J. Austral. Math. Soc. (*Series A*) **48** (1990), 397–401

GROUPS WITH MANY PERMUTABLE SUBGROUPS

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(Received 6 December 1988; revised 1 August 1989)

Communicated by H. Lausch

Abstract

We consider the influence on a group G of the condition that every infinite set of cyclic subgroups of G contains a pair that permute and prove (Theorem 1) that finitely generated soluble groups with this condition are centre-by-finite, and (Theorem 2) that torsion free groups satisfying the condition are abelian.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 20 F 16.

1. Introduction

In response to a question of Paul Erdös, B. H. Neumann proved in [9] that a group is centre-by-finite if and only if the subsets consisting of mutually non-commuting elements are finite, and so these non-commuting sets are boundedly finite if they are finite. Extensions of problems of this type are to be found in [7] and [2].

We investigate here the following rather similar class of groups. Recall that a group is Hamiltonian if all of its subgroups are normal and it is quasi-Hamiltonian if every pair of subgroups permute (as sets). By obvious analogy, we say that a group G is pseudo-Hamiltonian, or a PH-group, if the following conditions holds.

PH: Every infinite set of subgroups of G contains a pair that permute.

We shall also be interested in the following related condition.

PH* : Every infinite set of cyclic subgroups contains a pair that permute.

At first sight, this is a rather weaker condition than that mentioned above, and yet the group classes arising are not too different.

THEOREM 1. *Every finitely generated soluble* PH**-group is centre-by-finite.*

THEOREM 2. *All torsion-free* PH**-groups are abelian.*

Of course, all centre-by-finite groups satisfy PH, though the converse is false. For example, Napolitani [5] and Iwasawa [8] have constructed quasi-Hamiltonian groups as follows:

$$H_{n,q} = \langle a, b : a^{q^n} = 1, \; b^{q^{n-1}} = 1, \; a^b = a^{1+q} \rangle$$

where q is a prime and n a positive integer. Clearly, the direct product of any number of groups of this type of coprime orders is quasi-Hamiltonian and thus satisfies PH; and suitable constellations of the n, q give rise to PH-groups that are not nilpotent and not centre-by-finite. Indeed, some infinite analogues of the Iwasawa-Napolitani groups are not even FC-groups (see [5] and [8]).

A number of problems about PH-groups do not seem amenable to our methods. For instance, it should be the case that periodic PH-groups are locally finite, but the best we can say in this direction is the obvious fact that PH-groups of prime exponent are centre-by-finite, so that Tarski-Ol′shanskii monsters play no part in this subject. This is because the permutable product of two groups of the same prime order is commutative, so that the results of [9] apply. On the other hand, there are infinite groups of exponent p^2 with just $(p-1)$ elements of order p; for example, a suitable factor-group of Adian's group [1, page 269] is of this type, and is certainly not a PH-group.

We thank the referee for some useful comments.

2. Proofs

We break down the proof of Theorem 1 into several steps.

2.1. *Let G be a finitely generated soluble group such that for every cyclic subgroup H and every element x of G, there exists a positive integer i such that*

$$HH^{x^i} = H^{x^i}H.$$

Then G is nilpotent-by-finite.

PROOF. For arbitrary x, y in G, there must exist $i > 0$ such that

$$\langle y\rangle\langle y\rangle^{x^i} = \langle y\rangle^{x^i}\langle y\rangle.$$

Thus, by [3], the product $K = \langle y\rangle\langle y\rangle^{x^i}$ has a torsion-free abelian subgroup of finite index. Thus $[y^m, (y^m)^{x^i}] = 1$. for some $m \geq 1$, so

$$[x^i, y^m, y^m] = 1$$

for some $m \geq 1$ and some $i > 0$.

To complete the proof of 2.1, we need the following vital result.

2.2. *Let G be a finitely generated soluble group such that for all x, y in G there exist integers n, $i \geq 1$, such that $[x^i, y^n, y^n] = 1$. Then G is nilpotent-by-finite.*

Proof. By induction on the solubility length of G, we may assume that G is abelian-by-nilpotent-by-finite, and thus, ignoring the finite factor at the top, that G is abelian-by-nilpotent. Thus G is eremitic [6]; this means that there is an integer $e \geq 1$ depending only on G such that $[u, v^e] = 1$ whenever $[u, v^m] = 1$ for elements u, v of G and an integer $m > 1$. Let A be an abelian normal subgroup of G such that G/A is nilpotent. For $a \in A$ and $y \in G$, we have $[(ya)^i, y^n, y^n] = 1$ for some i, $n > 1$, so, after a small calculation, we have $[b, y^n, y^n] = 1$, where $b = a^{1+y+y^2+\cdots+y^{i-1}}$. But $\langle a, y\rangle$ is metabelian, so $[b, y^e, y^n] = 1$ and $[b, y^e, y^e] = 1$. Write A additively; then $a(1+y+\cdots+y^{i-1})(1-y^e)^2 = 0$, so that multiplying by $1-y$ gives $a(1-y^i)(1-y^e)^2 = 0$, and so by eremiticity, $a(1-y^e)^3 = 0$.

It will be enough if we show that G^e is nilpotent, since it is of finite index. All the elements of the form y^e act nilpotently on $A \cap G^e$, so $\langle A \cap G^e, y^e\rangle$ is nilpotent. But this subgroup is subnormal in G^e, so G^e is locally nilpotent and thus nilpotent since it is finitely generated.

Note that 2.2 remains true if the hypothesis is changed to $[x^i, y^n, \ldots, y^n] = 1$, where the repeated commutator has length depending on x and y only. We thus have a generalisation of the well-known fact that a finitely generated soluble Engel group is nilpotent.

Our final preliminary is

2.3. *Every torsion-free nilpotent* PH^**-group is abelian.*

Proof. We may assume that G is 2-generator, say $G = \langle x, y\rangle$. Let Z be the centre of G. Then G/Z is torsion-free and so by an obvious induction, it is abelian and hence G is of class 2. Setting $H_i = \langle xy^i\rangle$, $i \in \mathbb{Z}$, we have $H_nH_m = H_mH_n$ for some n, $m > 0$, $n \neq m$ so H_nH_m is a metabelian group by Itô's Theorem and by [3] has an abelian subgroup of finite index. But this group is also torsion-free nilpotent of class 2 and an easy argument

https://doi.org/10.1017/S1446788700029943 Published online by Cambridge University Press

shows that it is abelian. Hence xy^n and xy^m commute, and once again, the fact that G is torsion-free nilpotent of class 2 gives that x and y must commute.

We are now in a position to prove Theorem 1. Let G be a finitely generated soluble PH*-group. By 2.1, G is nilpotent-by-finite, so it has a torsion-free nilpotent subgroup A of finite index; by 2.3, A is abelian. We proceed by induction on $|G/A|$, all being well when $G = A$.

If $\langle A, x\rangle < G$ for all $x \in G$, then $\langle A, x\rangle$ is an FC-group for all x in G, so G is an FC-group. Finitely generated FC-groups are centre-by-finite and thus we may assume that $\langle A, x\rangle = G$ for some x. Let n be the order of x modulo A. If n is not a prime power, then $n = rs$ with $(r, s) = 1$. Since $\langle A, x^r\rangle$ and $\langle A, x^s\rangle$ are proper subgroups of G, the centralizers $C_G(x^r)$ and $C_G(x^s)$ are of finite index in G, as above, and $C_G(x) > C_G(x^r) \cap C_G(x^s)$.

We know now that $n = p^m$ for some prime p. By induction, $\langle A, x^p\rangle$ is centre-by-finite. Thus $[A, x^p]$ is finite; since A is torsion-free and normal, this means that $[A, x^p] = 1$. Thus the group $B := \langle A, x^p\rangle$ is abelian, and of course x^p is in the centre of G.

We can assume that x has infinitely many conjugates in G, else the centre $Z(G)$ has finite index, since it contains $A \cap C_G(x)$. Thus, there must exist an a such that $\langle x^{a^i}\rangle \neq \langle x^{a^j}\rangle$ if $i \neq j$. Property PH* now means that $y := \langle x\rangle\langle x^b\rangle = \langle x^b\rangle\langle x\rangle$ for some $b = a^i$. Modulo the central subgroup $\langle x^p\rangle$, Y has order p or p^2, so $[x, b]^p \in \langle x^p\rangle$; since $[x, b]^p = [x, b^p]$, we have the contradiction that $\langle x^{b^p}\rangle = \langle x\rangle$. Thus x has only finitely many conjugates after all, and G is centre-by-finite. This completes the proof of Theorem 1.

Proof of Theorem 2. We are required to show that a torsion-free PH*-group G is abelian. For this purpose it is sufficient to assume that $G = \langle g_1, \dots, g_k\rangle$ is finitely generated. Suppose that a, b are elements of G such that $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$. Since this product is metabelian, it follows from Theorem 1 and the fact that G is torsion free that $\langle a, b\rangle$ is abelian. Now for any pair x, y of elements of G, there exists $i > 0$ such that $\langle y\rangle\langle y^{x^i}\rangle = \langle y^{x^i}\rangle\langle y\rangle$. Thus $\langle y, y^{x^i}\rangle$ is abelian. Similarly, $\langle x, x^{y^j}\rangle$ is abelian for some $j > 0$. Hence $\langle x^i, y^j\rangle$ is nilpotent and hence abelian by Theorem 1. Since $[x^i, y, y] = 1$, we have $1 = [x^i, y^j] = [x^i, y]^j$, so $[x^i, y] = 1$. In particular, by considering the pairs (x, g_j), for $j = 1, \dots, k$, we get $[x^t, G] = 1$ for some $t > 0$. This shows that $G/Z(G)$ is periodic.

Obtain, if possible, a sequence $(a_1, a_2, \dots)$ of elements of G as follows. Pick any $a_1 \in G \setminus Z(G)$ and for $i \geq 2$, pick a_i from $G \setminus \bigcup_{j=1}^{i-1} C_G(a_j)$. If $\bigcup_{j=1}^{n} C_G(a_j) = G$ for some $n \in N$, then $C_G(a_i)$ is of finite index in G for some $i \leq n$. Set $A = \langle a_i^G \rangle$. Then A is in the FC-centre of G and $[A, G]$ is finite. But G is torsion-free, whence $A \leq Z(G)$, contradicting our choice of a_i. We conclude that in this case G is abelian.

The other alternative is the existence of an infinite sequence $(a_1, a_2, \dots)$ as constructed above. By hypothesis, $\langle a_j \rangle \langle a_i \rangle = \langle a_i \rangle \langle a_j \rangle$ for some $0 < i < j$. In this case $\langle a_i, a_j \rangle$ is abelian, as shown earlier in the proof. But then $a_j \in C_G(a_i)$, a contradiction. This completes the proof of Theorem 2.

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