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EQUIVALENT PRESENTATIONS OF MODULES OVER PRÜFER DOMAINS

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ABSTRACT. If F and F' are free R-modules, then $M \cong F/H$ and $M \cong F'/H'$ are viewed as equivalent presentations of the R-module M if there is an isomorphism $F \longrightarrow F'$ which carries the submodule H onto H'. We study when presentations of modules of projective dimension 1 over Prüfer domains of finite character are necessarily equivalent.

1. **Introduction.** Let R denote a commutative domain with 1; all R-modules are unital. In what follows, $\operatorname{rk} M$ will denote the rank and $\operatorname{gen} M$ the minimal cardinality of generating systems of the R-module M.

Let F and F' be free R-modules, H and H' submodules such that $F/H \cong F'/H'$. We say that F/H and F'/H' are *equivalent presentations* of the R-module $M \cong F/H$ if there is an isomorphism $\phi: F \longrightarrow F'$ which carries H onto H'.

Needless to say that, in general, there are no compelling reasons for the equivalence of two presentations of a module. Equivalent presentations of torsion-free abelian groups were investigated by J. Erdős [3]; his results were extended to the mixed case by Fuchs [4]. A more relevant study of presentations of abelian groups is due to Hill-Megibben [7]: they succeeded in giving a necessary and sufficient condition for the equivalence of two presentations. One of their numerous corollaries is the stacked bases theorem of Cohen-Gluck [2]. The results of [7] are extended to presentations over arbitrary valuation domains by L. Salce and P. Zanardo [unpublished].

The equivalence of presentations of finitely presented modules was established by Levy [9] and by Brewer-Klingler [1] over Prüfer domains of finite character (finite character means that every non-zero element is contained but in a finite number of maximal ideals) and over Prüfer domains of Krull dimension 1. Note that in the Prüfer case finite presentation is equivalent to finite generation plus having projective dimension ≤ 1. Accordingly, in the infinitely generated case, it is natural to concentrate on modules of projective dimension ≤ 1. It turns out that then the problem is still manageable, at least for torsion-free modules, though it is far from being a trivial generalization of the abelian group case. Let us note right away that over Prüfer domains torsion-freeness and flatness are equivalent.

An obvious necessary condition for the equivalence of the presentations F/H and F'/H' of an R-module M is that the ranks satisfy

$$(*) \qquad \operatorname{rk} F = \operatorname{rk} F' \quad \text{and} \quad \operatorname{rk} H = \operatorname{rk} H'.$$

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Our main purpose here is to show that if M is a flat R-module of projective dimension ≤ 1 (R is a Prüfer domain of finite character), then $(*)$ is a sufficient condition as well; moreover, the equality $\operatorname{rk} H = \operatorname{rk} H'$ alone implies that the presentations F/H and F'/H' are equivalent. (Observe that then $\operatorname{rk} F = \operatorname{rk} F'$ is automatically satisfied because of $\operatorname{rk} F = \operatorname{rk} H + \operatorname{rk} F/H = \operatorname{rk} H + \operatorname{rk} M$.) The main idea of the proof is borrowed from Erdős [3]; however, several essential modifications were needed to settle the problem in our case.

If the condition of M being flat is dropped, then we can establish only a sufficient condition for the equivalence of presentations of M. A main difficulty in obtaining a necessary and sufficient condition in the more general case lies in the fact that for the Hill-Megibben criterion the unique factorization of the integers seems to be a relevant feature. On the other hand, the hypothesis that the projective dimension of M is ≤ 1 is needed in order to assure that H is projective—this property plays an essential role in our considerations.

Our results provide an additional evidence to justify our old claim that the behavior of modules of projective dimensions ≤ 1 over Prüfer domains has a strong resemblance to modules over Dedekind domains (see [5]).

2. **Preliminary lemmas.** For the proof of our main results, we require a couple of preliminary lemmas.

LEMMA 1. *If R is a Prüfer domain and F is a projective R-module, then every finite rank pure submodule H of F is a summand of F.*

PROOF. Without loss of generality we may assume that F is a free R-module and H is contained in a finitely generated free summand F' of F. Then the factor module F'/H is a finitely generated flat R-module, so it is projective. Therefore, H is a summand of F' and hence of F. ∎

LEMMA 2. *A projective module of infinite rank over a Prüfer domain of finite character is free.*

PROOF. This follows at once from Kaplansky [8] and Heitmann-Levy [6]. ∎

The next two results are analogs of lemmas on abelian groups due to Erdős [3].

LEMMA 3. *A projective pure submodule H of a free R-module F over a Prüfer domain R of finite character contains a summand of F whose rank is the same as the rank of H. If H is of infinite rank, then this summand is free.*

PROOF. If H is of finite rank, then by Lemma 1 it is a summand of F, and we are done. So assume H is of infinite rank κ.

Let $B = \{b_\alpha\}$ be a basis of F, and consider finite subsets B_i of B such that $\langle B_i\rangle \cap H \neq 0$. Select a maximal pairwise disjoint set Σ of such subsets B_i, and a nonzero h_i in each $\langle B_i\rangle \cap H$. Let $\langle h_i\rangle_*$ denote the pure submodule generated by h_i, *i.e.*, $\langle h_i\rangle_*/\langle h_i\rangle$ is the torsion submodule of $H/\langle h_i\rangle$. Note that $\langle h_i\rangle_*$ is a summand of $\langle B_i\rangle$, and hence $G =$

$\oplus\langle h_i\rangle_*$ is a (projective) summand of F, and so of H. Write $F = \langle B_i \mid B_i \in \Sigma\rangle \oplus K$ where K is generated by the basis elements not in any member of Σ. Now $K \cap H \neq 0$ is impossible, because then the basis elements b_α occurring in a linear combination of a non-zero element in this intersection form a finite subset disjoint from every finite subset in Σ, contradicting the maximality of Σ. Therefore, $K \cap H = 0$. Manifestly, the cardinality of the set of all basis elements b_α occurring in members of Σ is the same as the cardinality of Σ. Hence $K \cap H = 0$ implies that $\operatorname{rk} G = \operatorname{rk}\langle B_i \mid B_i \in \Sigma\rangle = \operatorname{rk} F/K \geq \operatorname{rk} H = \kappa$. Now G is a projective module of infinite rank, so it is free by Lemma 2. ∎

The crucial lemma is the following.

LEMMA 4. *Let F be a free module of infinite rank over a Prüfer domain R of finite character, and H a projective pure submodule of F. Assume that S is a generating set of F/H whose cardinality is equal to $\operatorname{rk} F$, and T is a subset of F/H disjoint from S satisfying $|T| = |S|$. If $|S| = \operatorname{rk} H$, then F has a basis B which is $\operatorname{mod} H$ a complete set of representatives of $S \cup T$.*

PROOF. Suppose $|S| = \operatorname{rk} F = \operatorname{rk} H = \kappa$. In view of Lemma 3, H contains a free summand G of F with $\operatorname{rk} G = \kappa$. Choose a basis Y of G and extend it to a basis $C = \{b_\alpha\}$ of F. Next, well-order C in such a way that the elements of $Y = C \cap H$ precede the other basis elements in C. Moreover, we may assume that the well-ordering is done in such a way that Y has order type κ.

We are going to change the basis C to get one with the desired property. We use four steps in order to accomplish this goal.

STEP 1. We modify C such that the new basis C' will have the property that it contains Y and two elements of C' are congruent $\operatorname{mod} H$ if and only if both belong to H.

If a basis element b_β in C is in the same coset $\operatorname{mod} H$ as a basis element b_α with $\alpha < \beta$ in the well-ordering, then we replace b_β in the basis C by $b_\beta - b_\gamma$ with the first b_γ congruent to $b_\beta \operatorname{mod} H$.

STEP 2. We pass to a new basis C'' of F which contains κ elements of Y and every element of S is represented by exactly one basis element in C''.

Consider a set $S' = \{s_\rho\}$ $(\rho < \lambda \leq \kappa)$ of representatives of elements of S which have no representatives in the basis C'. If S' is empty, there is nothing to do. If it is not empty, then we proceed as follows. Without loss of generality we may assume that the representatives $s_\rho \in S'$ have been selected such that in their representations as linear combinations of the basis elements in C' no basis element from Y occurs. We split Y into two disjoint subsets: $Y = Y_1 \cup Y_2$ such that $|Y_1| = \kappa$ and there is a bijection $f\colon Y_2 \longrightarrow S'$. Using f, the basis elements $b_\rho \in Y_2$ are replaced by $b_\rho + f(b_\rho)$.

STEP 3. We find a new basis B' with the property that every element of S is represented by exactly one basis element in B', and all the other basis elements in B' (exactly κ of them) belong to H.

We concentrate on those basis elements $b_\alpha \in C''$ which do not belong either to H or to a coset in S. Since S generates $F \operatorname{mod} H$, to every $b_\alpha \in C''$ there is at least one

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linear combination x_α of the basis elements in C'' representing elements of S such that $b_\alpha - x_\alpha \in H$. For each $b_\alpha \in C''$ which is not in H or in a coset of S, select such an x_α and replace b_α in C'' by $b_\alpha - x_\alpha$.

STEP 4. Finally, we obtain a new basis B of F which is mod H a complete set of representatives of $S \cup T$.

We focus our attention on the set T. For each coset in T choose a representative $v_\beta \in F$, expressed in terms of basis elements in B' representing cosets in S. Owing to $|T| = \kappa = |B' \cap H|$, there is a bijection between the elements $\{b_\beta\}$ of B' not representing elements of S and the set $\{v_\beta + H\}$ of cosets (where we have the corresponding elements carrying the same indices). If in the basis B', the element b_β of B' is replaced by $b_\beta + v_\beta$, then we arrive at a basis with the desired properties.

This completes the proof. ∎

It is worth while observing that the set $S \cup T$ generates the module F/H, thus under the hypotheses of Lemma 4, *F has a basis whose elements are incongruent* mod H.

In some cases the condition stated in the preceding lemma is automatically satisfied. Indeed, we can verify the following simple fact valid over any domain R; this was proved by Hill-Megibben [7, Corollary 1.3] for abelian groups:

LEMMA 5. *If $M \cong F/H$ is a presentation of an R-module M such that* $\operatorname{rk} F > \operatorname{gen} M \geq \aleph_0$, *then the submodule H of F contains a summand G of F with* $\operatorname{rk} G = \operatorname{rk} F$.

PROOF. Let $\phi: F \longrightarrow M$ be the canonical epimorphism (with kernel H). Evidently, there is a summand F_1 of F with $\operatorname{rk} F_1 = \operatorname{gen} M$ which is mapped by ϕ onto M. Write $F = F_1 \oplus F_2$ and denote the restriction of ϕ to F_j by ϕ_j ($j = 1, 2$). As ϕ_1 is surjective and F_2 is projective, there is a map $\rho: F_2 \longrightarrow F_1$ such that $\phi_2 = \phi_1 \rho$. Then $G = \{x - \rho x \mid x \in F_2\}$ is a complement of F_1 in F contained in H whose rank is necessarily equal to $\operatorname{rk} F$. ∎

3. **The main result.** We are now ready to verify our main result which we have already mentioned in the Introduction.

THEOREM 6. *Let R be a Prüfer domain of finite character, and F, F' free R-modules. Two presentations, F/H and F'/H', of a flat (i.e. torsion-free) R-module M of projective dimension ≤ 1 are equivalent if and only if*

$$\operatorname{rk} H = \operatorname{rk} H'.$$

PROOF. Only sufficiency requires a proof. Suppose $\operatorname{rk} H = \operatorname{rk} H'$; as already noted above, this implies $\operatorname{rk} F = \operatorname{rk} F'$. Actually, we are going to prove a bit more than stated, viz. we will show that every isomorphism

$$\psi: M = F/H \longrightarrow F'/H' = M'$$

is induced by an isomorphism

$$\phi: F \longrightarrow F' \quad \text{with} \quad \phi(H) = H'.$$

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Note that the submodules H and H' are pure (since M is flat and R is Prüfer) and projective (since p.d. $M \leq 1$). Hence if H and H' are of finite rank, then by Lemma 1 they are summands of F and F', respectively. In this case M is projective, and the equivalence of the two presentations of M is obvious. Hence, in the balance of the proof we may suppose that $\operatorname{rk} H = \operatorname{rk} H'$ is infinite.

Choose a set S of generators of $M = F/H$ of minimal cardinality κ, and pick a subset T of M of the same cardinality, disjoint from S. This can be done as follows. If the characteristic of R is not 2, then after dropping from S one member of additive inverse pairs among the elements of S, we can choose T to consist of the additive inverses of elements of $S \setminus H$. If the characteristic of R is 2, then choose T to be $s_0 + s$ with a fixed element s_0 of S and s ranging over all elements of S after deleting from S generators of this form.

We clearly have $\kappa \leq \operatorname{rk} F$. Let S', T' denote the sets in M' corresponding to S, T under the isomorphism ψ. We distinguish three cases.

CASE I. $\operatorname{rk} H = \kappa$. Then $\operatorname{rk} H' = \kappa$ likewise. In view of Lemma 4, there exist a basis B of F and a basis B' of F' which are complete sets of representatives of $S \cup T \bmod H$ and $S' \cup T' \bmod H'$, respectively. (If S, T are chosen so as not to contain 0, then B will be disjoint from H.) The correspondence $B \longrightarrow B'$ which is well defined by mapping $b \in B$ upon $b' \in B'$ if and only if ψ maps the coset $b + H$ upon $b' + H'$ extends uniquely to an isomorphism $\phi: F \longrightarrow F'$ under which H' is clearly the image of H. Thus the two presentations are equivalent.

CASE II. $\operatorname{rk} H > \kappa$. Pick a free R-module G whose rank is $\operatorname{rk} H$, then replace F by $F \oplus G$ and F' by $F' \oplus G$, but keep H and H'. Application of Case I to the R-module $M \oplus G$ (with ψ extended by the identity map on G) implies the existence of an isomorphism $\phi: F \oplus G \longrightarrow F' \oplus G$ with $\phi H = H'$ inducing ψ. It is self-evident that $\phi F = F'$.

CASE III. $\operatorname{rk} H < \kappa$. There is a decomposition $F = F_1 \oplus F_2$ such that $H \leq F_1$ and $\operatorname{rk} H = \operatorname{rk} F_1 < \operatorname{rk} F_2 = \kappa$. Thus $M = F_1/H \oplus F_2$, and ψ yields a similar decomposition $M = F_1'/H' \oplus F_2'$. Case I guarantees the existence of an isomorphism $F_1 \longrightarrow F_1'$ mapping H upon H'; this along with $F_2 \longrightarrow F_2'$ (restriction of ψ) provides a desired isomorphism $\phi: F \longrightarrow F'$. ∎

REMARK. A careful examination of the proof reveals that the finite character of the Prüfer domain has been used only to guarantee that G of Lemma 3 is free whenever it is of infinite rank. Consequently, it is enough to require that every projective R-module of infinite rank κ contains a free summand of the same rank κ. It is straightforward to see that this is the case if and only if every projective R-module of countable rank contains a free summand of rank ≥ 1. This condition is satisfied, for instance, if R is *of countable character* in the sense that every non-zero element of R is contained in at most countably many maximal ideals. Thus Theorem 6 continues to hold for Prüfer domains of countable character.

We turn our attention to a more general situation, by dropping the condition of flatness. From the proofs of Lemma 4 and Theorem 6 it is easy to obtain a sufficient condition on the equivalence of presentations for arbitrary R-modules of projective dimension ≤ 1.

COROLLARY 7. *Let F and F' be free modules over a Prüfer domain R, and assume F/H and F'/H' are presentations of the R-module M of projective dimension 1. If*

(i) $\operatorname{rk} F = \operatorname{rk} F'$;

(ii) H contains a free summand of F of rank $\operatorname{gen} M$;

(iii) H' contains a free summand of F' of rank $\operatorname{gen} M$,

then every isomorphism $\psi: F/H \longrightarrow F'/H'$ is induced by an isomorphism $\phi: F \longrightarrow F'$ such that $\phi(H) = H'$.

PROOF. In the proofs above the flatness of M was used only to ascertain that conditions (ii) and (iii) were satisfied. Therefore, assuming (ii) and (iii), and choosing a generating set S of M of cardinality $\operatorname{gen} M$, the argument above establishes the present claim as well (in view of Remark above, the condition of R being of finite character is dropped). ∎

From the last corollary it follows at once:

COROLLARY 8. *Let R be a Prüfer domain, and $F/H, F'/H'$ two presentations of the R-module M of projective dimension 1 where F, F' are free R-modules. Then there is a free R-module G of rank $\leq \operatorname{gen} M$ such that*

$$(F \oplus G)/(H \oplus G) \quad \textit{and} \quad (F' \oplus G)/(H' \oplus G)$$

are equivalent presentations of M. ∎

4. **Application.** Finally, we mention an application of our results. This is an analog of one obtained by Erdős [3] for abelian groups.

COROLLARY 9. *Let R be a Prüfer domain of finite character, and N a submodule of an R-module M such that M/N is flat of projective dimension* 1. *If*

$$\aleph_0 \leq \operatorname{gen} M/N \quad \textit{and} \quad \operatorname{gen} N \leq \operatorname{gen} M/N,$$

then M has a generating system of cardinality $\operatorname{gen} M/N$ *whose elements are pairwise incongruent* $\bmod N$.

PROOF. Represent M as F/H with a free R-module F such that $\operatorname{rk} F = \operatorname{gen} M$. Then N will be of the form F'/H with a submodule F' of F containing H. Notice that F' is projective, since $F/F' \cong M/N$ has projective dimension ≤ 1. Furthermore, in view of $\operatorname{rk} F' = \operatorname{rk} H + \operatorname{gen} N \leq \operatorname{gen} M + \operatorname{gen} N = \operatorname{gen} M/N$ (the last equality is a consequence of the hypothesis $\operatorname{gen} N \leq \operatorname{gen} M/N$) we can choose a free R-module G such that $\operatorname{rk}(G \oplus F)/(G \oplus F') = \operatorname{rk}(G \oplus F')$. We now appeal to the remark made after Lemma 4 to conclude that the free R-module $G \oplus F$ has a basis B whose elements $\bmod G \oplus F'$ represent different elements of M/N. As $B \bmod H$ generates M, this yields a generating system for M of the desired kind. ∎

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