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Mutation-finite quivers with real weights

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Abstract

We classify all mutation-finite quivers with real weights. We show that every finite mutation class not originating from an integer skew-symmetrisable matrix has a geometric realisation by reflections. We also explore the structure of acyclic representatives in finite mutation classes and their relations to acute-angled simplicial domains in the corresponding reflection groups.

Contents

1. Introduction and main results

Mutations of quivers were introduced by Fomin and Zelevinsky in [14] in relation to cluster algebras. Mutations are involutive transformations decomposing the set of quivers into equivalence classes called *mutation classes*. Of special interest are quivers of *finite mutation type*, that is those whose mutation classes are finite, these quivers have shown up recently in various contexts. Most of such quivers are adjacency quivers of triangulations of marked bordered surfaces [11, 12, 13, 16], the complete classification of mutation-finite quivers was obtained in [6].

In this paper, we consider a more general notion of a quiver — we allow arrows of quivers to have *real weights* (and we refer to 'usual' quivers as to *integer quivers*). Quivers with real weights of arrows have been studied in [1], where the Markov constant was used to explore the mutation classes of quivers of rank 3. Quivers originating from noncrystallographic finite root systems were also considered in [18]. A categorification of mutations of quivers of noncrystallographic types $I_2(2n + 1), H_3, H_4$ was constructed in [3]. In [9], we constructed a geometric model for mutations of rank 3 quivers with real

Table 1.1. *Mutation-finite non-integer non-orbifold type quivers.*

	rank 3	rank 4	rank 5	rank 6
Finite type	H_3, H_3', H_3''	F_4, H_4, H_4', H_4'', H_4''', H_4''''		
Affine type	$\widetilde{G}_{2,n}$	$\widetilde{H}_3$, $\widetilde{H}_3'$	$\widetilde{F}_4$, $\widetilde{H}_4$	
Extended affine type		$\widetilde{G}_{2,2n}^{(*,+)}$, $\widetilde{G}_{2,2n}^{(*,*)}$, $\widetilde{G}_{2,2n+1}^{(*,*)}$	$H_3^{(1,1)}$	$F_4^{(*,+)}$, $F_4^{(*,*)}$, $H_4^{(1,1)}$

weights and classified all finite mutation classes. The main result of this paper is a classification of all finite-mutation quivers. More precisely, we prove the following theorem.

Theorem A. *For every mutation-finite non-integer quiver Q of rank $r \geq 3$*

- *either Q arises from a triangulated orbifold;*
- *or Q is mutation-equivalent to one of the F-type quivers shown in Figure 4.1;*
- *or Q is mutation-equivalent to one of the H-type quivers shown in Figure 4.2;*
- *or Q is mutation-equivalent to a representative of one of the three series of quivers shown in Figure 3.1.*

We list all non-orbifold mutation-finite classes and their sizes in Tables 1.1 and 1.2, respectively. Notice that the list above includes the appropriate *rescalings* (see [20, Section 7]) of all mutation-finite diagrams from [7]: quivers arising from triangulated orbifolds [8] and F-type quivers are explicitly mentioned in Theorem A, and G_2-type quivers obtained from the diagrams $G_2^{(*,+)}$ and $G_2^{(*,*)}$ belong to the series mentioned in the last line of Theorem A.

Our proof of Theorem A is based on the classification of mutation-finite rank 3 quivers [9] and the related geometry. In particular, all the weights of arrows of mutation-finite quivers should be of the form

Table 1.2. *Sizes of mutation classes of type H and type F quivers.*

H_3	H_3'	H_3''	H_4	H_4'	H_4''	H_4'''	H_4''''	$\widetilde{H}_3$	$\widetilde{H}_3'$	$\widetilde{H}_4$	$H_3^{(1,1)}$	$H_4^{(1,1)}$
6	6	10	18	23	32	60	30	36	28	524	8	179

F_4	$\widetilde{F}_4$	$F_4^{(*,+)}$	$F_4^{(*,*)}$
8	60	49	35

$2\cos(q\pi/d)$ for some integer q and d, and every rank 3 subquiver has to correspond to some spherical or Euclidean triangle (we recall the details in Section 2). We first show that all mutation-finite quivers of sufficiently high rank originate from orbifolds, and then treat quivers in low ranks, where we obtain three exceptional infinite families.

Next, we construct a geometric realisation for every finite mutation class of quivers except for mutation-cyclic ones originating from orbifolds (we conjecture, though, that mutation classes originating from unpunctured orbifolds also have geometric realisations by reflections). The realisation by reflections provides a quadratic form associated to a mutation class, and thus, we can define quivers of *finite type* when the corresponding quadratic form is positive definite. As in the integer case (see [15]), these correspond precisely to finite reflection groups. We say that a quiver has *affine* or *extended affine* type if its mutation class is realised in a semipositive quadratic space of corank 1 (at least 2, respectively). The result can be then formulated in the following theorem.

Theorem B. *Every non-integer finite mutation class (except for the quivers originating from orbifolds) has a geometric realisation by reflections. In particular,*

- *quivers in the top row of Table 1.1 are of finite type;*
- *quivers in the middle row of Table 1.1 are of affine type;*
- *quivers in the bottom row of Table 1.1 are of extended affine type.*

The paper is organised as follows. In Section 2, we recall some details from [9] on the classification of mutation-finite quivers in rank 3 which will be our main tool. In Section 3, we show that in rank greater than 4, the denominator d in the weight $2\cos(q\pi/d)$ of an arrow of a mutation-finite quiver is bounded by 5. Thus, we restrict our considerations to quivers with weights of arrows belonging to $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[(1+\sqrt{5})/2]$ which we consider in Section 4, and to quivers of rank 4 considered in Section 5. In Section 6, we show that mutations in finite mutation classes can be modelled by partial reflections in positive semidefinite quadratic spaces. Finally, in Section 7, we explore the relations between acyclic representatives in finite mutation classes and acute-angled simplices bounded by mirrors of reflections.

2. Classification in rank 3

In this section, we recall the results of [9] this paper is based on and deduce some straightforward corollaries we will use throughout the text. We start with reminding the reader of the notation we used in [9] and introducing some new ones.

Notation 2.1.

○ Given a quiver Q with vertices $1, \ldots, n$, and a subset $I \subset \{1, \ldots, n\}$, denote by Q_I the subquiver of Q spanned by vertices $\{i \in I\}$. In particular, the vertex labeled i will be denoted Q_i. For example, Q_{124} will denote a subquiver spanned by vertices Q_1, Q_2, Q_4.

○ While drawing quivers, we will use the following conventions:
  – given an arrow Q_{ij} of weight $2\cos\frac{\pi m}{d}$, we will label this arrow by $\frac{m}{d}$; we will also say that Q_{ij} is an arrow *marked* $\frac{m}{d}$;
  – arrows of weight 1 will be left unlabeled;
  – we draw double arrows instead of arrows of weight 2.

○ We say that an arrow Q_{ij} *vanishes* if Q_i and Q_j are not joined in Q.

○ By (a, b, c), $a, b, c > 0$, we denote a rank 3 cyclic quiver with arrows of weights a, b, c.

○ A rank 3 acyclic quiver with arrows of weights a, b looking in one direction and an arrow of weight c in the opposite direction will be denoted by $(a, b, -c)$, where some weights may equal 0.

○ Given a quiver Q, we will denote by Q^{op} the quiver obtained from Q by reversing all arrows. Q^{op} is also called a quiver *opposite* to Q.

Theorem 2.2 ([9], Theorem 6.11). *Let Q be a connected rank 3 quiver with real weights. Then Q is of finite mutation type if and only if it is mutation-equivalent to one of the following quivers:*

(1) $(2, 2, 2)$;

(2) $(2, 2\cos\frac{\pi}{d}, 2\cos\frac{\pi}{d})$, $d \in \mathbb{Z}_+$;

(3) $(1, 1, 0)$, $(1, \sqrt{2}, 0)$, $(1, 2\cos\frac{\pi}{5}, 0)$, $(2\cos\frac{\pi}{5}, 2\cos\frac{2\pi}{5}, 0)$, $(1, 2\cos\frac{2\pi}{5}, 0)$.

Below we list some corollaries of Theorem 2.2 and related geometric constructions proved in [9].

Corollary 2.3. *Let Q be a connected mutation-finite rank 3 quiver with real weights. Then*

(1) *all weights of Q are of the form $2\cos\frac{\pi m}{d}$ with $m, d \in \mathbb{Z}$, $m \leq d/2$;*

(2) *if Q contains an arrow marked $\frac{m}{d}$ with $d > 5$, $\gcd(m, d) = 1$, then Q is mutation-equivalent to $(2, 2\cos\frac{\pi}{d}, 2\cos\frac{\pi}{d})$;*

(3) *if Q contains an edge of weight 2, then Q is a cyclic quiver which coincides with either $(2, 2, 2)$ or $(2, 2\cos\frac{\pi m}{d}, 2\cos\frac{\pi m}{d})$, $d, m, \in \mathbb{N}$, $0 < m \leq d/2$;*

(4) *if $Q = (a, b, -c)$, $a, b, c > 0$ is an acyclic quiver, then*
  - *$Q = (2\cos\frac{\pi m}{d}, 2\cos\frac{\pi s}{d}, -2\cos\frac{\pi t}{d})$ for some $d, m, s, t \in \mathbb{N}$, $m, s, t \leq d/2$, such that $\frac{m}{d} + \frac{s}{d} + \frac{t}{d} \geq 1$;*
  - *if in addition at least one of a, b, c equals $2\cos\frac{\pi m}{d}$ with $d > 5$, $\gcd(m, d) = 1$, then $\frac{m}{d} + \frac{s}{d} + \frac{t}{d} = 1$;*

(5) *if $Q = (a, b, c)$, $a, b, c > 0$ is a cyclic quiver, then*
  - *$Q = (2\cos\frac{\pi m}{d}, 2\cos\frac{\pi s}{d}, 2\cos\frac{\pi t}{d})$ for some $d, m, s, t \in \mathbb{N}$, $m, s, t \leq d/2$, such that $\frac{m}{d} + \frac{s}{d} + (1 - \frac{t}{d}) \geq 1$ (up to permutation of m, s, t);*
  - *if in addition at least one of a, b, c equals to $2\cos\frac{\pi m}{d}$ with $d > 5$, $\gcd(m, d) = 1$, then $\frac{m}{d} + \frac{s}{d} + (1 - \frac{t}{d}) = 1$.*

The equalities (4)–(5) in Corollary 2.3 have a geometric interpretation: for every mutation-finite quiver, there is a spherical or Euclidean triangle with the corresponding angles (the triangle is acute-angled if Q is acyclic, and has an obtuse angle otherwise). We also remind the reader that the mutations can be modeled by partial reflections (see [9]).

3. High denominators in ranks 5 and higher

In this section, we show that there are no high denominator quivers of rank higher than 4 (Theorem 3.8). To prove the theorem, we start with several technical lemmas (Lemma 3.2–3.7) about rank 3 and 4 quivers.

Definition 3.1. Given a quiver Q, we say that $d \in \mathbb{N}$ is the *highest denominator* in the mutation class of Q if all weights of quivers in the mutation class of Q are either 2 or of the form $2\cos\frac{p'}{d'}$ with $d' \leq d$, and there exists a quiver Q' in the mutation class of Q with an arrow of weight $2\cos\frac{p}{d}$, $\gcd(p, d) = 1$. Abusing notation, we will say that Q is a *denominator d quiver*.

Lemma 3.2. *No connected mutation-finite quiver contains the Markov quiver $(2, 2, 2)$ as a proper subquiver.*

Proof. Suppose the contrary, that is $Q = Q_{1234}$ with $Q_{123} = (2, 2, 2)$ is a mutation-finite quiver. By Corollary 2.3(3), all rank 3 subquivers of Q should be cyclic, which is clearly impossible. □

Lemma 3.3. *Let Q be a connected mutation-finite quiver of rank 4. Suppose that $d > 5$ is the highest denominator of weights of arrows in the mutation class of Q. Then either Q or Q^{op} is mutation-equivalent to one of the quivers listed in Figure 3.1.*
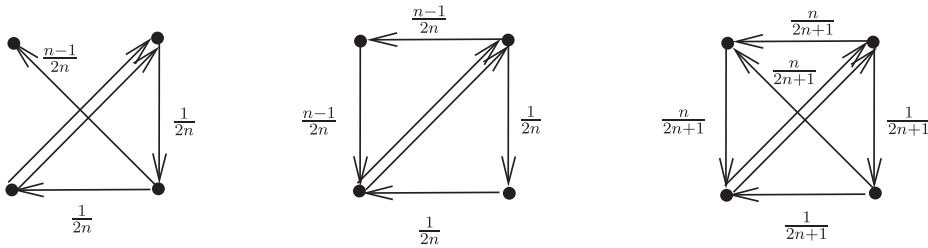
Figure 3.1. *Three infinite series of quivers. Following in Section 2, the arrows of weight $2\cos\frac{\pi p}{q}$ are labeled by $\frac{p}{q}$, and the arrows of weight 2 are shown by double arrows.*
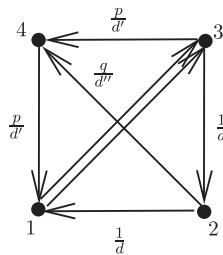

Figure 3.2. *To the proof of Lemma 3.3.*

Proof. Consider such a quiver Q. We can assume that Q has an arrow of weight $2\cos\frac{\pi m}{d}$, $d > 5$ and m and d are coprime. Let Q_{123} be a rank 3 connected subquiver of Q containing an arrow of weight $2\cos\frac{\pi m}{d}$. Then Q_{123} corresponds to a Euclidean triangle, and hence, the mutation class of Q_{123} contains an oriented subquiver with weights $(2, 2\cos\frac{\pi}{d}, 2\cos\frac{\pi}{d})$ (without loss of generality, we can assume that this is the subquiver Q_{123} itself, and Q_{13} is the double arrow). Moreover, as a mutation-finite rank 3 subquiver Q_{134} with a double arrow cannot be a Markov quiver (see Lemma 3.2), Q_{134} should be a cyclic quiver with the weights $(2, 2\cos\frac{\pi p}{d'}, 2\cos\frac{\pi p}{d'}))$ for some $p \le d'/2$, $d' \le d$ (see Corollary 2.3(4)). We conclude that Q is the quiver shown in Figure 3.2, where the weight of Q_{24} is $2\cos\frac{\pi q}{d''}$ with $q \le d''/2$, $d'' \le d$ (note that the arrow Q_{24} may be oriented in the opposite way — this would mean the quiver in Figure 3.2 is the opposite quiver Q^{op}).

Consider the acyclic subquiver Q_{234}: as it contains an arrow Q_{23} of weight $2\cos\frac{\pi}{d}$ with $d > 5$, Corollary 2.3(4) implies

$$\frac{1}{d} + \frac{p}{d'} + \frac{q}{d''} = 1. \tag{3.1}$$

We will consider three cases: either $p/d' = 1/2$ (i.e. the arrows Q_{34} and Q_{14} vanish), $q/d'' = 1/2$ (the arrow Q_{24} vanishes) or, otherwise, all six arrows are present in Q.

Case 1. If $\frac{p}{d'} = \frac{1}{2}$, then $\frac{q}{d''} = \frac{1}{2} - \frac{1}{d}$. Since $d'' \le d$, we conclude that $d = 2n$ for some $n \in \mathbb{N}$, which implies $d'' = 2n$, $q = n - 1$ and Q is the quiver shown in the middle of Figure 3.1.
Case 2. If $\frac{q}{d''} = \frac{1}{2}$, then $\frac{p}{d'} = \frac{1}{2} - \frac{1}{d}$, so that $d = d' = 2n$, and $p = n - 1$ (where $n \in \mathbb{N}$), which produces the quiver on the left of Figure 3.1.
Case 3. Suppose that $\frac{p}{d'} \ne \frac{1}{2} \ne \frac{q}{d''}$. This implies that $\frac{p}{d'} < \frac{1}{2}$ and $\frac{q}{d''} < \frac{1}{2}$. Moreover, as $d' \le d$ and $d'' \le d$, we see that $\frac{1}{2} - \frac{p}{d'} \ge \frac{1}{2d}$ and $\frac{1}{2} - \frac{q}{d''} \ge \frac{1}{2d}$. In view of (3.1), this implies $\frac{1}{2} - \frac{p}{d'} = \frac{1}{2d}$ and $\frac{1}{2} - \frac{q}{d''} = \frac{1}{2d}$. Hence, $d = d' = d'' = 2n + 1$ for some $n \in \mathbb{N}$, $p = q = n$ and Q is the quiver shown on the right of Figure 3.1. $\qquad\square$

Remark 3.4. Notice that Lemma 3.3 gives a *necessary* condition for a quiver of rank 4 with large denominator to be mutation-finite. We will show that this condition is also sufficient (i.e. all quivers in Figure 3.1 are indeed mutation-finite) in Section 5.

The following lemma can be verified by a straightforward computation.

Lemma 3.5. *Let $d = 2n + 1$, where $n \geq 2$, $n \in \mathbb{N}$. Let Q be an acyclic quiver of rank 3 and μ be a nonsink/source mutation of Q. Then*

(a) *if $Q = (2\cos\frac{\pi}{d}, 2\cos\frac{\pi n}{d}, -2\cos\frac{\pi n}{d})$, then $\mu(Q) = (2\cos\frac{\pi n}{d}, 2\cos\frac{\pi n}{d}, 2\cos\frac{\pi(n-1)}{d})$;*
(b) *if $Q = (2\cos\frac{\pi n}{d}, 2\cos\frac{\pi n}{d}, -2\cos\frac{\pi}{d})$, then $\mu(Q) = (2\cos\frac{\pi n}{d}, 2\cos\frac{\pi n}{d}, 2)$.*

Remark 3.6. The quiver Q in Lemma 3.5 corresponds to an acute-angled Euclidean triangle T with angles $(\frac{\pi}{d}, \frac{\pi n}{d}, \frac{\pi n}{d})$, so the statement can also be easily checked by applying partial reflections.

Lemma 3.7. *Let $Q = Q_{1234}$ be an acyclic connected rank 4 quiver. Suppose that the vertex Q_3 is not joined with Q_1 in Q. If the weight of Q_{12} is $2\cos\frac{\pi m}{d}$ with $d > 5$, $\gcd(m, d) = 1$, then Q is mutation-infinite.*

Proof. Suppose that Q is mutation-finite, and assume first that Q_3 is neither a sink nor a source. In particular, it is connected to both Q_2 and Q_4. Then, as Q is acyclic, the mutation μ_3 at vertex Q_3 changes the weight of the arrow Q_{24} but does not change its direction.

Now, consider the subquiver Q_{124}. Since the arrow incident to Q_{12} has the weight $2\cos\frac{\pi m}{d}$ with $d > 5$, we conclude that the acyclic subquiver Q_{124} can be modeled by an acute-angled Euclidean triangle, and moreover, the weight of the arrow Q_{24} is uniquely determined by the weights of Q_{12} and Q_{14}. Since Q_3 is not joined with Q_1, the mutation μ_3 preserves the weights and directions of arrows Q_{12} and Q_{14}. Since μ_3 also preserves the direction of Q_{24}, this implies that the subquiver Q'_{124} of the mutated quiver $Q' = \mu_3(Q)$ is still acyclic and satisfies the same properties as Q_{124}: it is modeled by an acute-angled Euclidean triangle. Hence, the weight of the new arrow Q'_{24} should coincide with the weight of the old arrow Q_{24}. This contradicts the result of the paragraph above. The contradiction shows that Q is mutation-infinite.

Assume now that Q_3 is either a sink or a source. We will now show that by applying sink/source mutations only, we can make Q_3 neither a sink nor a source, and thus reduce the case to the one already being considered.

Indeed, without loss of generality, we can assume Q_3 is a sink. By applying, if necessary, a source mutation in Q_1, we can assume that Q_1 is not a source. Since Q is acyclic, it contains a source, and thus, either Q_2 or Q_4 is a source. After mutating at a source, the vertex Q_3 is neither a sink nor a source anymore, so we are in the assumptions of the first case. □

Theorem 3.8. *There is no rank 5 connected mutation-finite quiver with an arrow of weight $2\cos\frac{\pi m}{d}$ with $d > 5$, $m \leq d/2$, $\gcd(m, d) = 1$.*

Proof. Suppose that Q is a mutation-finite connected rank 5 quiver, and assume that Q_{1234} is a connected subquiver containing an arrow of weight $2\cos\frac{\pi m}{d}$, $d > 5$, $\gcd(m, d) = 1$. We can assume that d is the highest denominator in the mutation class of Q. Then by Lemma 3.3, Q_{1234} is mutation-equivalent to one of the quivers in Figure 3.1 or its opposite. Without loss of generality, we may assume that the subquiver Q_{1234} of Q itself is one of the quivers in Figure 3.1. We consider these three series of quivers separately.

Case 1: Odd denominator. $d = 2n + 1$. Then Q_{1234} is the subquiver shown in Figure 3.1 on the right (we can assume Q_{13} is the double arrow and Q_2 is the vertex incident to two arrows marked $\frac{1}{d}$). By reasoning as in Case 3 of the proof of Lemma 3.3, we see that the subquiver Q_{1235} looks identical to Q_{1234} modulo the direction of the arrow Q_{25} which can point either way (see Figure 3.3(a) and (b)). By applying Corollary 2.3(4) to acyclic subquiver Q_{145}, we see that the arrow Q_{45} should be marked $\frac{1}{d}$. In the case shown in Figure 3.3(b), we also see that the arrow Q_{45} is directed from Q_4 to Q_5 (as the weights
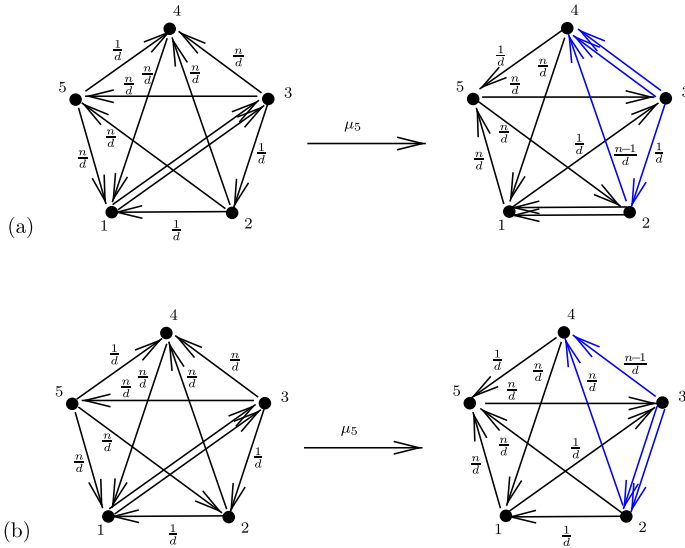
Figure 3.3. *Quiver of rank 5 with an arrow of weight* $2\cos\frac{\pi m}{d}$ *for odd* $d = 2n + 1$.
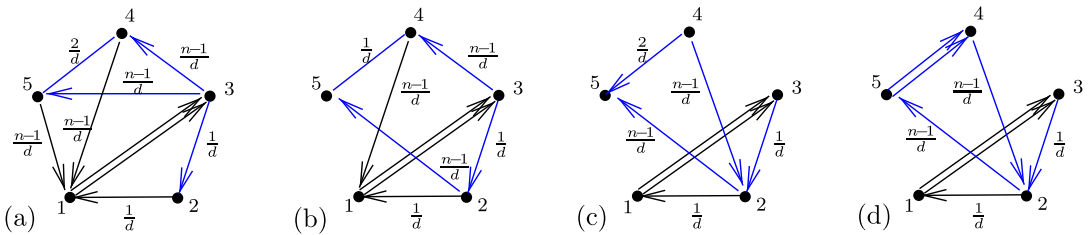

Figure 3.4. *Quiver of rank 5 with an arrow of weight* $2\cos\frac{\pi m}{d}$ *for even* $d = 2n$.

of arrows in the subquiver Q_{245} require this subquiver to be acyclic); in the case shown in Figure 3.3(a), the vertices Q_4 and Q_5 are completely symmetric, so we can also assume Q_{45} is directed from Q_4 to Q_5. Therefore, the quiver Q is one of the two quivers shown on the left of Figure 3.3. By applying mutation μ_5 in vertex Q_5, we obtain the quiver $Q' = \mu_5(Q)$ shown on the right of Figure 3.3 (we use Lemma 3.5 to compute the new weights of arrows). However, the subquiver Q'_{234} of Q' is an acyclic subquiver with arrow of weight 2, which is impossible by Corollary 2.3(3).

Case 2: Even denominator. $d = 2n$. In this case, there are two possibilities for each of the subquivers Q_{1234} and Q_{1235} (see Figure 3.1), which, up to symmetry and taking Q^{op} (and sink/source mutations), give rise to four forms of the quiver Q shown in Figure 3.4. In each of the four possibilities, the weight of the arrow Q_{45} is determined uniquely from subquivers Q_{145} or Q_{245}.

Notice that in cases (a), (b) and (c), the subquiver Q_{2345} is acyclic, having a vertex (Q_2, Q_3 and Q_3 in the three cases, respectively) which is not joined with Q_4 and incident to the arrow Q_{23} of weight $2\cos\frac{\pi}{d}$, $d > 5$. So, by Lemma 3.7, Q_{2345} (and, hence, Q) is mutation-infinite.

We are left to consider the case (d). By applying mutations in vertices Q_2 and the Q_1, we obtain the quiver Q' shown in Figure 3.5. Its subquiver Q'_{245} is acyclic, has a denominator $d > 5$ arrow Q'_{25} but does not correspond to a Euclidean triangle, so it is mutation-infinite.

□

Remark 3.9. In view of Theorem 3.8, in rank 5 and higher, we only need to consider the quivers with arrows marked $\frac{p}{d}$, $p < d \leq 5$. This will be done in Section 4.
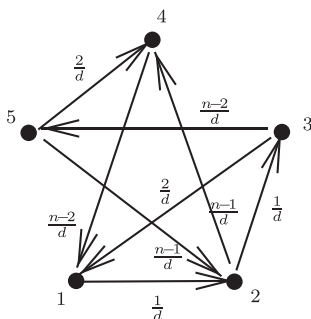
Figure 3.5. *A mutation of the quiver shown in Figure 3.4(d).*

4. Low denominator quivers

By *low denominator* quivers, we mean quivers without arrows marked $\frac{m}{d}$, where $m \leq d/2$, $\gcd(m, d) = 1$ and $d > 5$. There are finitely many of low denominator quivers in each rank, so one can classify mutation-finite low denominator quivers of small ranks checking them case by case.

4.1. Denominator 4 *mutation classes*

For every skew-symmetrisable integer matrix B, one can construct a skew-symmetrisation B' of it by putting $b'_{ij} = \operatorname{sgn} b_{ij} \sqrt{-b_{ij}b_{ji}}$. Matrix B' gives rise to a (possibly non-integer) quiver Q' whose mutations agree with mutations of the diagram of B (see [15]). Notice that not every non-integer denominator 4 quiver corresponds to a diagram of an integer skew-symmetrisable matrix: to have a corresponding skew-symmetrisable matrix, the number of arrows of weight $\sqrt{2}$ in every (not obligatory oriented) cycle must be even (cf. [17, Exercise 2.1]). However, it is easy to check that any chordless cycle with odd number of arrows of weight $\sqrt{2}$ is mutation-infinite, and thus, we can conclude that the finite mutation classes of denominator 4 quivers are the same as the ones described in [7].

Remark 4.1. Denominator 3 and 2 quivers are actually integer, so we do not need to consider them.

Corollary 4.2. *Any mutation-finite quiver with highest denominator 4 is mutation-equivalent to a symmetrisation of one of the integer diagrams, that is either it arises from a triangulated orbifold or is one of the exceptional quivers listed in Figure 4.1 (we call these F-type quivers).*

Remark 4.3. Notice that the diagrams $G_2^{(*,+)}$ and $G_2^{(*,*)}$ from classification in [7] correspond to denominator 6 quivers and arise as elements of series shown in Figure 3.1 for $n = 3$.

4.2. Denominator 5: *Separating* 4 *from* 5

Proposition 4.4. *Let Q be a quiver of finite mutation type with the highest denominator $d = 5$ in the mutation class. Then no quiver in the mutation class of Q contains a denominator 4 arrow.*

Proof. If some quiver in the mutation class of Q does not contain arrows with denominators 4, then the whole mutation class has no such arrows: this is immediate from the mutation rule (as $\sqrt{2} \notin \mathbb{Q}(\sqrt{5})$). Therefore, we can assume that every quiver in the mutation class of Q contains both denominator 5 and denominator 4 arrows. Without loss of generality, we can also assume that Q is of smallest possible rank with this property. Let n be the rank of Q. In view of classification of mutation-finite rank 3 quivers, we see that $n \geq 4$. Suppose that $Q_{n,n-1}$ is a denominator 5 arrow. By the minimality of Q, none of the arrows $Q_{i,n}$ has denominator 4. This implies that a denominator 4 arrow is contained in $Q_{1,\dots,n-2}$. Without loss of generality, we can assume that the arrow Q_{12} has denominator 4.

Consider the shortest path $\mathcal{P}$ connecting (one of the endpoints of) Q_{12} to (one of the endpoints of) $Q_{n-1,n}$, we can assume that $\mathcal{P}$ connects Q_2 to Q_{n-1}. Since Q is minimal and $\mathcal{P}$ is shortest, the support of
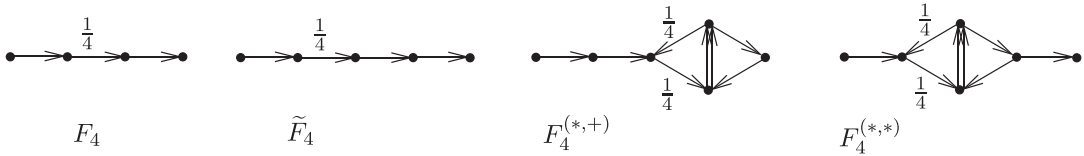
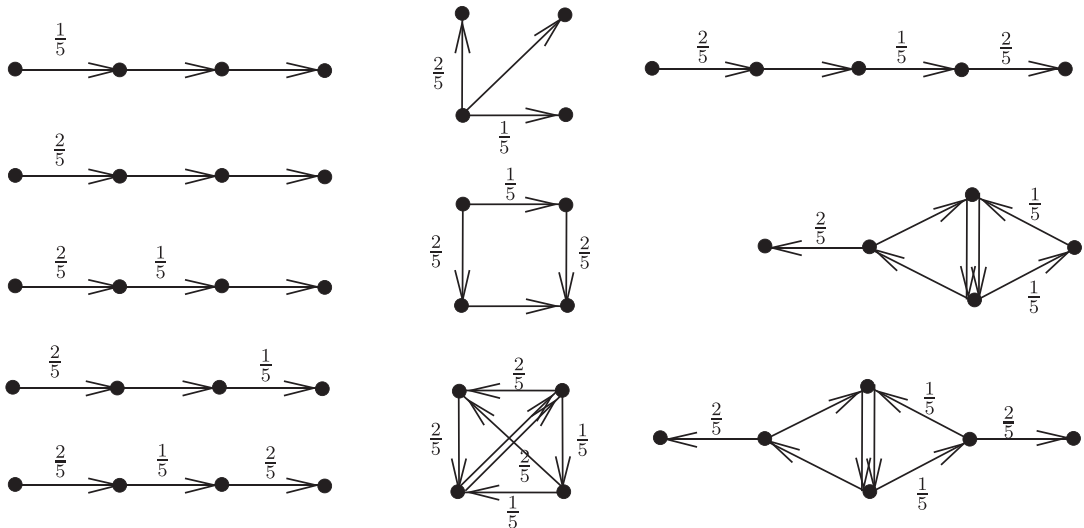**Figure 4.1.** *Exceptional denominator 4 quivers.*

Figure 4.2. *Denominator 5 quivers of finite mutation type.*

$\mathcal{P}$ coincides with $Q_{2,\dots,n-1}$ and is a linear graph containing all vertices of Q except for Q_1 and Q_n. Thus, we can assume that the subquiver $Q_{2,\dots,n-1}$ only contains arrows $Q_{i,i+1}$, and each of these arrows is of weight 1 or 2. Furthermore, besides the arrows in $Q_{2,\dots,n-1}$, denominator 4 arrow Q_{12} and denominator 5 arrow $Q_{n,n+1}$, the quiver Q may only contain two other arrows: Q_{13} and $Q_{n-2,n}$.

Notice that Q_{13} cannot have denominator 4, as this would contradict the minimality of Q. Also, Q_{13} cannot have weight 1 or 2, as in that case, the subquiver Q_{123} would not be mutation-finite. Thus, there is no arrow between vertices Q_1 and Q_3. If Q_{23} has weight 2, then Q_{123} is already mutation-infinite, so we can assume that Q_{23} has weight 1. By applying (if needed) mutation μ_1, we can assume that Q_2 is neither a sink nor a source, so by applying mutation μ_2, we will create a denominator 4 arrow Q'_{13} (and this will not affect the rest of the quiver). The subquiver spanned by all vertices but Q_2 will now contain both denominator 4 and denominator 5 arrows, which contradicts the minimality of Q. □

4.3. Denominator 5 mutation classes

In this section, we classify denominator 5 mutation classes (i.e. low denominator quivers containing arrows marked $\frac{1}{5}$ or $\frac{2}{5}$). In view of Proposition 4.4, such a quiver only contains arrows marked $\frac{1}{2}$ (such arrows are absent), $\frac{1}{3}$ (simple arrows), $\frac{1}{5}$, $\frac{2}{5}$ and 2 (double arrows).

The classification can be now achieved by a short (computer assisted) case-by-case study which we organise as follows.

All rank 3 mutation-finite classes are listed in Theorem 2.2 (there are only three mutation classes containing arrows of denominator 5). The fourth vertex may be added to a representative of each of these three mutation classes in 9^3 ways. Most of the obtained quivers are mutation-infinite, so this will produce eight mutation-finite classes listed in the left and middle columns of Figure 4.2. Then one can

add the fifth vertex to get two mutation classes of rank 5. Adding the sixth vertex, we get exactly one mutation class, while adding one more vertex to that one does not give any new mutation-finite quivers.

We can now summarise the results of the computation described above.

Theorem 4.5. *A denominator* 5 *quiver of finite mutation type is mutation-equivalent to one of the quivers shown in Figure 4.2.*

5. Rank 4 quivers with high denominators

In view of Theorems 3.8 and 4.5, we are left to classify mutation-finite quivers of rank 4 with the largest denominator $d > 5$. By Lemma 3.3, every such quiver is mutation-equivalent to one of the quivers shown in Figure 3.1. In other words, we are left to study three infinite series of rank 4 quivers. Below, we show that each mutation class in these three families is mutation-finite.

Note that all three series in Figure 3.1 are infinite (as $n \in \{2, 3, \dots\}$), and computing the mutations classes for relatively small n, one can observe the size of the mutation classes grows with n. We will show by induction that all quivers in each of these mutation classes satisfy certain conditions, which will imply mutation-finiteness as the conditions describe a finite set of quivers for every given n. The three types of quivers shown in Figure 3.1 will be treated separately (but in a very similar way).

Lemma 5.1. *The quiver shown on the right of Figure 3.1 is mutation-finite for every $n \in \{2, 3, \dots\}$.*

Proof. We will show by induction (on the number of mutations applied) that every quiver in the mutation class can be presented in a standard form shown in Figure 5.1 with some parameters $k, q, m, s \in \{0, 1, 2, \dots n\}$ satisfying the following conditions:

(1) $k + q \in \{n, n + 1\}$;
(2) $k > \frac{n}{2} \geq q$;
(3) $s + m + k + q = 2n + 1$;
(4) $q \leq s, m \leq n - q$ and $0 < s, m$.

The mutation-finiteness then follows immediately.

Base of induction. Reordering the vertices, one can redraw the quiver shown on the right of Figure 3.1 as in Figure 5.2. In this case, $q = 0$, $k = m = n$ and $s = 1$, which clearly satisfies conditions (1)–(4).

Step of induction. Our aim is now to show that the class of quivers described in Figure 5.1 with the conditions (1)–(4) is closed under mutations. *A priori*, we need to check four mutations for that (one mutation in each of the four directions). However, taking into account the symmetry of the conditions above and considering the quivers up to the opposite allow us to reduce the work to checking the two mutations in the two vertices Q_1 and Q_2 (see Figure 5.1). Indeed, taking Q to Q^{op} and renumbering vertices according to permutation $(14)(23)$ result in the same quiver Q with the label m swapped with s (and $m + q$ swapped with $s + q$). Now observe that taking the opposite commutes with mutations, and
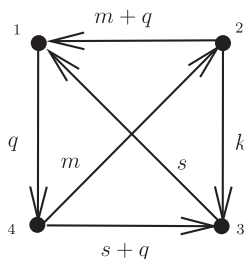

Figure 5.1. *Standard form for quivers in the mutation classes shown in Figure 3.1. We label the arrow of weight* $2 \cos \frac{k\pi}{d}$ *by k (with $d = 2n + 1$ or $d = 2n$ for all arrows).*
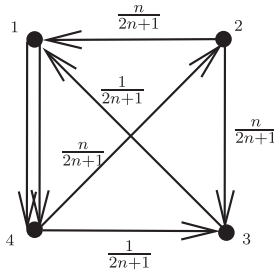
Figure 5.2. *Base of induction: The quiver from the right of Figure 3.1 in the standard form.*
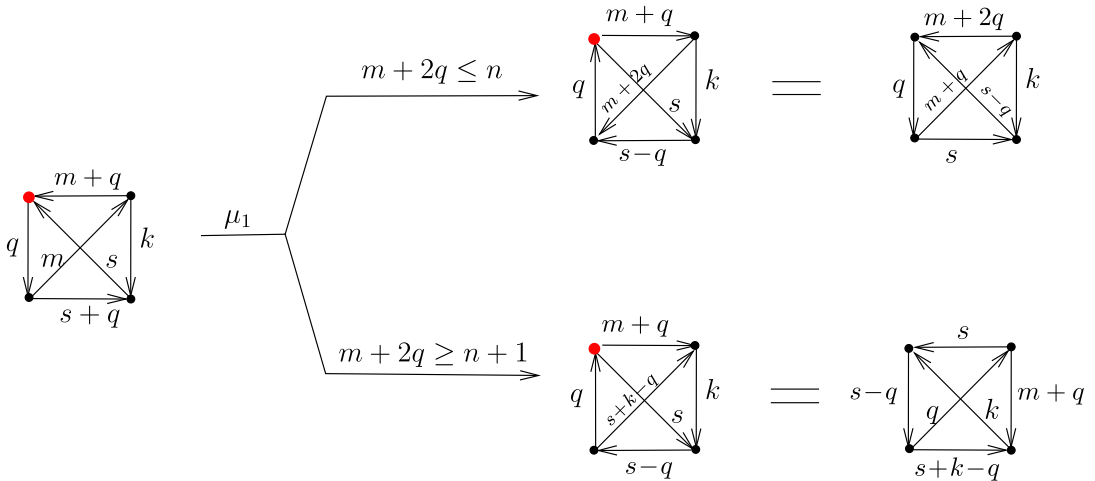

Figure 5.3. *First mutation.*

Q^{op} satisfies the conditions (1)–(4) if and only if Q does. Therefore, checking the mutation in, say, Q_3 is equivalent to checking the mutation in Q_1.

1. Mutation in Q_1. We will first check the mutation $\mu_1(Q)$. Depending on various values of k, q, m, s and n, the quiver obtained is of one of the two forms shown in Figure 5.3 (in the figure, we first show the mutation and then redraw the same quiver in the standard form). In computing the new weights of arrows, we apply parts (4) and (5) of Corollary 2.3 and use the assumption $s + m + k + q = 2n + 1$. Notice also that we obtain a weight $s - q$ (and not $q - s$) as $s \geq q$ in view of assumption (4).

Case 1a: $m + 2q \leq n$. As follows from Figure 5.3, the result of this mutation is still a quiver having the standard form shown in Figure 5.1 with the new values of labels

$$k' = k, \qquad q' = q, \qquad s' = s - q, \qquad m' = m + q.$$

We now need to check properties (1)–(4) for k', q', s', m' (using the ones for k, q, s, m). We denote by (1)', (2)' etc the corresponding conditions for the mutated quiver.

The properties (1)'–(3)' for k', q', s', m' follow immediately from the ones for k, q, s, m.

Now, we need to check (4)'. First, $s', m' > 0$ (otherwise, $s = q$, so $m + 2q = m + s + q = 2n + 1 - k > n$ which contradicts the assumption of the Case 1a). Next, we rewrite (4)' for k', q', s', m' in terms of the old values:

$$q \leq s - q, \; m + q \leq n - q$$

and prove these four inequalities.

It is clear that $q \leq m + q$ and $s - q \leq n - q$. The inequality $m + q \leq n - q$ also holds as $m + 2q \leq n$ by the assumption of Case 1a. Finally, to prove $q \leq s - q$, assume the contrary, that is $s - q < q$, and hence, $s < 2q$. This implies $s + m < 2q + m \leq n$ (again, by the assumption of Case 1a), that is $s + m < n$. However, (1) and (3) imply that $s + m = 2n + 1 - (k + q) = n$ or $n + 1$, so we come to a contradiction.

Case 1b: $m + 2q \geq n + 1$. The new values of the labels are

$$k' = m + q, \qquad q' = s - q, \qquad s' = k, \qquad m' = q.$$

Now, we verify properties (1)'–(4)' for k', q', s', m':

(1)': $k' + q' = m + s = 2n + 1 - (k + q)$, and hence is equal to either n or $n + 1$.
(2)': We need to check that $m + q > \frac{n}{2} \geq s - q$. The first of these inequalities follows from

$$m + q = m + 2q - q \geq n + 1 - q \geq n + 1 - \frac{n}{2} = \frac{n}{2} + 1,$$

while the second one follows from

$$s - q = (s + m) - (m + q) \leq n + 1 - (\frac{n}{2} + 1) = \frac{n}{2}.$$

(3)': $s' + m' + q' + k' = m + s + q + k = 2n + 1$.
(4)': First, $s', m' > 0$ as $q > 0$ in view of the assumption $m + 2q \geq n + 1$ and property (4) for k, q, s, m. Next, we check that

$$s - q \leq k, q \leq n - (s - q)$$

as follows:
(α) As shown in the proof of (2)', $s - q \leq \frac{n}{2}$. Thus, $s - q \leq \frac{n}{2} \leq k$.
(β) If $s - q > q$, then $s > 2q$, which implies $m + s > m + 2q \geq n + 1$ in contradiction to (3). Hence, $s - q \leq q$.
(γ) To show $k \leq n - (s - q)$, consider two cases: $k = n - q$ and $k = n + 1 - q$ (one of them holds by (1)).
    If $k = n - q$, then $k + 2q = n + q$, and hence, $k + s \leq k + 2q = n + q$ (as $s \leq 2q$ in view of part (β) above), which implies $k \leq n - (s - q)$.
    If $k = n + 1 - q$, then $s < 2q$ (otherwise, $m + s \geq m + 2q \geq n + 1$ by the assumption of Case 1b, this would imply $k + q = n$ in contradiction to the assumption $k = n + 1 - q$)). Therefore, $k + s < k + 2q = n + 1 + q$. This means $k < n - (s - q) + 1$, and thus, $k \leq n - (s - q)$ as required.
(δ) The inequality $q \leq n - (s - q)$ follows from (γ) and $q \leq k$.

2. Mutation in Q_2. Now, we need to check the mutation $\mu_2(Q)$. We follow the same scheme as before: consider two cases as shown in Figure 5.4 (again, we apply Corollary 2.3 and the assumption that $s + m + k + q = 2n + 1$ to compute the new weights of some arrows). Notice also that we obtain a weight $k - m$ (rather than $m - k$), as otherwise, $m > k$ would by (1) imply $m + q > k + q \geq n$, and hence, $m > n - q$ which contradicts (4).

Case 2a: $2m + q \leq n$. The new weights of arrows in the standard form are

$$k' = s, \qquad q' = m, \qquad s' = m + q, \qquad m' = k - m.$$

The conditions (1)'–(4)' are verified as follows:

(1)' $k' + q' = s + m = n + 1 - (k + q)$ equals either n or $n + 1$, as it should.
(2)' We need to show $s > \frac{n}{2} \geq m$.
    We start with the latter by noticing that the assumption of Case 2a implies $2m \leq n - q$, and hence, $2m \leq n$, that is $m \leq \frac{n}{2}$.
    Now, $s + m \geq n$ (see the proof of (1)'), so we see that $s \geq n - m \geq \frac{n}{2}$ (where the second inequality makes use of $m \leq \frac{n}{2}$ shown above). If $s > \frac{n}{2}$, we are done, otherwise, $s = \frac{n}{2}$ which
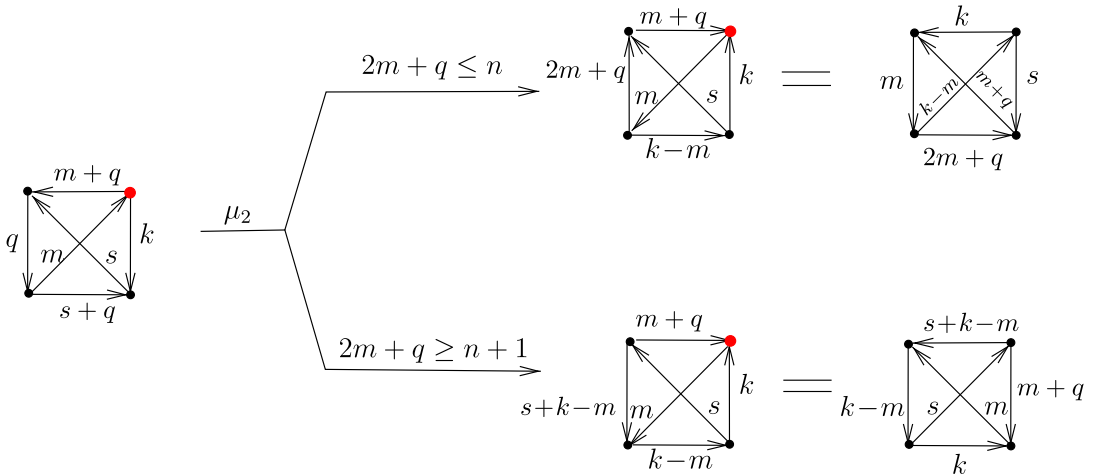
Figure 5.4. *Second mutation.*

implies $m = \frac{n}{2}$ and $q = 0$ (as $2m \leq n - q$). Hence, $k = n$, which by (3) means $s + m = n + 1$ in contradiction to $s + m = \frac{n}{2} + \frac{n}{2} = n$.

(3)' $s' + m' = k + q$ equals either n or $n + 1$ by (1).

(4)' The conditions $s', m' > 0$ hold as $s' = m + q \geq m > 0$ by (4) and $m' = k - m > 0$ as $k > n/2$ by (1) and $m \leq n/2$ by (2)'. Another set of inequalities rewrites as

$$m \leq m + q, k - m \leq n - m$$

and can be proved as follows:

(α) Clearly, $m \leq m + q$.

(β) By the assumption of Case 2a, $2m + q \leq n$, hence, $m + q \leq n - m$.

(γ) $k - m \leq n - m$ as $k \leq n$.

(δ) We need to show $m \leq k - m$ which is equivalent to $2m \leq k$. Suppose the contrary, that is $k < 2m$, then by applying (1) and the assumption of Case 2a, we have $n \leq k + q < 2m + q \leq n$, which is impossible.

Case 2b: $2m + q \geq n + 1$. The new weights of arrows in the standard form are

$$k' = m + q, \qquad q' = k - m, \qquad s' = m, \qquad m' = s.$$

The computations in this case are a bit more involved than before:

(1)' $k' + q' = k + q$, hence, it is still equal to either n or $n + 1$.

(2)' We need to show $m + q > \frac{n}{2} \geq k - m$.

The first of these inequalities is obtained (using the assumption of Case 2b) as follows:

$$m + q = \frac{2m + 2q}{2} \geq \frac{2m + q}{2} \geq \frac{n + 1}{2} > \frac{n}{2}.$$

To prove the second inequality, we apply the assumption of Case 2b again and compute

$$k - m \leq k - \frac{n + 1 - q}{2} = k + \frac{q}{2} - \frac{n + 1}{2} =$$

$$= k + q - \frac{n + 1}{2} - \frac{q}{2} \leq n + 1 - \frac{n + 1}{2} - \frac{q}{2} = \frac{n}{2} + \frac{1}{2} - \frac{q}{2},$$

which gives the required inequality if $q > 0$.

Table 5.1. *Conditions for three families.*

	denominator=$2n$		denominator= $2n+1$
(1)	$k+q \in \{n-1, n+1\}$	$k+q=n$	$k+q \in \{n, n+1\}$
(2)	$k \geq [\frac{n}{2}] \geq q$	$k \geq [\frac{n}{2}] \geq q$	$k > \frac{n}{2} \geq q$
(3)	$s+m+k+q=2n$	$s+m+k+q=2n$	$s+m+k+q=2n+1$
(4)	$q \leq s, m \leq n-q$	$q \leq s, m \leq n-q$	$q \leq s, m \leq n-q$
	$0 < s, m$	$0 < s, m$	$0 < s, m$

If $q = 0$, then by (1) we have $k = n$. Also, the assumption of Case 2b then reads as $m \geq \frac{n+1}{2}$. Therefore,

$$k - m \leq n - \frac{n+1}{2} = \frac{n-1}{2} < \frac{n}{2},$$

as required.

(3)' $(m + q) + (k - m) + m + s = k + q + m + s = 2n + 1$, as required.

(4)' The inequalities $s', m' > 0$ hold as $s' = m > 0$, $m' = s > 0$. The inequalities

$$k - m \leq m, s \leq n - k + m$$

can be checked as follows:

(α) $m \leq n - k + m$ as $k \leq n$.

(β) $k - m \leq m$, as otherwise, $k > 2m$, which (together with the assumption of Case 2b) implies $k + q > 2m + q \geq n + 1$, which contradicts (1).

(γ) $k - m \leq s$, as otherwise, $s < k - m$, implying $s + m < k$; by (1) and (3), this means $n \leq s + m < k$, that is $n < k$, which is impossible.

(δ) To show $s \leq n - k + m$, assume the contrary, that is $k > m + n - s$, then

$$k + q > m + n + q - s = (2m + q) - m + n - s \geq n + 1 - (m + s) + n,$$

which implies that $k + q + m + s > 2n + 1$ in contradiction to (3).

As neither of the two mutations takes the quiver away from the (finite) set of quivers having the standard form described by Figure 5.1 and conditions (1)–(4), we conclude that the mutation class is finite. □

Using very similar computations, we prove the following lemma.

Lemma 5.2. *The quivers shown on the left and in the middle of Figure 3.1 are mutation-finite for every $n \in \{2, 3, \ldots\}$.*

To prove this lemma, we use exactly the same standard form of the quivers (see Figure 5.1) together with a marginal variation of the set of conditions (see Table 5.1). These variations (as well as different shapes of quivers) are due to different parity of the denominator: there is one mutation class for every $n \geq 2$ with odd denominator $2n + 1$, while in the case of the even denominator $2n$, the set of quivers splits into two mutation classes for every n (see also Proposition 5.3).

Proposition 5.3. *If Q is a quiver in the standard form (as in Figure 5.1) satisfying the conditions as in Table 5.1, then Q is mutation-finite. Moreover, two such quivers belong to the same mutation class if and only if they have the same denominator and satisfy the same set of conditions.*

Proof. From the proof of Lemmas 5.1 and 5.2, we see that by applying mutations to any quiver Q represented in the standard form and satisfying one of the three versions of the conditions, we always obtain quivers of the same family. As each family is finite for any given n, this shows mutation-finiteness of Q.

We are left to discuss which quivers belong to the same class. It is clear that quivers with different denominators (or with the same even denominator but different sets of conditions) belong to different mutation classes. On the other hand, by Theorem 3.3, every mutation-finite high denominator quiver is mutation-equivalent to one of the quivers in Figure 3.1. So, quivers with the same invariants (i.e. the same denominator and the same set of conditions) are mutation-equivalent, while quivers with different invariants are not. □

This concludes the proof of Theorem A.

6. Geometric realisation for finite mutation classes

In this section, we will show that every non-integer mutation-finite mutation class (except, possibly, for ones of the orbifold type) admits a geometric realisation by reflections in some positive semidefinite quadratic space V. This will allow us to define the finite, affine and extended affine types of quivers.

6.1. Definitions and results

First, we recall the necessary details from [9, 10].

Definition 6.1. Let B be an $n \times n$ skew-symmetric matrix corresponding to a quiver Q, and let V be a real quadratic space. We say that a tuple of vectors $\mathbf{v} = (v_1, \ldots, v_n)$ is a *geometric realisation* of Q if the following conditions hold:

(1) $(v_i, v_i) = 2$ for $i = 1, \ldots, n$, $|(v_i, v_j)| = |b_{ij}|$ for $1 \leq i < j \leq n$;
(2) if Q_{i_1, i_2, i_3} is a cycle, then the number of pairs (j, k), such that $(v_{i_j}, v_{i_k}) > 0$ is even if Q_{i_1, i_2, i_3} is acyclic and odd if Q_{i_1, i_2, i_3} is cyclic.

A *mutation* μ_k of $\mathbf{v}$ is defined by partial reflection:

$$\mu_k(v_j) = \begin{cases} v_j - (v_j, v_k)v_k & \text{if } b_{jk} > 0, \\ -v_k & \text{if } j = k, \\ v_j & \text{otherwise.} \end{cases}$$

We say that $\mathbf{v}$ provides a *realisation by reflections* of the mutation class of Q if the mutations of $\mathbf{v}$ agree with the mutations of Q, that is if conditions (1)–(2) are satisfied after every sequence of mutations.

We recall that every acyclic integer quiver admits a realisation by reflections [22, 23]. Following [21], we give the following definition.

Definition 6.2. A geometric realisation of a quiver Q by vectors $\mathbf{v} = \{v_1, \ldots, v_n\}$ is *admissible* if for every chordless oriented cycle $Q_{i_1}, \ldots, Q_{i_s}$ of Q, the number of positive scalar products $(v_{i_j}, v_{i_{j+1}})$ is odd, while in every chordless nonoriented cycle, such a number is even. A geometric realisation of a mutation class is admissible if the realisation of every quiver is admissible.

We will start by showing that every non-orbifold finite mutation class of non-integer quivers has a representative possessing an admissible geometric realisation.

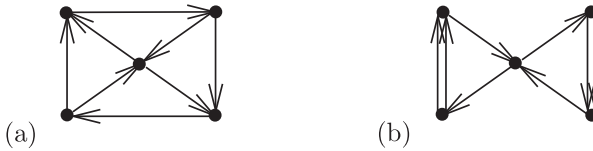**Lemma 6.3.** *Every quiver shown in Table 1.1 has an admissible geometric realisation.*

Figure 6.1. *The surface quiver (a) has no admissible geometric realisation.*

Proof. Every quiver listed in Table 1.1 is of one of the following three types:

- either the rank is 3 (and then it is $\widetilde{G}_{2,n}$ or H_3);
- or it is acyclic;
- or it has a double arrow, and by removing either end of the double arrow, we obtain an acyclic quiver (the two acyclic quivers are the same up to one source/sink mutation).

Quiver $\widetilde{G}_{2,n}$ is mutation-acyclic of rank 3, and thus has an admissible realisation by [9].

For an acyclic quiver Q, we define inner product on vectors $v_1, \ldots, v_n$ by $(v_i, v_i) = 2$, $(v_i, v_j) = -w_{ij}$, where w_{ij} is the weight of the arrow Q_{ij}. Clearly, $\mathbf{v} = \{v_1, \ldots, v_n\}$ is an admissible realisation of Q.

For the last type of quivers, assume that the ends of the double arrow are Q_1 and Q_n. We take the acyclic subquiver Q' obtained by removing Q_n, define inner product on vectors $v_1, \ldots, v_{n-1}$ for the acyclic subquiver Q' as described above and then define $(v_n, v_n) = 2$, $(v_n, v_i) = (v_1, v_i)$ for $i < n$. Then $\mathbf{v} = \{v_1, \ldots, v_n\}$ is an admissible realisation of Q.   □

Remark 6.4. The condition that Q is not of orbifold type is necessary for Lemma 6.3: it is easy to check that already the surface quiver shown in Figure 6.1(a) has no admissible geometric realisation. This quiver defines a triangulation of a once punctured annulus, another representative of the same mutation class is shown in Figure 6.1(b).

Theorem 6.5. *Let Q be a real mutation-finite quiver of rank higher than 3 not originating from an orbifold. Then the mutation class of Q admits a geometric realisation by reflections in a positive or semipositive quadratic space V. In particular, the quadratic form has the kernel of dimension*

- *zero for quivers in the top row of Table 1.1;*
- *one for quivers in the middle row of Table 1.1;*
- *two for quivers in the bottom row of Table 1.1.*

Proof. In Lemma 6.3, we have constructed geometric realisations for representatives of required mutation classes, so we only need to show that these geometric realisations can be extended to the whole of these mutation classes. For rank 3 mutation classes, we know the result from [9]. For the three series in rank 4, this will be done in Section 6.2. Other mutation classes are treated case-by-case.

The case-by-case check is done via a code which verifies that the realisation of the initial quiver propagates as a realisation of the whole mutation class. The algorithm is the following: we apply a mutation to a quiver and the partial reflections to the corresponding set of vectors (i.e. mutate the Gram matrix according to the rules prescribed in [2]), and then verify that the mutated Gram matrix provides an admissible realisation of the mutated quiver. Notice that the mutated Gram matrix only depends on the initial Gram matrix and the directions of arrows in the corresponding quiver before the mutation but does not depend on the actual vectors $v_1, \ldots, v_n$. The code checks that in each of the (nonserial) finite mutation classes, the pair (Gram matrix; exchange matrix) takes only finitely many values and the entries in the Gram matrix and the exchange matrix agree, that is $|(v_i, v_j)| = |b_{ij}|$ for $i \neq j$.

The dimension of the kernel can be easily seen from the initial construction in Lemma 6.3.   □

Remark 6.6. It follows from Remark 6.4 that already the mutation class of a quiver originating from a punctured annulus (see Figure 6.1) does not have an admissible realisation by reflections. This implies that most non-acyclic mutation classes of punctured surfaces or orbifolds do not possess admissible realisations by reflections.
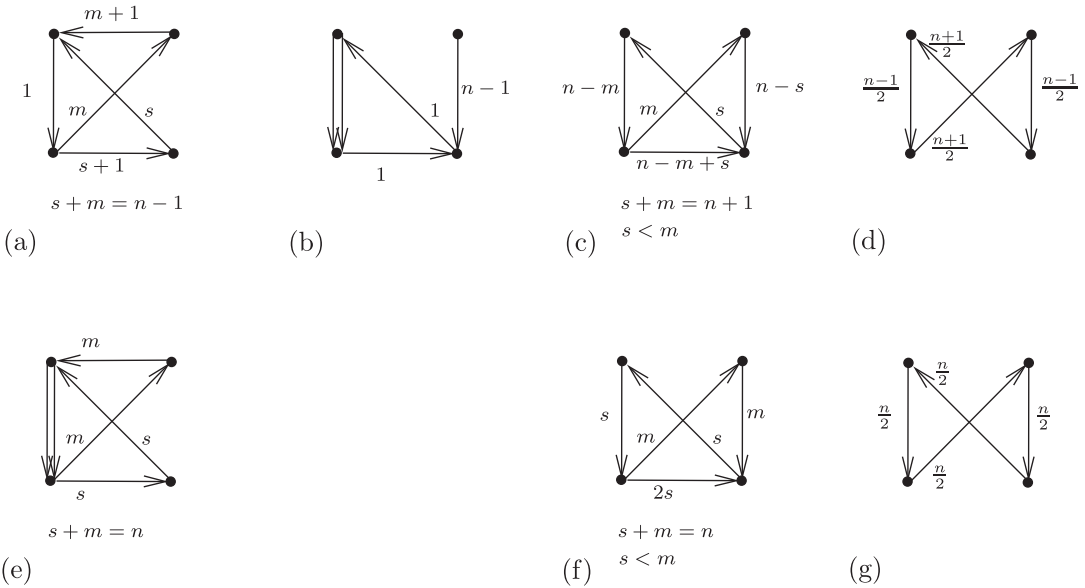
Figure 6.2. *Quivers with vanishing arrows: (a)–(d) and (e)–(g) belong to two different mutation classes, respectively (cf. Table 5.1).*

Geometric realisations by reflections of all mutation classes of quivers originating from unpunctured surfaces were constructed in [5]. There is strong evidence for the following conjecture.

Conjecture 6.7. *Every mutation class originating from an unpunctured orbifold admits a realisation by reflections.*

6.2. *Geometric realisations for rank* 4 *series*

In rank 4, we have infinitely many finite mutation classes whose sizes are not uniformly bounded, so we are not able to apply a computer verification. We start with proving the following technical lemma.

Lemma 6.8. *Let Q be a quiver mutation-equivalent to one of the quivers in Figure 3.1. If Q has a vanishing arrow, then Q is one of the quivers shown in Figure 6.2.*

Proof. Q can have a vanishing arrow in the only case when the highest denominator of Q is even, and hence, Q is mutation-equivalent to the quiver on the left or in the middle of Figure 3.1. For each of these quivers (considered in the standard form), we check which arrows can vanish (we use the conditions shown in Table 5.1 for that; an arrow marked $x/2n$ vanishes if and only if $x = n$). In particular, condition (2) implies that $q \neq n$.

Further, if $k = n$, then the conditions imply that no other arrow vanishes, and moreover, the quiver is as on Figure 6.2(a) or (e). If $k \neq n$, we check the case $m = n$ and get the quiver on Figure 6.2(b) (there are no such quivers in the other mutations class). The same (up to symmetry) will happen if $s = n$. Finally, if $k, m, s \neq n$ and $m + q = n$, we obtain the quivers in Figure 6.2(c) and (f). If, in addition, we require $s + q = n$, we get the quivers shown in Figure 6.2(d) and (g). □

Lemma 6.9. *Let Q be a quiver in its standard form (see Figure 5.1) and $\mathbf{v} = \{v_1, \ldots, v_4\} \subset V$ its admissible geometric realisation. Then for every $i \in \{1, \ldots, 4\}$, the collections of vectors $\mu_i(\mathbf{v})$ provide geometric realisations of $\mu_i(Q)$.*

Proof. The proof is by induction on the number of mutations needed to reach a given quiver Q from the initial quiver $\hat{Q}$ shown in Figure 3.1. We start with a quiver shown in Figure 3.1 and consider its admissible geometric realisation $\hat{\mathbf{v}}$ constructed in Lemma 6.3. Given a quiver Q in the mutation class
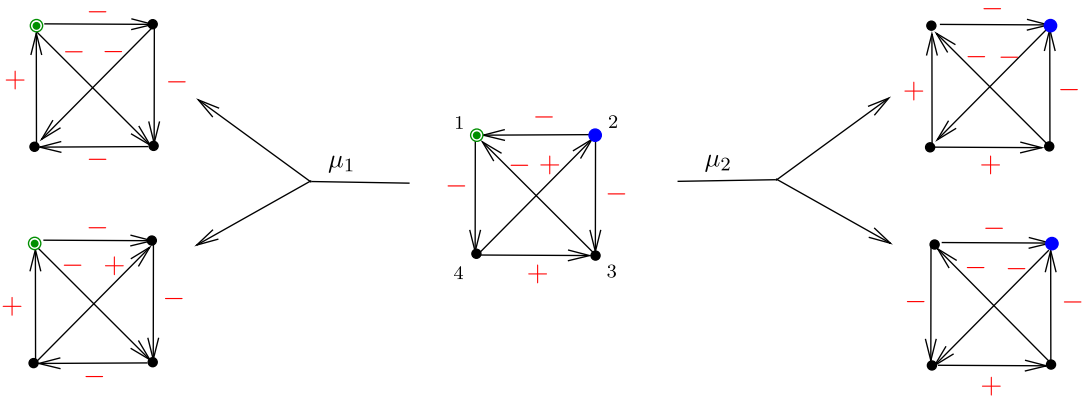
Figure 6.3. *Mutating signs for odd denominators. For each mutation, we consider two cases (depending on the weights of arrows, see Figures 5.3 and 5.4).*

and its admissible realisation **v**, we will apply all four possible mutations (mutating the set of vectors using partial reflections) and check that the mutated set of vectors $\mathbf{v}' = \mu(\mathbf{v})$ provides an admissible geometric realisation for the mutated quiver $Q' = \mu(Q)$ (note that, as in Lemma 5.1, we actually need to check only two mutations, the other two follow from a symmetry of the quiver provided by taking Q^{op} with a permutation of vertices). Since **v** is an admissible realisation for Q, we conclude that $\mathbf{v}'$ is a geometric realisation for Q' (see [2, Proposition 3.2]), and we only need to show that $\mathbf{v}'$ is an *admissible* geometric realisation for Q'.

We start checking the admissibility of $\mathbf{v}'$ by considering the case of odd denominator: this will be the easiest case, as no quiver in the mutation class has vanishing arrows.

Case 1: Odd denominators. We label an arrow Q_{ij} of Q by '+' (respectively, '−') if (v_i, v_j) is positive (respectively, negative).

By applying a mutation μ_i, we compute the new sign labels as follows. First, we compute the new labels of all arrows Q'_{ij} incident to Q_i: these labels easily follow from the mutation rule (which says that either both vectors are reflected in v_i and, hence, the sign is preserved, or only v_i is substituted by its negative, and then the sign changes to the opposite). The label for an arrow Q'_{lj} nonincident to i is computed from the triangle Q'_{ijl}: namely, the number of arrows labeled by '+' in Q'_{ijl} should be even if and only if Q'_{ijl} is cyclic by [9]. When all labels are computed, we check the rank 3 subquiver $Q' \setminus Q'_i$ and see that the labeling is also admissible on this rank 3 subquiver (see Figure 6.3). This implies that $\mathbf{v}'$ is an admissible realisation for Q' (indeed, in the assumption of odd denominator, we only need to check cycles of length 3). Notice that as all quivers in the mutation class are ones in the standard form and no arrow vanishes from it, mutations considered in Figure 6.3 exhaust all possibilities for the case of odd denominator (here, we use the two possible forms of mutated quiver explored in the proof of Lemma 5.1, see Figures 5.3 and 5.4).

Case 2: Even denominators. We follow exactly the same plan as for odd denominator, however, we need to consider the quivers with some vanishing arrows separately (as in this case, we need to take additional care while mutating the sign labels).

In Lemma 6.8, we list the quivers with vanishing arrows appearing in the considered mutation classes. Given a quiver Q from one of the two series, we need to do the following:

(1) if Q has no vanishing arrows and **v** is an admissible geometric realisation of Q, then we need to check the admissibility of the realisation $\mu(\mathbf{v})$;
(2) if Q has vanishing arrows, **v** is an admissible geometric realisation of Q and $Q' = \mu(Q)$, then we need to check whether $\mathbf{v}' = \mu(\mathbf{v})$ is a geometric realisation of Q', and whether it is admissible.
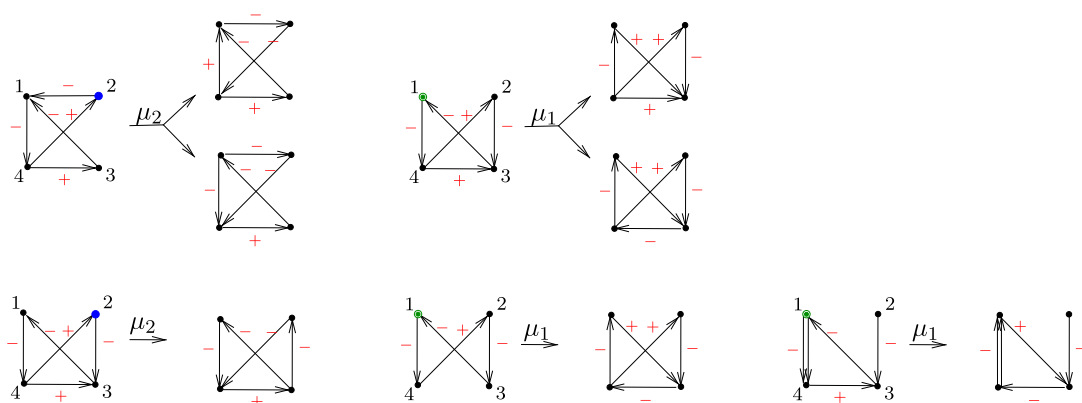
Figure 6.4. *Mutating signs: Additional mutations for even denominators.*

In the first of these checks, the condition for cycles of length 3 is verified by the same computation as before, however, we need to check also cycles of length 4 (as $\mu(Q)$ may have vanishing arrows). As we can see from the list in Figure 6.2, a length 4 chordless cycle is always oriented (for quivers in these mutation classes). As one can check in Figure 6.3, an oriented cycle of length 4 always receives an odd number of labels '+', even when this cycle is not a chordless one. This verifies the condition for cycles of length 4, and hence, we can assume that Q has at least one vanishing arrow.

To complete the second check above, we need to mutate the quiver. As before, we label the arrows of Q with '+' and '−' in an admissible way (which exists by the induction assumption). Note that for each of the quivers in the list, there is a unique way to choose such a labeling (up to changing signs of some of vectors $v_1, \ldots, v_4 \in \mathbf{v}$). We will only need to look at the mutations in the directions of vertices incident to some vanishing arrows (all other mutations are treated in exactly the same way as before). Also, we do not need to check the mutations with respect to sink or source, as they do not change signs in any oriented cycle and change exactly two signs in a nonoriented one. Furthermore, we will use symmetries of quivers (and the symmetry up to taking Q^{op}) to reduce the computations. This reduces the list of cases to the one in Figure 6.4.

To compute the signs after mutation μ_i, we do the following. First, we compute the signs of all arrows incident to Q_i as before. Then, we compute the signs of all arrows Q'_{jl}, where both Q_j and Q_l are connected to Q_i by a nonvanishing arrow. All the other signs remain intact (indeed, if both Q_j and Q_l are not joined to Q_i then $v_j = v'_j$ and $v_l = v'_l$; if the arrow Q_{ij} vanishes and Q_{il} does not, then $v_j = v'_j$, while v'_l may either coincide with v_i or computes as $v'_l = v_l - (v_i, v_l)v_i$, in both cases, we have $(v'_j, v'_l) = (v_j, v_l)$). The computation shows that $\mathbf{v}'$ is an admissible realisation of Q'.   □
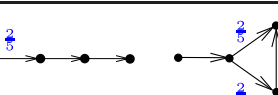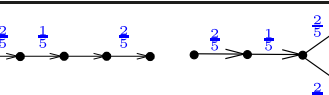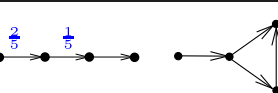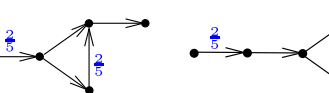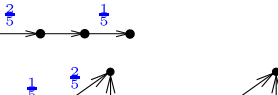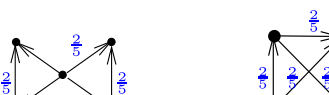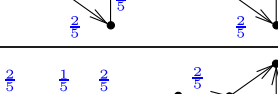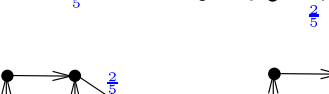
Remark 6.10. Once we have geometric realisation of a mutation class by reflections, we can consider the set of all mirrors of reflections obtained by iterated mutations of the initial tuple of vectors. Then it is easy to see that a quiver is of finite type if and only if the corresponding set of mirrors is finite (and coincides with the hyperplane arrangement associated to a finite Coxeter group).

7. Acyclic quivers and acute-angled simplices

For non-integer quivers of finite or affine type, geometric realisation constructed in Lemma 6.3 defines an acute-angled simplex bounded by the mirrors of reflections of the corresponding finite or affine Coxeter group (B_n, F_4, H_2, H_3, H_4 or G_2^n and affine versions of them). It is natural to ask the following two questions:

(1) Given a mutation-finite acyclic quiver, does it always correspond to an acute-angled simplex defined by the geometric realisation (up to a change of signs of some of the vectors v_i)?

Table 7.1. *Acyclic quivers in finite mutation classes containing more than one acyclic representative (up to sink/source mutations).*

(2) Given an acute-angled simplex defined by some roots of a root system in *V*, does it give rise to a realisation of a mutation-finite quiver?

Notice that in rank 3, the answers to both of these questions are positive (see [9]). Moreover, for integer quivers, this also holds: in finite (respectively, affine) types *A*, *B*, *C*, *D*, *E*, *F* and *G*, every acyclic quiver defines an acute-angled spherical (respectively, Euclidean) Coxeter simplex, and any acyclic orientation of a Coxeter diagram of a Coxeter simplex, gives rise to a mutation-finite quiver.

We will now see that the situation in the general case is more involved.
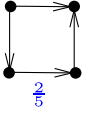
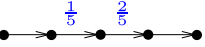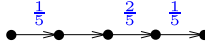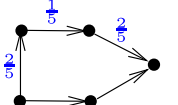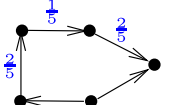### 7.1. Acute-angled simplices for all acyclic representatives
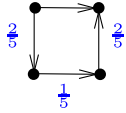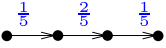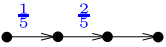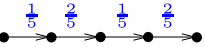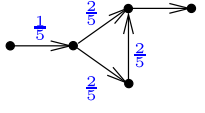
The answer to the first question is positive also in the general non-integer case. We have checked it case-by-case (but we have no conceptual proof at the moment). In Table 7.1, we list all acyclic quivers (up to sink-source mutations) in the mutation classes containing more than one acyclic representative.

7.2. Not all acute-angled simplices give rise to mutation-finite acyclic quivers

It turns out that the answer to the second question is negative.

By a *diagram* of a simplex, we will mean a counterpart of a Coxeter diagram, that is a weighted graph, where vertices correspond to the facets of the simplex and the weights *m*/*d* of the edges denote

Table 7.2. *Mutation-infinite acyclic quivers from acute-angled H-simplices (up to sink/source mutations).*

the dihedral angles $\pi m/d$ (the edges with label $1/2$ are omitted, the edges corresponding to $\pi/3$ are unlabeled). We have written out the complete list of diagrams of acute-angled simplices in root systems $\widetilde{H}_3$, H_4 and $\widetilde{H}_4$ and checked that most of these simplices appear as geometric realisations of some mutation-finite acyclic mutation classes. However, there is a number of exceptions: in Table 7.2, we list all (up to sink/source mutations) mutation-infinite acyclic quivers appearing as orientations of diagrams of acute-angled simplices in $\widetilde{H}_3$, H_4 and $\widetilde{H}_4$.

It is currently not clear to us what distinguishes the acute-angled simplices appearing in Tables 7.2 from ones defining mutation-finite quivers, and we think it would be an interesting question to understand the source of this difference.

Remark 7.1. Finally, we list some observations concerning the acute-angled simplices and the corresponding quivers.

(a) Every acute-angled simplex in H_3, $\widetilde{H}_3$, H_4 or $\widetilde{H}_4$ either is decomposable (i.e. its diagram is disconnected), is a spherical Coxeter simplex of the type H_3 or H_4 or has a diagram whose orientation appears either in Table 7.1 or in Table 7.2 (or in both: two distinct acyclic orientations of the same simplex diagram may not be simultaneously mutation finite/infinite).

(b) Notice that when a diagram of a simplex has a cycle of length more than 3, there are two acyclic orientations of such a diagram up to sink/source mutations. All other diagrams arising from acute-angled simplices have a unique acyclic orientation up to sink/source mutations.

(c) Some of the $\widetilde{H}_4$-quivers in Table 7.2 are mutation-equivalent. In particular, this is the case for the two quivers shown in the first, second and the third rows (see Figure 7.1 for the sequences of mutations).

(f) Acute-angled simplices in finite types are also listed in [4].

(g) The affine extensions of Coxeter groups of type H were described in [19]. In particular, the diagram of the simplex giving rise to the top left $\widetilde{H}_4$ quiver in Table 7.2 was used to define the group $\widetilde{H}_4$. We note that one can start with any of the simplices whose diagrams are listed in the $\widetilde{H}_4$ parts of Tables 7.1 and 7.2 to get the same group.
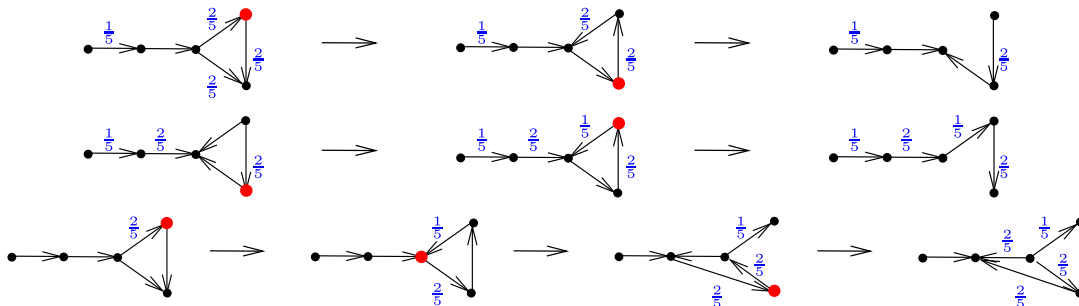
Figure 7.1. *Mutation equivalences between exceptional mutation-infinite quivers.*

Conflict of Interest. The authors have no conflict of interest to declare.

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