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ON UNIFORM CONVERGENCE OF CONTINUOUS FUNCTIONS AND TOPOLOGICAL CONVERGENCE OF SETS

BY
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ABSTRACT. Let X and Y be metric spaces. This paper considers the relationship between uniform convergence in $C(X, Y)$ and topological convergence of functions in $C(X, Y)$, viewed as subsets of $X\times Y$. In general, uniform convergence in $C(X, Y)$ implies Hausdorff metric convergence which, in turn, implies topological convergence, but if X and Y are compact, then all three notions are equivalent. If $C([0, 1], Y)$ is nontrivial and topological convergence in $C(X, Y)$ implies uniform convergence then X is compact. Theorem: Let X be compact and Y be locally compact but noncompact. Then topological convergence in $C(X, Y)$ implies uniform convergence if and only if X has finitely many components. We also sharpen a related result of Naimpally.

Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces and let $C(X, Y)$ denote the collection of continuous functions from X to Y. The purpose of this note is to describe circumstances under which the uniform convergence of a sequence of functions $\{f_n\}$ in $C(X, Y)$ to a continuous function f is equivalent to the so-called topological convergence of $\{f_n\}$ to f, where we view each function as a subset of $X\times Y$. The notion of topological convergence of a sequence of sets rests on two primitive notions: the *lower* and *upper limits* of a sequence of sets [7].

DEFINITION. Let $\{C_n\}$ be a sequence of sets in a metric space. Then Li C_n (resp. Ls C_n) is the set of all points y each neighborhood of which meets all but finitely (resp. infinitely) many sets C_n.

Both Li C_n and Ls C_n are closed sets. It is clear that $y\in \text{Li } C_n$ if and only if there are points $y_n\in C_n$, $n=1, 2, 3, \ldots$, such that $\{y_n\}\to y$. Similarly, $y\in \text{Ls } C_n$ if and only if there is an increasing sequence $\{n_k\}$ in Z^+ and points $y_k\in C_{n_k}$, $k=1, 2, 3, \ldots$, such that $\{y_k\}\to y$. We say that $\{C_n\}$ *converges topologically* to a (possibly empty) set C if $\text{Li } C_n=\text{Ls } C_n=C$. From the above remarks, we can characterize topological convergence in $C(X, Y)$ locally.

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LEMMA 1. *Let X and Y be metric spaces and let $f, f_1, f_2, f_3, \ldots$ be functions in $C(X, Y)$. Then $\{f_n\}$ converges topologically to f if and only if at each x in X*

(1) *Whenever $\{(x_k, f_{n_k}(x_k)\}$ converges to (x, y), then $y = f(x)$*

(2) *There exists a sequence $\{x_n\}$ convergent to x for which $\lim_{n\to\infty} f_n(x_n) = f(x)$.*

Proof. Condition (1) says that $\mathrm{Ls}\, f_n \subset f$ and condition (2) says that $f \subset \mathrm{Li}\, f_n$. Since we always have $\mathrm{Li}\, f_n \subset \mathrm{Ls}\, f_n$, these two statements in conjunction are equivalent to the statement $f = \mathrm{Li}\, f_n = \mathrm{Ls}\, f_n$.

To see the connection between topological convergence of graphs and uniform convergence, we next consider the notion of Hausdorff distance between sets. Let C be a set in a metric space with metric d, and let $B_\varepsilon[C]$ denote the union of all closed balls of radius $\varepsilon > 0$ whose centers run over C. If K is another set in the metric space and there exists $\varepsilon > 0$ for which both $B_\varepsilon[C] \supset K$ and $B_\varepsilon[K] \supset C$, then the *Hausdorff distance* δ_d between C and K is given by

$$\delta_d(C, K) = \inf\{\varepsilon : B_\varepsilon[C] \supset K \text{ and } B_\varepsilon[K] \supset C\}$$

If no such ε exists, we set $\delta_d(C, K) = \infty$. If we identify sets with the same closure, then δ_d is well defined on the equivalence classes so induced, and δ_d determines an extended real valued metric on the class of nonempty closed subsets of the space. Now if $\{C_n\}$ is a sequence of closed sets in the space convergent in the Hausdorff metric to a closed set C, it is evident that $C = \mathrm{Li}\, C_n = \mathrm{Ls}\, C_n$. The converse is false. For example, in the line, if we let $C_n = [0, 1] \cup \{n\}$, then $\{C_n\}$ converges topologically to $[0, 1]$ but for each $n > 1$ $\delta_d(C_n, [0, 1]) = n - 1$. When the underlying space is compact, everything becomes quite nice. In this context topological convergence does imply convergence in the Hausdorff metric [8]. Moreover, both the collection of nonempty closed sets and the collection of nonempty closed connected sets in the space are compact when equipped with this metric; a nice proof of these results can be constructed using the Ascoli theorem [1].

We now look at the Hausdorff metric as applied to $C(X, Y)$, viewed as a collection of closed subsets of $X \times Y$. We first need a metric on $X \times Y$ to induce the Hausdorff metric. For definiteness and computational simplicity, we take ρ defined by $\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. By our above remarks, convergence in $C(X, Y)$ with respect to δ_ρ implies topological convergence. It is also clear that uniform convergence in $C(X, Y)$ implies δ_ρ-convergence, for if $\sup_{x\in X} d_Y[f(x), g(x)] < \varepsilon$, then for each x in X $\rho[(x, f(x)), (x, g(x))] < \varepsilon$, whence both $f \subset B_\varepsilon[g]$ and $g \subset B_\varepsilon[f]$. As a consequence, for general X and Y uniform convergence in $C(X, Y)$ implies topological convergence. If Y is at all interesting, a necessary condition for the converse to hold is the compactness of X.

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EXAMPLE 1. Let Y be a metric space whose path components are not all singletons, i.e., a space for which $C([0, 1], Y)$ is nontrivial. Let X be a noncompact metric space. We produce functions $f, f_1, f_2, f_3, \ldots,$ in $C(X, Y)$ such that $\{f_n\}$ converges topologically to f, yet $\{f_n\}$ fails to converge to f uniformly. Choose a continuous functon $\phi:[0, 1] \to Y$ such that $\phi(0) \neq \phi(1)$, and choose a sequence $\{z_n\}$ in X with no convergent subsequence. We can find a sequence $\{\lambda_n\}$ of positive numbers such that for each n $\lambda_n < 1/n$, and the collection of balls $\{B_{\lambda_n}[z_n]: n \in Z^+\}$ is pairwise disjoint. Define for each $n \in Z^+$ $f_n: X \to Y$ by

$$f_n(x) = \begin{cases} \phi\left[\frac{1}{\lambda_n} d(x, z_n)\right] & \text{if} \quad d(x, z_n) < \lambda_n \\ \phi(1) & \text{otherwise} \end{cases}$$

We claim that $\{f_n\}$ converges topologically to the function f that is identically equal to $\phi(1)$ on X. Since at each x $\{f_n(x)\}$ is $\phi(1)$ eventually, condition (2) of Lemma 1 is satisfied. On the other hand, if $\{(x_k, f_{n_k}(x_k))\}$ converges to a point (x, y), then eventually $x_k \notin B_{\lambda_{n_k}}[z_{n_k}]$ because $\text{Ls}\, B_{\lambda_k}[z_k]$ is empty. Thus, $y = \phi(1)$, and condition (1) holds. Since for each n, $f_n(z_n) = \phi(0)$, the sequence does not converge uniformly to f.

In light of Example 1 we now restrict our attention to compact X. Naimpally [9] first observed that in this context δ_ρ-convergence in $C(X, Y)$ implies uniform convergence. Thus, if Y is also compact, then since topological convergence in $C(X, Y)$ now implies δ_ρ-convergence, it also implies uniform convergence. As an application of this result we present a novel proof of Dini's theorem: Let $\{f_n\}$ be a decreasing sequence of continuous real functions defined on a compact metric space X. If $f = \inf f_n$ is a continuous real valued function, then $\{f_n\}$ converges uniformly to f. First, since $\{f_n\}$ converges pointwise to f, condition (2) of Lemma 1 is satisfied, i.e., $\text{Li}\, f_n \supset f$. Since each term of the sequence lies in the *epigraph* of f, the closed set

$$\text{epi}\, f = \{(x, y) : y \geqq f(x)\}$$

it is clear that $\text{Ls}\, f_n \subset \text{epi}\, f$. If $(x, y) \in \text{epi}\, f - f$, choose N so large that $f_N(x) < y$. There exists $y^* < y$ and $\lambda > 0$ such that if $w \in B_\lambda[x]$ then $f_N(w) < y^*$. It follows that $B_\lambda[x] \times [y^*, \infty)$ is a neighborhood of (x, y) that fails to meet $\bigcup_{n=N}^{\infty} f_n$ so that $(x, y) \notin \text{Ls}\, f_n$. Hence $\text{Ls}\, f_n \subset f$, and we have $\text{Ls}\, f_n = \text{Li}\, f_n = f$. Since each term of $\{f_n\}$ lies in the compact set $\{(x, y) : x \in X \text{ and } f(x) \leq y \leq f_1(x)\}$, the sequence converges uniformly to f.

If Y is noncompact, then topological convergence in $C(X, Y)$ need not imply uniform convergence.

EXAMPLE 2. Let $X = \{0\} \cup \{1/n: n \in Z^+\}$, viewed as a metric space subspace

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of the line. Let $f \in C(X, R)$ denote the zero function, and for each $n \in Z^+$ let

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x > 1/n \\ n & \text{if} \quad 0 < x \le 1/n \end{cases}$$

Each function above is continuous, $\text{Li}\, f_n = \text{Ls}\, f_n = f$, but $\{f_n\}$ fails to converge in the Hausdorff metric to f or even pointwise to f.

One might guess from Example 2 that a connectivity requirement on X would patch things up. However, the problem is more involved: if X is a metric continuum and Y is complete, topological convergence in $C(X, Y)$ does not force uniform convergence.

EXAMPLE 3. Let l_2 denote the Hilbert space of square summable real sequences and let $\{e_n : n \in Z^+\}$ denote the standard orthonormal basis for the space. For each $n \in Z^+$ define $f_n : [0, 1] \to l_2$ by

$$f_n(x) = \begin{cases} xe_1 + (1 - nx)e_n & \text{if} \quad 0 < x \le \dfrac{1}{n} \\ xe_1 & \text{if} \quad \dfrac{1}{n} < x < 1 \end{cases}$$

The graph of f_n in $[0, 1] \times l_2$ consists of the segment joining $(0, e_n)$ to $(1/n, (1/n)e_1)$ plus the segment joining $(1/n, (1/n)e_1)$ to $(1, e_1)$. If $f : [0, 1] \to l_2$ is defined by $f(x) = xe_1$, then $\text{Li}\, f_n = \text{Ls}\, f_n = f$. However, the Hausdorff distance from f_n to f is one for each index n; so, $\{f_n\}$ does not converge uniformly to f.

If X is a metric continuum and Y is locally compact rather than complete (a stronger requirement [6; pg. 294]), then topological convergence in $C(X, Y)$ does imply uniform convergence. Our main result says much more on this matter.

THEOREM 1. *Let X be a compact metric space. The following are equivalent.*

(1) *X has finitely many components.*

(2) *For each locally compact metric space Y the topological convergence of sequences of functions in $C(X, Y)$ implies their uniform convergence.*

Proof. (1) $\to$ (2) Let Y be a locally compact metric space. Suppose $\{f, f_1, f_2, \ldots\} \subset C(X, Y)$ and $\text{Li}\, f_n = \text{Ls}\, f_n = f$. By Naimpally's result it suffices to show that $\{f_n\}$ δ_ρ-converges to f. To accomplish this it suffices to show that all but finitely many terms of $\{f_n\}$ lie in some common compact subset of $X \times Y$. Since $f \subset X \times Y$ is a compact set and $X \times Y$ is locally compact, there exists a finite subset $\{x_1, \ldots, x_n\}$ of X and positive numbers $\{\lambda_1, \ldots, \lambda_n\}$ such that for each i the ball $B_{\lambda_i}[(x_i, f(x_i))]$ is compact and $f \subset \text{int} \bigcup_{i=1}^n B_{\lambda_i}[(x_i, f(x_i))]$. Denote the set on the right hand side of the above inclusion by E, and let $\varepsilon = \frac{1}{2}\min\{\rho[(x, f(x)), (z, y)] : x \in X \text{ and } (z, y) \in X \times Y - E\}$. Since f is compact, the set $B_\varepsilon[f]$ is closed. Since $\text{cl}\, E$ is compact, and $B_\varepsilon[f] \subset \text{cl}\, E$, the set $B_\varepsilon[f]$ is also

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compact. We claim that all but finitely many terms of $\{f_n\}$ lie in $B_\varepsilon[f]$. If not, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and points $\{x_k\}$ in X such that for each k $(x_k, f_{n_k}(x_k)) \notin B_\varepsilon[f]$. Since X has finitely many components, we can assume w.l.o.g. that each x_k lies in the same component C of X. Since the components of X are compact sets, there exists $\lambda > 0$ for which $B_\lambda[C]$ meets no other component of X. Let $z \in C$ be fixed. By Lemma 1 there exists a sequence $\{z_k\}$ in C convergent to z for which $\lim_{k\to\infty} f_{n_k}(z_k) = f(z)$. By the connectedness of (the graph of) $f_{n_k} \mid C$, for each sufficiently large index k

$$f_{n_k} \mid C \cap \{(x, y) : \rho[(x, y), f] = \varepsilon\} \neq \emptyset$$

Since $\{(x, y)$: $\rho[(x, y), f] = \varepsilon\}$ is a closed subset of $B_\varepsilon[f]$, it, too, is compact. Hence this boundary set must contain a point of $\mathrm{Ls}\, f_n$, in contradiction to $\mathrm{Ls}\, f_n = f$. This establishes the claim, whence $\{f_n\}$ converges uniformly to f.

(2) $\to$ (1) Suppose that X has infinitely many components. Let $\{C_n\}$ be an arbitrary sequence of distinct components. By the compactness of the closed connected nonempty subsets of X with respect to δ_{d_X}, we can assume that $\{C_n\}$ converges to a nonempty closed connected set C. By throwing out at most one term from our sequence, we can assume that for each n $C_n \cap C = \emptyset$. Choose $\varepsilon_n < 1/n$ such that $B_{\varepsilon_n}[C_n] \cap C = \emptyset$, and let F_n be the closed set $X - \mathrm{int}\,(B_{\varepsilon_n}[C_n])$. Since X is compact, by Theorem IV-5.6 of [10] there exist for each positive integer n disjoint open sets V_n and W_n such that (i) $C_n \subset V_n$ (ii) $F_n \subset W_n$ (iii) $W_n \cup V_n = X$. Let Y be a noncompact space and let $\{y_n\}$ be a sequence in Y with no convergent subsequence. For each $n \in Z^+$ let f_n be the following continuous function:

$$f_n(x) = \begin{cases} y_1 & \text{if} \quad x \in W_n \\ y_n & \text{if} \quad x \in V_n \end{cases}$$

We claim $\mathrm{Ls}\, f_n = \mathrm{Li}\, f_n = f$, where f is the function identically equal to y_1 on X. Again we show that the local conditions of Lemma 1 are satisfied. Condition (1) holds because if $\{(x_k, f_{n_k}(x_k)\}$ is ρ-convergent, then eventually $f_{n_k}(x_k) = y_1$, or else $\{y_n\}$ would have a convergent subsequence. To see that condition (2) holds, we consider two cases: (i) $x \in C$ (ii) $x \notin C$. If $x \in C$ then for each n $f_n(x) = y_1$, and we are done. On the other hand, if $x \notin C$ then since $\lim_{n\to\infty} \delta_{d_X}(B_{\varepsilon_n}[C_n], C) = 0$ and $V_n \subset B_{\varepsilon_n}[C_n]$, there exists an index N such that for each $n > N$ the point x lies in W_n. Hence, for each $n > N$ $f_n(x) = y_1$. Thus, condition (2) holds; in fact, $\{f_n\}$ converges pointwise to f. However, the convergence can't be uniform, or else $\{y_n\}$ would converge to y_1.

We now look at Naimpally's result more closely. Even if $Y = \boldsymbol{R}$, the equality of the topology of uniform convergence and the Hausdorff metric topology on $C(X, Y)$ does not imply that X is compact. For example, equality holds for any set X metrized by the discrete metric. However, the equality of these topologies does characterize the compact metric spaces among the metric spaces with finitely many components.

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THEOREM 2. *Let X be a metric space with finitely many components. Then the topology of uniform convergence and the Hausdorff metric topology on $C(X, R)$ are equal if and only if X is compact.*

Proof. By Naimpally's theorem, one direction is immediate. Conversely, suppose X is noncompact. Let $\{z_n\}$ be a sequence in some component C of X with no convergent subsequence. Since the components of X are open, there exists for each positive integer n $\lambda_n < 1/n$ such that the balls $\{B_{\lambda_n}[z_n] : n \in Z^+\}$ are pairwise disjoint, and each ball lies in C. Define $f : X \to R$ as follows:

$$f(x) = \begin{cases} 1 - (d_X(x, z_n)/\lambda_n) & \text{if for some } n \ 0 \le d_X(x, z_n) \le \lambda_n \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in Z^+$ define $g_n : B_{\lambda_n}[z_n] \to R$ by

$$g_n(x) = \begin{cases} 1 - (2d_X(x, z_n)/\lambda_n) & \text{if} \quad 0 \le d_X(x, z_n) \le \lambda_n/2 \\ 0 & \text{if} \quad \lambda_n/2 < d_X(x, z_n) \le \lambda_n \end{cases}$$

and define $f_n : X \to R$ by

$$f_n(x) = \begin{cases} g_n(x) & \text{if} \quad d_X(x, z_n) \le \lambda_n \\ f(x) & \text{otherwise} \end{cases}$$

We claim that for each n $\delta_\rho[f_n, f] \le 2/n$. To see this first note that if $x \notin B_{\lambda_n}[z_n]$, then $\rho[(x, f(x)), (x, f_n(x))] = 0$. Now suppose $x \in B_{\lambda_n}[z_n]$. Since C is connected, for each $\lambda \le \lambda_n$ there exists $x_\lambda \in B_{\lambda_n}[z_n]$ for which $d_X(x_\lambda, z_n) = \lambda$. Thus, both f_n and f map $B_{\lambda_n}[z_n]$ onto $[0, 1]$. In particular there are points w_1 and w_2 in the ball for which $f(x) = f_n(w_1)$ and $f_n(x) = f(w_2)$. Thus, $\rho[(x, f(x)), (w_1, f_n(w_1))] \le 2/n$ and $\rho[(x, f_n(x)), (w_2, f(w_2))] \le 2/n$, and the claim is established. Now if for each n we choose p_n such that $d_X(p_n, z_n) = \lambda_n/2$, the definition of g_n implies that $|f_n(p_n) - f(p_n)| = \frac{1}{2}$, so that $\{f_n\}$ cannot converge uniformly to f.

We note that Theorem 2 remains true if R is replaced by any metric space Y for which $C([0, 1], Y)$ is nontrivial. We mention that the compact spaces are also characterized among the spaces with finitely many components as follows: X is uniformly locally compact and each component of X is uniformly chainable [2].

The graph of a function $f : X \to Y$ is not always the only set in $X \times Y$ that one can naturally associate with the function. For example, if X is a normed linear space, then it is customary [11] to study the real valued convex functions on X by identifying each function with its epigraph in $X \times R$, the convex set of points on or above its graph. Because of its importance in applied mathematics and nonlinear analysis, the topological convergence of epigraphs, first considered by Wijsman [12] and, most recently, by Bergstrom and McLinden [5], has received much greater attention in the literature than has the topological convergence of graphs. We mention in closing that convergence in this sense

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can be described locally as follows: at each x in X (i) whenever $\{x_n\} \to x$, then $\liminf_{n\to\infty} f_n(x_n) \geq f(x)$ (ii) there exists $\{x_n\}$ convergent to x for which $\lim_{n\to\infty} f_n(x_n) = f(x)$. For the relationship between topological convergence of epigraphs, Hausdorff metric convergence of graphs and epigraphs, uniform convergence, pointwise convergence, and convergence in measure for real valued functions on compact spaces, the reader can consult [3] and [4].

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