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RIEMANNIAN STRUCTURES SUBORDINATE TO CERTAIN ALMOST TANGENT STRUCTURES

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1. **Introduction.** Almost tangent structures have been studied by Eliopoulos [1] and certain Riemannian structures subordinate to almost tangent structures have been studied by Closs [2]. In this paper we investigate those subordinate Riemannian structures for which the underlying almost tangent structure is without torsion and those for which the fundamental form is closed.

Similar studies have been carried out with respect to Riemannian structures subordinate to almost complex structures by Lichnerowicz [4] and with respect to Riemannian structures subordinate to almost product structures by Legrand [5].

2. **Almost tangent structures.** Let V_{2n} be a differentiable manifold of class C^∞ and dimension $2n$. We assume that a field J, of class C^∞, of linear operators is given on V_{2n} such that, at each point x of V_{2n}, J_x maps the tangent vector space to V_{2n} at x into itself. We also assume that J is of rank n and satisfies the relation $J^2=0$. We then say that V_{2n} is endowed with an almost tangent structure.

In an almost tangent structure there always exist bases, called AT-adapted bases, such that

$$J = (F_i^j) = (\delta_i^{j+n})$$

relative to such a basis. Here and in the sequel any Latin index $i, j, k, \ldots$ takes the values $1, 2, \ldots, 2n$ and any Greek index $\alpha, \beta, \lambda, \ldots$ takes the values $1, 2, \ldots, n$. Also we define $\alpha^* = \alpha + n$.

The results in [3] can be applied to almost tangent structures by putting $r=1$. We may thus define a tensor T with components t^i_{jk} by setting

$$t^{\alpha}_{\beta^*\lambda^*} = c^{\alpha^*}_{\beta\lambda}$$

and all other components equal to zero, where

$$d\theta^i = \tfrac{1}{2}c^i_{jk}\theta^j \wedge \theta^k, \quad c^i_{jk} + c^i_{kj} = 0$$

and (θ^i) is a dual basis for the tangent vector space at x. T is known as the torsion tensor of the almost tangent structure.

We are also able to define a tensor of structure S with components s^i_{jk} by setting

$$\begin{aligned}
s^{\lambda^*}_{\alpha^*\beta^*} &= c^{\lambda^*}_{\alpha\beta} \\
s^{\lambda^*}_{\alpha\beta} &= s^{\lambda^*}_{\alpha^*\beta} = s^{\lambda^*}_{\alpha\beta^*} = 0 \\
s^{\lambda}_{\alpha^*\beta^*} &= c^{\lambda}_{\alpha\beta} - c^{\lambda^*}_{\alpha\beta^*} - c^{\lambda^*}_{\alpha^*\beta} \\
s^{\lambda}_{\alpha^*\beta} &= -c^{\lambda^*}_{\alpha\beta} \\
s^{\lambda}_{\alpha\beta^*} &= -c^{\lambda^*}_{\alpha\beta}
\end{aligned}$$

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where the c^i_{jk} are the same constants of structure as above. This tensor S is equivalent to the Nijenhuis tensor.

In [6] it is shown that an almost tangent structure is integrable if and only if the Nijenhuis tensor is equal to zero. Hence if the almost tangent structure is integrable then S vanishes and by comparison we see also that T vanishes. We conclude that if an almost tangent structure is integrable it is necessarily without torsion.

3. **Almost Hermitian structures.** In addition to an almost tangent structure on V_{2n} let us be given a Riemannian metric of class C^∞, that is, a real symmetric tensor $G=(g_{ij})$ for which the determinant is everywhere nonzero and for which the components in a system of local coordinates (x^i) are functions of class C^∞ of the (x^i). By imposing the compatibility condition

$$G{}^tJ+JG = 0$$

where ${}^tJ$ is the transpose of the matrix J we endow V_{2n} with a real almost hermitian structure subordinate to the given almost tangent structure. It is then necessary that $n=2m$ for some integer m. We adopt the following convention:

$$\begin{aligned}\alpha(1) &= 1, 2, \ldots, m\\ \alpha(2) &= \alpha(1)+m\\ \alpha(1)^* &= \alpha(1)+n\\ \alpha(2)^* &= \alpha(2)+n = \alpha(1)+m+n.\end{aligned}$$

With respect to any AT-adapted basis we have

$$g_{\alpha\beta} = 0 \quad \text{and} \quad g_{\alpha\beta^*} = -g_{\alpha\beta^*}.$$

In an almost hermitian structure there always exist special AT-adapted bases, called H-adapted bases, such that with respect to such a basis we have

$$(g_{ij}) = \begin{bmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & -I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{bmatrix}$$

where I_m is the (m, m) identity matrix.

If we put $F_{ij}=g_{jk}F^k_i$ then F_{ij} are the components of a real exterior 2-form F which we call the fundamental form of the almost hermitian structure.

Since an almost hermitian structure has the structure of a Riemannian manifold then from the fundamental theorem of Riemannian geometry there exists a unique euclidean connection with zero torsion. This connection will be called the Riemannian connection of the manifold V_{4m}. We will denote by (ω^i_j) the forms defining the Riemannian connection and with respect to some dual basis (θ^k) we put

$$\omega^i_j = \gamma^i_{jk}\theta^k.$$

Let DF^i_j be the absolute differential of the tensor F^i_j in the Riemannian connection. Then the covariant derivative $D_kF^i_j$ is given by:

$$D_kF^i_j = \partial_kF^i_j+\gamma^i_{hk}F^h_j-\gamma^h_{jk}F^i_h.$$

Relative to the AT-adapted bases we obtain:

$$\begin{aligned} D_kF^\alpha_\beta &= -\gamma^{\alpha^*}_{\beta k} \\ D_kF^\alpha_{\beta^*} &= \gamma^\alpha_{\beta k}-\gamma^{\alpha^*}_{\beta^* k} \\ D_kF^{\alpha^*}_\beta &= 0 \\ D_kF^{\alpha^*}_{\beta^*} &= \gamma^{\alpha^*}_{\beta k}. \end{aligned}$$

Since the Riemannian connection has zero torsion we have

$$d\theta^i = \theta^j \wedge \omega^i_j = \gamma^i_{jk}\theta^j \wedge \theta^k.$$

It follows that the nonzero components of the torsion tensor of the almost tangent structure underlying the almost hermitian structure are given by

$$\begin{aligned} t^\alpha_{\beta^*\lambda^*} &= \gamma^{\alpha^*}_{\beta\lambda}-\gamma^{\alpha^*}_{\lambda\beta} \\ &= D_\lambda F^{\alpha^*}_{\beta^*}-D_\beta F^{\alpha^*}_{\lambda^*} \\ &= D_\beta F^\alpha_\lambda - D_\lambda F^\alpha_\beta. \end{aligned}$$

The Riemannian connection being Euclidean, we can write

$$D_hF_{ij} = D_h(F^k_i g_{kj}) = g_{kj}D_hF^k_i.$$

Then, relative to the AT-adapted bases it is easy to show that:

$$\begin{aligned} D_hF_{\alpha\beta} &= 0 \\ D_hF_{\alpha^*\beta} &= g_{\lambda^*\beta}\gamma^{\lambda^*}_{\alpha h} = D_hF_{\alpha\beta^*}. \end{aligned}$$

We then have

THEOREM 1. *In an almost hermitian structure subordinate to an almost tangent structure*

$$D_hF_{\alpha\beta} = 0$$

and

$$D_hF_{\alpha\beta^*} = D_hF_{\alpha^*\beta}$$

with respect to the AT-*adapted bases, where D_h denotes covariant derivation in the Riemannian connection and (F_{ij}) is the fundamental form of the almost hermitian structure.*

4. **Hermitian and pseudohermitian structures.** Given an almost hermitian structure on V_{2n} we say that it is hermitian if the underlying almost tangent structure is integrable. The underlying almost tangent structure is then necessarily without torsion.

Given an almost hermitian structure we say that it is pseudohermitian if the underlying almost tangent structure is without torsion. A hermitian structure is thus always pseudohermitian.

Let us now consider an H-adapted basis; relative to this basis we have

$$\begin{aligned}\gamma^{\alpha^*}_{\beta\lambda} &= -D_\lambda F^\alpha_\beta \\ &= -D_\lambda(F_{\beta\mu}g^{\mu\alpha}+F_{\beta\mu^*}g^{\mu^*\alpha})\end{aligned}$$

where $(g^{ij})=(g_{ij})$ so that we get

$$\gamma^{\alpha^*}_{\beta\lambda} = -g^{\mu^*\alpha}D_\lambda F_{\beta\mu^*}.$$

It follows that

$$\begin{aligned}\gamma^{\alpha(1)^*}_{\beta\lambda} &= -D_\lambda F_{\beta\alpha(2)^*} \\ \gamma^{\alpha(2)^*}_{\beta\lambda} &= -D_\lambda F_{\beta\alpha(1)^*}.\end{aligned}$$

Thus

$$\begin{aligned}t^{\alpha(1)}_{\beta^*\lambda^*} &= \gamma^{\alpha(1)^*}_{\beta\lambda}-\gamma^{\alpha(1)^*}_{\lambda\beta} \\ &= D_\beta F_{\lambda\alpha(2)^*}-D_\lambda F_{\beta\alpha(2)^*}\end{aligned}$$

and

$$\begin{aligned}t^{\alpha(2)}_{\beta^*\lambda^*} &= \gamma^{\alpha(2)^*}_{\beta\lambda}-\gamma^{\alpha(2)^*}_{\lambda\beta} \\ &= -(D_\beta F_{\lambda\alpha(1)^*}-D_\lambda F_{\beta\alpha(1)^*}).\end{aligned}$$

We see from the above that the almost tangent structure underlying an almost hermitian structure is without torsion if and only if $D_\beta F_{\lambda\alpha^*}$ is symmetric in β and λ. But since $D_\beta F_{\lambda\alpha^*}$ is antisymmetric in λ and α^* and since $D_\beta F_{\lambda\alpha^*}=D_\beta F_{\lambda^*\alpha}$, such a symmetry condition implies that $D_\beta F_{\lambda\alpha^*}=0$.

THEOREM 2. *In order that an almost hermitian structure be pseudohermitian it is necessary and sufficient that relative to* H-*adapted bases one has*

$$D_\lambda F_{\alpha\beta^*} = 0$$

or equivalently, $\gamma^{\beta^*}_{\lambda\alpha}=0$.

We may also characterize the pseudohermitian structures by means of a tensor.

THEOREM 3. *With respect to any basis whatever, the pseudohermitian structures are able to be characterized by the relation*

$$F^a_i({}^tF)^b_jF^c_kD_aF_{bc} = 0$$

where $({}^tF)^b_j=F^j_b$.

To prove this let

$$p_{ijk} = F^a_i({}^tF)^b_jF^c_kD_aF_{bc};$$

then the p_{ijk} are the components of a tensor. Relative to an H-adapted basis we have

$$\begin{aligned}p_{\lambda jk} &= 0 \\ p_{ij\beta} &= 0 \\ p_{i\alpha^*k} &= 0 \\ p_{\lambda^*\alpha\beta^*} &= D_\lambda F_{\alpha\beta^*}.\end{aligned}$$

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Hence (p_{ijk}) vanishes if and only if the given almost hermitian structure is pseudo-hermitian.

5. **Almost Kahlerian structures.** An almost hermitian structure will be called almost Kahlerian if the fundamental form F is closed, that is, $dF=0$. This condition may be written

$$D_iF_{jk}+D_jF_{ki}+D_kF_{ij} = 0$$

where the relation is to hold at any point of V_{2n} for any i, j, k.

Relative to an H-adapted basis, the relation assumes the form:

$$D_\lambda F_{\alpha\beta^*}+D_\alpha F_{\beta^*\lambda} = 0 \tag{1}$$

$$D_\lambda F_{\alpha^*\beta^*}+D_{\alpha^*}F_{\beta^*\lambda}+D_{\beta^*}F_{\lambda\alpha^*} = 0 \tag{2}$$

$$D_{\lambda^*}F_{\alpha^*\beta^*}+D_{\alpha^*}F_{\beta^*\lambda^*}+D_{\beta^*}F_{\lambda^*\alpha^*} = 0. \tag{3}$$

The equation (1) can be written

$$D_\lambda F_{\alpha\beta^*}-D_\alpha F_{\lambda\beta^*} = 0$$

or

$$D_\lambda F_{\alpha\beta^*} = D_\alpha F_{\lambda\beta^*};$$

that is, $D_\lambda F_{\alpha\beta^*}$ is symmetric with respect to λ and α. But, as we have seen before, this is equivalent to the condition

$$D_\lambda F_{\alpha\beta^*} = 0. \tag{4}$$

THEOREM 4. *Any almost Kahlerian structure is pseudohermitian. Any pseudo-hermitian structure satisfying* (2) *and* (3) *is almost Kahlerian.*

Let us consider a pseudohermitian structure for which we define

$$\begin{aligned} a_{ijk} &= F_i^h D_h F_{jk} \\ b_{ijk} &= F_i^h D_j F_{kh} \\ c_{ijk} &= F_i^h D_k F_{hj} \\ q_{ijk} &= ({}^tF)_i^a({}^tF)_j^b({}^tF)_k^c D_a F_{bc}. \end{aligned}$$

The a_{ijk}, b_{ijk}, c_{ijk}, and q_{ijk} are the components of respective tensors. Let us put

$$r_{ijk} = q_{ijk}+q_{jki}+q_{kij}+a_{ijk}+b_{ijk}+c_{ijk}.$$

Then the r_{ijk} are also components of a tensor. With respect to an H-adapted basis we have

$$\begin{aligned} r_{\lambda\alpha\beta} &= D_{\lambda^*}F_{\alpha^*\beta^*}+D_{\alpha^*}F_{\beta^*\lambda^*}+D_{\beta^*}F_{\lambda^*\alpha^*} \\ r_{\lambda^*\alpha^*\beta^*} &= D_\lambda F_{\alpha^*\beta^*}+D_{\alpha^*}F_{\beta^*\lambda}+D_{\beta^*}F_{\lambda\alpha^*} \end{aligned}$$

and all other components equal to zero. Thus (r_{ijk}) vanishes if and only if (2) and (3) are satisfied and hence if and only if the given pseudohermitian structure is almost Kahlerian.

Let us now consider an almost hermitian structure and let us define

$$u_{ijk} = r_{ijk} - p_{ijk}.$$

The u_{ijk} are again the components of a tensor. We note that

$$\begin{aligned}
u_{\lambda\alpha\beta} &= D_{\lambda^*}F_{\alpha^*\beta^*} + D_{\alpha^*}F_{\beta^*\lambda^*} + D_{\beta^*}F_{\lambda^*\alpha^*} \\
u_{\lambda^*\alpha^*\beta} &= D_{\lambda}F_{\alpha^*\beta} + D_{\beta}F_{\lambda\alpha^*} \\
u_{\lambda^*\alpha\beta^*} &= D_{\alpha}F_{\beta^*\lambda} \\
u_{\lambda^*\alpha^*\beta^*} &= D_{\lambda}F_{\alpha^*\beta^*} + D_{\alpha^*}F_{\beta^*\lambda} + D_{\beta^*}F_{\lambda\alpha^*}
\end{aligned}$$

and the remaining components of u_{ijk} vanish. One sees that the vanishing of $u_{\lambda^*\alpha\beta^*}$ is equivalent to (4) or that the vanishing of $u_{\lambda^*\alpha^*\beta}$ is equivalent to (1); also the vanishing of $u_{\lambda\alpha\beta}$ is equivalent to (3) and the vanishing of $u_{\lambda^*\alpha^*\beta^*}$ is equivalent to (2). It then follows that the tensor u_{ijk} vanishes if and only if the given almost hermitian structure is almost Kahlerian. We then have proven

THEOREM 5. *With respect to any basis whatever, the almost Kahlerian structures are able to be characterized by the relation*:

$$\begin{aligned}
&({}^tF)_i^a({}^tF)_j^b({}^tF)_k^c D_aF_{bc} + ({}^tF)_j^a({}^tF)_k^b({}^tF)_i^c D_aF_{bc} + ({}^tF)_k^a({}^tF)_i^b({}^tF)_j^c D_aF_{bc} \\
&\qquad + F_i^h D_hF_{jk} + F_i^h D_jF_{kh} + F_i^h D_kF_{hj} - F_i^a({}^tF)_j^b F_k^c D_aF_{bc} = 0.
\end{aligned}$$

An almost hermitian structure will be called Kahlerian if the underlying almost tangent structure is integrable and the fundamental form F is closed. It will be called pseudo-Kahlerian if the underlying almost tangent structure is without torsion and F is closed. In other words an almost hermitian structure is Kahlerian if it is at the same time hermitian and almost Kahlerian. Also, an almost hermitian structure is pseudo-Kahlerian if it is at the same time pseudohermitian and almost Kahlerian. Since any almost Kahlerian structure is necessarily pseudohermitian the pseudo-Kahlerian structures coincide with the almost Kahlerian structures. The relationship between these various structures is shown in the diagram below.

$$\begin{tikzcd}
\text{Almost Hermitian} \arrow[r, "T=0"] \arrow[d, "dF=0"] & \text{Pseudohermitian} \arrow[r, "S=0"] \arrow[d, "dF=0"] & \text{Hermitian} \arrow[d, "dF=0"] \\
\text{Almost Kahlerian} \arrow[r, phantom, "\equiv"] & \text{Pseudo-Kahlerian} \arrow[r, "S=0"] & \text{Kahlerian}
\end{tikzcd}$$

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