ is an approximation of $\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$. ∎

§6. BCM test ideal with respect to a big Cohen–Macaulay algebra constructed via ultraproducts

Throughout this section, we assume that $(R,\mathfrak{m})$ is a normal local domain essentially of finite type over $\mathbb{C}$. Fix a canonical divisor K_R such that $R \subseteq \omega_R := R(K_R) \subseteq \mathrm{Frac}(R)$. Let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that $\operatorname{div} f = n(K_R+\Delta)$ for $f \in R^\circ$, $n \in \mathbb{N}$. Let B_p be a BCM R_p^+-algebra for almost all p and $B := \mathrm{ulim}_p B_p$. We use $\widehat{R}$ to denote the completion of R with respect to $\mathfrak{m}$ and $\widehat{\Delta}$ to denote the flat pullback of Δ by $\operatorname{Spec}\widehat{R} \to \operatorname{Spec} R$.

Proposition 6.1. *In the setting as above, we have*

$$\mathcal{J}(\widehat{R},\widehat{\Delta}) \subseteq \tau_{\widehat{B}}(\widehat{R},\widehat{\Delta}).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} 0^{B,K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)} & \longrightarrow & \mathrm{ulim}_p\, 0^{B_p,K_{R_p}+\Delta_p}_{H^d_{\mathfrak{m}_p}(\omega_{R_p})} \\ \downarrow & & \downarrow \\ H^d_{\mathfrak{m}}(\omega_R) & \longrightarrow & \mathrm{ulim}_p\, H^d_{\mathfrak{m}_p}(\omega_{R_p}) \\ \psi \downarrow & & \downarrow \\ H^d_{\mathfrak{m}}(B) & \longrightarrow & \mathrm{ulim}_p\, H^d_{\mathfrak{m}_p}(B_p) \end{array}.$$

By Proposition 2.12, we have

$$0^{B_p,K_{R_p}+\Delta_p}_{H^d_{\mathfrak{m}_p}(\omega_{R_p})} = 0^{*\Delta_p}_{H^d_{\mathfrak{m}_p}(\omega_{R_p})}$$

for almost all p. Let $x_1,\dots,x_d$ be a system of parameters, and let $x = x_1 \cdots x_d$ be the product of them. Take $a \in \mathcal{J}(R,\Delta) = \mathrm{ulim}_p\, \tau(R_p,\Delta_p) \cap R$ and $[\frac{z}{x^t}] \in 0^{B,K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)}$. Let J be a divisorial ideal which is isomorphic to ω_R and $g \in R^\circ$ an element such that $\omega_R \xrightarrow{\cdot g} J$ is an isomorphism. As in Proof of [29, Th. 2.8], we have $g_p z_p x_p^t \in ((x_{1p}^{2t},\dots,x_{dp}^{2t})J_p)^{*\Delta_p}$ for almost all p. Hence, $a_p g_p z_p x_p^t \in (x_{1p}^{2t},\dots,x_{dp}^{2t})J_p$ for almost all p. Therefore, $agzx^t \in (x_1^{2t},\dots,x_d^{2t})J$ and $[\frac{az}{x^t}] = 0$ in $H^d_{\mathfrak{m}}(\omega_R)$. Hence, we have $a \in \mathrm{Ann}_R\, 0^{B,K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)}$. In conclusion, we have $\mathcal{J}(R,\Delta)\widehat{R} \subseteq \tau_{\widehat{B}}(\widehat{R},\widehat{\Delta})$. ∎

Lemma 6.2 [29, Th. 2.13]. *Let $(R,\mathfrak{m})$ be an F-finite normal local domain of characteristic $p > 0$ and $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor on $X := \operatorname{Spec} R$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism with X normal. Suppose that $Z := f^{-1}(\mathfrak{m})$ and $\delta : H^d_{\mathfrak{m}}(R(K_X)) \to H^d_Z(Y, \mathcal{O}_Y(\lfloor f^*(K_X+\Delta)\rfloor)$ is the Matlis dual of the natural inclusion map $H^0(Y,\mathcal{O}_Y(\lceil K_Y - f^*(K_X+\Delta)\rceil)) \hookrightarrow R$. Then $\operatorname{Ker}\delta \subseteq 0^{*\Delta}_E$, where E is the injective hull of the residue field $R/\mathfrak{m}$ of R.*

Proof. By [29, Th. 2.13], we have $\tau(R,\Delta) \subseteq H^0(Y,\mathcal{O}_Y(\lceil K_Y - f^*(K_X+\Delta)\rceil))$. Hence,

$$\begin{aligned}\operatorname{Ker}\delta &= \mathrm{Ann}_E\, H^0(Y,\mathcal{O}_Y(\lceil K_Y - f^*(K_X+\Delta)\rceil)) \\ &\subseteq \mathrm{Ann}_E\, \tau(R,\Delta) \\ &= \mathrm{Ann}_E\, \tau(R,\Delta)\widehat{R}\end{aligned}$$

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$$
\begin{aligned}
&= \operatorname{Ann}_E \tau(\widehat{R}, \widehat{\Delta}) \\
&= \operatorname{Ann}_E \operatorname{Ann}_{\widehat{R}} 0_E^{*\Delta} \\
&= 0_E^{*\Delta}.
\end{aligned}
$$

∎

REMARK 6.3. Moreover, we have $\operatorname{Ker}\delta = 0_E^{*\Delta}$ if f is a reduction of a log resolution in characteristic zero modulo $p \gg 0$ by [29, Th. 3.2].

THEOREM 6.4. *Let R be a normal local domain essentially of finite type over $\mathbb{C}$. Fix an effective canonical divisor $K_R \geqslant 0$ on $\operatorname{Spec} R$. Let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier and B_p is a BCM R_p^+-algebra for almost all p. Suppose that $n(K_R + \Delta) = \operatorname{div}(f)$ for $f \in R^\circ, n \in \mathbb{N}$. Then we have*

$$\tau_{\widehat{B}}(\widehat{R}, \widehat{\Delta}) = \mathcal{J}(\widehat{R}, \widehat{\Delta}).$$

Proof. Thanks to Proposition 6.1, it suffices to prove $\tau_{\widehat{B}}(\widehat{R}, \widehat{\Delta}) \subseteq \mathcal{J}(\widehat{R}, \widehat{\Delta})$. Let $\mu : Y \to X := \operatorname{Spec} R$ be a log resolution of (X, Δ), and let $Z := \mu^{-1}(\mathfrak{m})$. Considering approximations, we have a corresponding morphisms $\mu_p : Y_p \to X_p := \operatorname{Spec} R_p$, $Z_p = \mu_p^{-1}(\mathfrak{m}_p)$ for almost all p. Then we have a commutative diagram

$$
\begin{array}{ccccc}
 & & H^d_{\mathfrak{m}}(\omega_R) & & \\
 & & \downarrow{\scriptstyle \gamma} & \overset{\delta}{\searrow} & \\
H^{d-1}(Y, \mathcal{L}) & \longrightarrow & H^{d-1}(Y \setminus Z, \mathcal{L}|_{Y \setminus Z}) & \longrightarrow & H_Z(\mathcal{L}) \\
\downarrow & & \downarrow{\scriptstyle u^{d-1}} & & \\
H^{d-1}_\infty(Y, \mathcal{L}) & \xrightarrow{\rho^{d-1}_\infty} & H^{d-1}_\infty(Y \setminus Z, \mathcal{L}|_{Y \setminus Z}) & &
\end{array}
,
$$

where $\mathcal{L} := \mathcal{O}_Y(\lfloor \mu^*(K_X + \Delta) \rfloor)$ and the middle row is exact. Similarly, we have the following commutative diagram for almost all p:

$$
\begin{array}{ccccc}
 & & H^d_{\mathfrak{m}_p}(\omega_{R_p}) & & \\
 & & \downarrow{\scriptstyle \gamma_p} & \overset{\delta_p}{\searrow} & \\
H^{d-1}(Y_p, \mathcal{L}_p) & \xrightarrow{\rho^{d-1}_p} & H^{d-1}(Y_p \setminus Z_p, \mathcal{L}_p|_{Y_p \setminus Z_p}) & \longrightarrow & H^d_{Z_p}(\mathcal{L}_p)
\end{array}
,
$$

where the middle row is exact. Assume that $\eta \in \operatorname{Ker}\delta$. Then $u^{d-1}(\gamma(\eta)) \in \operatorname{Im}\rho^{d-1}_\infty$. Therefore, $\gamma_p(\eta_p) \in \operatorname{Im}\rho^{d-1}_p$ for almost all p. Hence, $\eta_p \in \operatorname{Ker}\delta_p$ for almost all p. By Lemma 6.2, $\eta_p \in 0^{*\Delta_p}_{H^d_{\mathfrak{m}_p}(\omega_{R_p})}$ for almost all p. Hence, by Propositon 2.12, we have $\eta_p \in 0^{B_p, K_{R_p}+\Delta_p}_{H^d_{m_p}(\omega_{R_p})}$ for almost all p. We have a commutative diagram

$$
\begin{array}{ccc}
H^d_{\mathfrak{m}}(\omega_R) & \longrightarrow & \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(\omega_{R_p}) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_\infty := \operatorname{ulim}_p \psi_p} \\
H^d_{\mathfrak{m}}(B) & \longrightarrow & \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(B_p)
\end{array}
,
$$

where ψ, ψ_p are the morphisms as in Definition 2.11. Since $\psi_\infty(\operatorname{ulim}_p \eta_p) = 0$ and $H^d_{\mathfrak{m}}(B) \to \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(B_p)$ is injective by Lemma 4.7, we have $\psi(\eta) = 0$ in $H^d_{\mathfrak{m}}(B)$. Hence, $\eta \in 0^{B, K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)}$. Therefore, we have

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$$\begin{aligned}\tau_{\widehat{B}}(\widehat{R},\widehat{\Delta}) &\subseteq \operatorname{Ann}_{\widehat{R}}(\operatorname{Ker}\delta)\\ &= \operatorname{Ann}_{\widehat{R}}\operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)}\mathcal{J}(R,\Delta)\\ &= \mathcal{J}(\widehat{R},\widehat{\Delta}).\end{aligned}$$

∎

REMARK 6.5. We can generalize the notion of ultra-test ideals in [31, Def. 5.5] to the pair (R,Δ). Using Lemma 6.2 instead of [11, Th. 6.9], we can show that generalized ultra-test ideals are equal to multiplier ideals.

§7. Generalized module closures and applications

We introduce the notion of generalized module closures inspired by [20]. Using the generalized module closures, we will generalize [31, Cor. 5.30]. We also use [19, §6.1] as reference in the following arguments.

SETTING 7.1. Suppose that R is a normal local domain essentially of finite type over $\mathbb{C}$ of dimension d, $K_R \geqslant 0$ is a fixed effective canonical divisor and $\Delta \geqslant 0$ is an effective $\mathbb{Q}$-Weil divisor such that $K_R+\Delta$ is $\mathbb{Q}$-Cartier. Moreover, we assume that B_p is a BCM R_p^+-algebra for almost all p, $B := \operatorname{ulim}_p B_p$ and $r(K_R+\Delta) = \operatorname{div} f$ for $f \in R$, $r \in \mathbb{N}$. Let $R' \subseteq R^+$ be an integrally closed finite extension of R such that $f^{1/r} \in R'$ and $\pi^*\Delta$ is Weil divisor, where $\pi : \operatorname{Spec} R' \to \operatorname{Spec} R$.

DEFINITION 7.2. Assume Setting 7.1 and let $g \in R^\circ$ and $t > 0$ be a positive rational number. We use $\widehat{B_\Delta}$ to denote

$$B \otimes_{R'} R'(\pi^*\Delta) \otimes_R \widehat{R}.$$

For any $\widehat{R}$-modules $N \subseteq M$, we define $N_M^{\mathrm{cl}_{\widehat{B_\Delta}},g^t}$ as follows: $x \in N_M^{\mathrm{cl}_{\widehat{B_\Delta}},g^t}$ if and only if $g^t \otimes x \in \operatorname{Im}(\widehat{B_\Delta} \otimes_{\widehat{R}} N \to \widehat{B_\Delta} \otimes_{\widehat{R}} M)$. We use $\tau_{\mathrm{cl}_{\widehat{B_\Delta},g^t}}(\widehat{R})$ to denote

$$\bigcap_{N\subseteq M} (N :_{\widehat{R}} N_M^{\mathrm{cl}_{\widehat{B_\Delta}},g^t}),$$

where M runs through all $\widehat{R}$-modules and N runs through all $\widehat{R}$-submodules of M.

PROPOSITION 7.3. *In Setting 7.1, if $g \in R^\circ$ and $t > 0$ is a positive rational number, then we have*

$$\tau_{\mathrm{cl}_{\widehat{B_\Delta},g^t}}(\widehat{R}) = \bigcap_M \operatorname{Ann}_{\widehat{R}} 0_M^{\mathrm{cl}_{\widehat{B_\Delta}},g^t} = \operatorname{Ann}_{\widehat{R}} 0_E^{\mathrm{cl}_{\widehat{B_\Delta}},g^t},$$

where M runs through all $\widehat{R}$-modules and E is the injective hull of the residue field of R.

Proof. We can prove this by arguments similar to [20, Lem. 3.3 and Prop. 3.9]. ∎

PROPOSITION 7.4. *In Setting 7.1, if $g \in R^\circ$ and $t > 0$ is a positive rational number, then we have*

$$0_E^{B,K_R+\Delta+t\operatorname{div} g} = 0_E^{\mathrm{cl}_{\widehat{B_\Delta}},g^t}.$$

Proof. Since the reflexive hull $(R'(\pi^*\Delta)\otimes_R \omega_R)^{**}$ is equal to $R'(\operatorname{div}(f^{\frac{1}{r}}))$, we have $H^d_{\mathfrak{m}}(R'(\pi^*\Delta)\otimes_R \omega_R) \cong H^d_{\mathfrak{m}}(R'(\operatorname{div}(f^{\frac{1}{r}})))$. Hence, we have

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$$\begin{aligned}\widehat{B_\Delta}\otimes_{\widehat{R}} E &\cong B\otimes_{R'} H^d_{\mathfrak{m}}(R'(\pi^*\Delta)\otimes_R \omega_R)\\ &\cong B\otimes_{R'} H^d_{\mathfrak{m}}(R'(\operatorname{div}(f^{\frac{1}{r}}))).\end{aligned}$$

Then there exists a commutative diagram

$$\begin{array}{ccc}
E \cong H^d_{\mathfrak{m}}(\omega_R) & \longrightarrow & \widehat{B_\Delta}\otimes_{\widehat{R}} E \\
\Big\downarrow & & \Big\downarrow{\scriptstyle g^t\otimes 1} \\
 & & \widehat{B_\Delta}\otimes_{\widehat{R}} E \\
 & & \Big\downarrow{\scriptstyle \cong} \\
 & & B\otimes_{R'} H^d_{\mathfrak{m}}(R'(\operatorname{div}(f^{\frac{1}{r}}))) \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes(\cdot f^{1/r})} \\
 & & B\otimes_{R'} H^d_{\mathfrak{m}}(R') \\
 & & \Big\downarrow{\scriptstyle \cong} \\
H^d_{\mathfrak{m}}(B\otimes_R \omega_R) & \xrightarrow[\psi]{} & H^d_{\mathfrak{m}}(B)
\end{array},$$

where ψ is the second map of

$$\cdot f^{\frac{1}{r}} g^t : H^d_{\mathfrak{m}}(B) \to H^d_{\mathfrak{m}}(B\otimes_R \omega_R) \to H^d_{\mathfrak{m}}(B).$$

The result follows by the above commutative diagram. ∎

Definition 7.5. Let $R \hookrightarrow S$ be an injective local homomorphism of normal local domains essentially of finite type over $\mathbb{C}$. Fix $K_R, K_S \geqslant 0$ effective canonical divisors on $\operatorname{Spec} R$ and on $\operatorname{Spec} S$, respectively. Let $\Delta_R, \Delta_S \geqslant 0$ be effective $\mathbb{Q}$-Weil divisors on $\operatorname{Spec} R$ and on $\operatorname{Spec} S$, respectively, such that $K_R+\Delta_R$, $K_S+\Delta_S$ are $\mathbb{Q}$-Cartier. Let $\mathfrak{a}\subseteq R$ be a nonzero ideal and $t>0$ be a positive rational number. Suppose that $\widehat{B_{\Delta_R}}$ and $\widehat{B_{\Delta_S}}$ are defined as in Definition 7.2. Then, for an $\widehat{R}$-module M and an $\widehat{S}$-module N, we define $0_M^{\mathrm{cl}_{\widehat{B_{\Delta_R}},\mathfrak{a}^t}}$, $0_N^{\mathrm{cl}_{\widehat{B_{\Delta_S}}\mathfrak{a}^t}}$ by

$$0_M^{\mathrm{cl}_{\widehat{B_{\Delta_R}},\mathfrak{a}^t}} := \bigcap_{n\in\mathbb{N}} \bigcap_{g\in\mathfrak{a}^{\lceil nt\rceil}} 0_M^{\mathrm{cl}_{\widehat{B_{\Delta_R}},g^{\frac{1}{n}}}},$$

$$0_N^{\mathrm{cl}_{\widehat{B_{\Delta_S}}\mathfrak{a}^t}} := \bigcap_{n\in\mathbb{N}} \bigcap_{g\in\mathfrak{a}^{\lceil nt\rceil}} 0_N^{\mathrm{cl}_{\widehat{B_{\Delta_S}}g^{\frac{1}{n}}}}.$$

We use $\tau_{\mathrm{cl}_{\widehat{B_{\Delta_R}},\mathfrak{a}^t}}(\widehat{R})$, $\tau_{\mathrm{cl}_{\widehat{B_{\Delta_S}},\mathfrak{a}^t}}(\widehat{S})$ to denote

$$\bigcap_M \operatorname{Ann}_{\widehat{R}} 0_M^{\mathrm{cl}_{\widehat{B_{\Delta_R}},\mathfrak{a}^t}},$$

$$\bigcap_N \operatorname{Ann}_{\widehat{S}} 0_N^{\mathrm{cl}_{\widehat{B_{\Delta_S}},\mathfrak{a}^t}},$$

where M runs through all $\widehat{R}$-modules and N runs through all $\widehat{S}$-modules.

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PROPOSITION 7.6. *In the setting of Definition 7.5, we have*

$$\mathrm{Ann}_{\widehat{R}} 0_{E_R}^{\mathrm{cl}_{B_{\widehat{\Delta_R}}, \mathfrak{a}^t}} = \bigcap_M \mathrm{Ann}_{\widehat{R}} 0_M^{\mathrm{cl}_{B_{\widehat{\Delta_R}}, \mathfrak{a}^t}},$$

$$\mathrm{Ann}_{\widehat{S}} 0_{E_S}^{\mathrm{cl}_{B_{\widehat{\Delta_S}}, \mathfrak{a}^t}} = \bigcap_N \mathrm{Ann}_{\widehat{S}} 0_N^{\mathrm{cl}_{B_{\widehat{\Delta_S}}, \mathfrak{a}^t}},$$

where M, N run through all $\widehat{R}$-modules and $\widehat{S}$-modules, respectively, and E_R, E_S are the injective hulls of the residue fields of R and S, respectively.

Proof. We can show this by arguments similar to Proposition 7.3. ∎

PROPOSITION 7.7. *In the setting of Definition 7.5, we have*

$$\tau_{\mathrm{cl}_{B_{\widehat{\Delta_R}}, \mathfrak{a}^t}}(\widehat{R}) = \mathcal{J}(\widehat{R}, \widehat{\Delta}, (\mathfrak{a}\widehat{R})^t).$$

Proof. Let E be the injective hull of the residue field of R. Then

$$\begin{aligned} 0_E^{\mathrm{cl}_{B_{\widehat{\Delta_R}}, \mathfrak{a}^t}} &= \bigcap_{n \in \mathbb{N}} \bigcap_{g \in \mathfrak{a}^{\lceil nt \rceil}} 0_E^{\mathrm{cl}_{B_{\widehat{\Delta_R}}, g^{\frac{1}{n}}}} \\ &= \bigcap_{n \in \mathbb{N}} \bigcap_{g \in \mathfrak{a}^{\lceil nt \rceil}} \mathrm{Ann}_E \mathcal{J}(\widehat{R}, \widehat{\Delta}, g^{\frac{1}{n}}) \\ &= \mathrm{Ann}_E \sum_{n \in \mathbb{N}} \sum_{g \in \mathfrak{a}^{\lceil nt \rceil}} \mathcal{J}(\widehat{R}, \widehat{\Delta}, g^{\frac{1}{n}}) \\ &= \mathrm{Ann}_E \mathcal{J}(\widehat{R}, \widehat{\Delta}, (\mathfrak{a}\widehat{R})^t), \end{aligned}$$

where the second equality follows from Theorem 6.4. Hence, we have

$$\mathrm{Ann}_{\widehat{R}} 0_E^{\mathrm{cl}_{B_{\widehat{\Delta_R}}, \mathfrak{a}^t}} = \mathcal{J}(\widehat{R}, \widehat{\Delta}, (\mathfrak{a}\widehat{R})^t).$$ ∎

The next lemma is a generalization of [30, Th. 3.2].

LEMMA 7.8. *Let R be a normal local domain essentially of finite type over $\mathbb{C}$, and let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subseteq R$ be nonzero ideals, and let $t > 0$ be a positive rational number. Then we have*

$$\mathcal{J}(R, \Delta, (\mathfrak{a}_1 + \cdots + \mathfrak{a}_n)^t) = \sum_{\lambda_1 + \cdots + \lambda_n = t} \mathcal{J}(R, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_n^{\lambda_n}).$$

LEMMA 7.9. *In the setting of Definition 7.5, we have*

$$\sum_{n \in \mathbb{N}} \sum_{g \in \mathfrak{a}^{\lceil nt \rceil}} \mathcal{J}(S, \Delta_S, g^{\frac{1}{n}}) = \mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t).$$

Proof. $\sum_{n \in \mathbb{N}} \sum_{g \in \mathfrak{a}^{\lceil nt \rceil}} \mathcal{J}(S, \Delta_S, g^{1/n}) \subseteq \mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t)$ is clear. If $t = q/p$, $p, q > 0$ and $\mathfrak{a} = (g_1, \ldots, g_l)$, then

$$\begin{aligned} \sum_{n \in \mathbb{N}} \sum_{g \in \mathfrak{a}^{\lceil nt \rceil}} \mathcal{J}(S, \Delta_S, g^{\frac{1}{n}}) &\supseteq \sum_{n \in \mathbb{N}} \sum_{i_1 + \cdots + i_l = nq} \mathcal{J}(S, \Delta_S, (g_1^{i_1} \cdots g_l^{i_l})^{\frac{1}{np}}) \\ &= \mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t), \end{aligned}$$

by the above lemma. ∎

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THEOREM 7.10. *Let $R \hookrightarrow S$ be a pure local homomorphism of normal local domains essentially of finite type over $\mathbb{C}$. Fix effective canonical divisors K_R and K_S on $\operatorname{Spec} R$ and $\operatorname{Spec} S$, respectively. Let $\Delta_R, \Delta_S \geqslant 0$ be effective $\mathbb{Q}$-Weil divisors on $\operatorname{Spec} R$, $\operatorname{Spec} S$ such that $K_R + \Delta_R$, $K_S + \Delta_S$ are $\mathbb{Q}$-Cartier. Take normal domains R', S' and morphisms π_R, π_S as in Setting 7.1. Moreover, let $\mathfrak{a} \subseteq R$ be a nonzero ideal, and let $t > 0$ be a positive rational number. If $R'(\pi_R^* \Delta_R) \subseteq S'(\pi_S^* \Delta_S)$, then we have*

$$\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \Delta_R, \mathfrak{a}^t).$$

Proof. Since $R \hookrightarrow S$ is pure, $\widehat{R} \hookrightarrow \widehat{S}$ is pure (see [6, Cor. 3.2.1]). Since $R \to \widehat{R}$, $S \to \widehat{S}$ are pure, it is enough to show

$$\mathcal{J}(\widehat{S}, \widehat{\Delta_S}, (\mathfrak{a}\widehat{S})^t) \cap \widehat{R} \subseteq \mathcal{J}(\widehat{R}, \widehat{\Delta_R}, (\mathfrak{a}\widehat{R})^t).$$

Let $\mathcal{B}(R)$, $\mathcal{B}(S)$ be the canonical BCM algebras. Let $\widehat{\mathcal{B}_{\Delta_R}} := \widehat{\mathcal{B}(R)_{\Delta_R}}$ and $\widehat{\mathcal{B}_{\Delta_S}} := \widehat{\mathcal{B}(S)_{\Delta_S}}$. Take an $\widehat{R}$-module M. Then we have a commutative diagram

$$\begin{array}{ccc} \widehat{R} & \xhookrightarrow{\text{pure}} & \widehat{S} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{B}_{\Delta_R}} & \longrightarrow & \widehat{\mathcal{B}_{\Delta_S}} \end{array} .$$

Tensoring the commutative diagram with M, we have

$$\begin{array}{ccc} M & \hookrightarrow & \widehat{S} \otimes_{\widehat{R}} M \\ \downarrow & & \downarrow \\ \widehat{\mathcal{B}_{\Delta_R}} \otimes_{\widehat{R}} M & \longrightarrow & \widehat{\mathcal{B}_{\Delta_S}} \otimes_{\widehat{R}} M \end{array} .$$

Hence, we have

$$0_M^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_R}}, \mathfrak{a}^t}} \subseteq 0_{\widehat{S} \otimes_{\widehat{R}} M}^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_S}}, \mathfrak{a}^t}}.$$

Then we have

$$\begin{aligned} \mathcal{J}(\widehat{R}, \widehat{\Delta_R}, \mathfrak{a}^t) &= \bigcap_M \operatorname{Ann}_{\widehat{R}} 0_M^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_R}}, \mathfrak{a}^t}} \\ &\supseteq \bigcap_M \operatorname{Ann}_{\widehat{R}} 0_{M \otimes \widehat{S}}^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_S}}, \mathfrak{a}^t}} \\ &\supseteq \bigcap_N \operatorname{Ann}_{\widehat{R}} 0_N^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_S}}, \mathfrak{a}^t}} \\ &= \bigcap_N (\operatorname{Ann}_{\widehat{S}} 0_N^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_S}}, \mathfrak{a}^t}} \cap \widehat{R}) \\ &= (\operatorname{Ann}_{\widehat{S}} 0_{E_S}^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_S}}, \mathfrak{a}^t}}) \cap \widehat{R} \\ &= (\operatorname{Ann}_{\widehat{S}} \bigcap_{n \in \mathbb{N}} \bigcap_{g \in \mathfrak{a}^{\lceil nt \rceil}} 0_{E_S}^{\mathrm{cl}_{\widehat{\mathcal{B}_{\Delta_S}}, g^{\frac{1}{n}}}}) \cap \widehat{R} \end{aligned}$$

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$$= (\mathrm{Ann}_{\widehat{S}} \mathrm{Ann}_{E_S} \sum_{n \in \mathbb{N}} \sum_{g \in \mathfrak{a}^{\lceil nt \rceil}} \mathcal{J}(\widehat{S}, \widehat{\Delta_S}, g^{\frac{1}{n}})) \cap \widehat{R}$$
$$= \mathcal{J}(\widehat{S}, \widehat{\Delta_S}, (\mathfrak{a}\widehat{S})^t)) \cap \widehat{R},$$

where M runs through all $\widehat{R}$-modules, N runs through all $\widehat{S}$-modules, and E_S is the injective hull of the residue field of S. ∎

As a corollary, we have a generalization of [31, Cor. 5.30] to the case that $\mathfrak{a}$ is not necessarily a principal ideal.

COROLLARY 7.11. *Let $R \hookrightarrow S$ be a pure local homomorphism of normal local domains essentially of finite type over $\mathbb{C}$. Suppose that R is $\mathbb{Q}$-Gorenstein. Fix effective canonical divisors K_R and K_S on* $\operatorname{Spec} R$ *and* $\operatorname{Spec} S$*, respectively. Let Δ_S be an effective $\mathbb{Q}$-Weil divisor on* $\operatorname{Spec} S$ *such that $K_S + \Delta_S$ is $\mathbb{Q}$-Cartier. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $t > 0$ a positive rational number. Then we have*

$$\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \mathfrak{a}^t).$$

Proof. Let R' be the integral closure of $R[f^{1/r}]$ in R^+. Then the result follows from Theorem 7.10. ∎

§8. $\mathcal{B}$-regularity

As another application of the main theorem, we will give a partial answer to [24, Rem. 3.10]. For this, we will review the definition of $\mathcal{B}$-regularity.

DEFINITION 8.1 [23, Def. 4.3]. Let R be a normal $\mathbb{Q}$-Gorenstein local domain essentially of finite type over $\mathbb{C}$.

(1) R is said to be *weakly $\mathcal{B}$-regular* if $R \to \mathcal{B}(R)$ is cyclically pure.
(2) R is said to be *$\mathcal{B}$-regular* if every localization of R at a prime ideal is weakly $\mathcal{B}$-regular.

THEOREM 8.2. *Let R be a normal $\mathbb{Q}$-Gorenstein local domain. Then the following are equivalent:*

(1) *R has log-terminal singularities.*
(2) *R is ultra-F-regular.*
(3) *R is weakly generically F-regular.*
(4) *R is generically F-regular.*
(5) *R is weakly $\mathcal{B}$-regular.*
(6) *R is $\mathcal{B}$-regular.*
(7) *$\widehat{R}$ is $BCM_{\widehat{\mathcal{B}(R)}}$-regular.*

Proof. The equivalence of (1) and (2) follows from Proposition 3.54 and the equivalence of (1) and (7) follows from Theorem 6.4. Since, if R has log-terminal singularities, then every localization of R at a prime ideal is log-terminal, it is enough to show the equivalence of (1), (3), and (5). (1) is equivalent to (3) by [31, Th. 5.24 and Proof of Th. 5.25]. Lastly, we will show the equivalence of (5) and (7). Let E be the injective hull of the residue field of R. By Proposition 7.4, we have $0_E^{\mathrm{cl}_{\mathcal{B}(R) \otimes_R \widehat{R}}} = 0_E^{\mathcal{B}(R), K_R}$. Hence, $E \to \mathcal{B}(R) \otimes_R E$ is injective if and only if $\widehat{R}$ is $\mathrm{BCM}_{\widehat{\mathcal{B}(R)}}$-regular. $R \to \mathcal{B}(R)$ is pure if and only if $E \to \mathcal{B}(R) \otimes_R E$ is injective by [15, Lem. 2.1(e)]. $R \to \mathcal{B}(R)$ is pure if and only if $R \to \mathcal{B}(R)$ is cyclically pure by [12, Th. 1.7]. Therefore, (5) is equivalent to (7). ∎

REMARK 8.3. For the equivalence of (5) and (7) (see [19, Prop. 6.14]).

§9. Further questions and remarks

In this section, we will consider whether R is BCM-rational if R has rational singularities. The next question is a variant of [7, Quest. 2.7].

QUESTION 1. Let R be a local domain essentially of finite type over $\mathbb{C}$, and let B be a BCM R-algebra. If S is finitely generated R-algebra such that the following diagram commutes:

$$\begin{array}{ccc} R & \longrightarrow & B \\ \downarrow & \nearrow & \\ S & & \end{array}$$

then does there exist a BCM R_p-algebra for almost all p which fits into the following commutative diagram:

$$\begin{array}{ccc} R_p & \longrightarrow & B_p \\ \downarrow & \nearrow & \\ S_p & & \end{array}$$

where S_p is an R-approximation of S?

PROPOSITION 9.1 (Cf. [19, Conj. 3.9]). *Let R be a normal local domain essentially of finite type over $\mathbb{C}$ of dimension d. Suppose that R has rational singularities. If Question 1 has an affirmative answer, then R is BCM-rational.*

Proof. Let B be a BCM R^+-algebra. Suppose that $\eta \in \operatorname{Ker}(H^d_m(R) \to H^d_{\mathfrak{m}}(B))$. Then there exists a finitely generated R-subalgebra of B such that the image of η in $H^d_m(S)$ is zero. If Question 1 has an affirmative answer, we can take S_p and B_p as in Question 1. Then we have a commutative diagram

$$\begin{array}{ccc} H^d_{\mathfrak{m}}(R) & \longrightarrow & \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(R_p) \\ \downarrow & & \downarrow \\ H^d_{\mathfrak{m}}(S) & \longrightarrow & \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(S_p) \\ \downarrow & & \downarrow \\ H^d_m(B) & & \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(B_p) \end{array}$$

By the proof of Proposition 4.8, $\operatorname{ulim}_p H^d_{\mathfrak{m}_p}(R_p) \to \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(S_p)$ is injective. Therefore, the image of η in $\operatorname{ulim}_p H^d_{\mathfrak{m}_p}(R_p)$ is zero. Suppose that $\eta = [\frac{y}{x^t}]$, where $y \in R$, $t \in \mathbb{N}$ and x is the product of a system of parameters $x_1, \ldots, x_d$ of R. Since R_p is Cohen–Macaulay for almost all p, $y_p \in (x^t_{1p}, \ldots, x^t_{dp})$ for almost all p. Hence, $y \in (x^t_1, \ldots, x^t_d)$ and $\eta = 0$ in $H^d_{\mathfrak{m}}(R)$. Thus, $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B)$ is injective. ∎

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The next result follows from a similar argument.

PROPOSITION 9.2. *Let R be a normal local domain essentially of finite type over $\mathbb{C}$ of dimension d. Fix an effective canonical divisor K_R on* $\operatorname{Spec} R$*. Let $\Delta \geqslant 0$ be an effective $\mathbb{Q}$-Weil divisor on* $\operatorname{Spec} R$ *such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that C is a BCM R^+-algebra. If Question 1 has an affirmative answer, then we have*

$$\mathcal{J}(R,\Delta) \subseteq \tau_{\widehat{C}}(\widehat{R},\widehat{\Delta}).$$

DEFINITION 9.3 (Cf. [19, Def. 6.9]). Let R be a normal local domain essentially of finite type over $\mathbb{C}$. Fix an effective canonical divisor K_R on $\operatorname{Spec} R$. Let $\Delta \geqslant 0$ be a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that $n(K_R + \Delta) = \operatorname{div}(f)$ for $f \in R^\circ$, $n \in \mathbb{N}$. We define

$$0^{\mathscr{B},K_R+\Delta}_{H^d_{\mathfrak{m}}(R)} := \{\eta \in H^d_{\mathfrak{m}}(R) \mid \quad \exists C \text{ BCM } R^+\text{-algebra such that } f^{\frac{1}{n}}\eta = 0 \text{ in } H^d_{\mathfrak{m}}(C)\}.$$

We define *the BCM test ideal*$\tau_{\mathscr{B}}(R,\Delta)$ *of* $(\widehat{R},\widehat{\Delta})$ by

$$\tau_{\mathscr{B}}(\widehat{R},\widehat{\Delta}) := \operatorname{Ann}_{\omega_{\widehat{R}}} 0^{\mathscr{B},K_R+\Delta}_{H^d_{\mathfrak{m}}(R)}.$$

COROLLARY 9.4 (Cf. [19, Th. 6.21]). *In the setting of the above proposition, if Question 1 has an affirmative answer, then we have*

$$\tau_{\mathscr{B}}(\widehat{R},\widehat{\Delta}) = \mathcal{J}(\widehat{R},\widehat{\Delta}).$$

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