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BEATTY SEQUENCES, CONTINUED FRACTIONS, AND CERTAIN SHIFT OPERATORS

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ABSTRACT. Let $\theta=\theta(k)$ be the positive root of $\theta^2+(k-2)\theta-k=0$. Let $f(n)=[(n+1)\theta]-[n\theta]$ for positive integers n, where $[x]$ denotes the greatest integer in x. Then the elements of the infinite sequence $(f(1), f(2), f(3), \ldots)$ can be rapidly generated from the finite sequence $(f(1), f(2), \ldots, f(k))$ by means of certain shift operators. For $k=1$ we can generate (the characteristic function of) the sequence $[n\theta]$ itself in this manner.

1. **Introduction.** Since the "golden mean" $\alpha=\frac{1}{2}(1+\sqrt{5})$ has the expansion $1+\frac{1}{1+}\frac{1}{1+}\cdots$ as a simple continued fraction, it is readily associated with the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$. We shall show there is an apparently very different way in which α is associated with this sequence. Let $101010\cdots$ be the sequence consisting of the **2** numbers $0, 1$ repeated infinitely often (in that order). If we repeat the first **3** numbers of this sequence infinitely often we get $101101101\cdots$. If we similarly repeat the first **5** numbers here, we get $1011010110\cdots$; for **8** we get $1011010110110101\cdots$. It is clear that this process "converges" (see §2 for a precise definition) to a sequence $a_1a_2a_3\cdots$ where a_i is 0 or 1.

THEOREM 1. *Let $[x]$ denote the largest integer not exceeding x. Then $a_n=1$ if $n=[m\alpha]$ for some integer m, and $a_n=0$ otherwise.*

We shall find a similar phenomenon (Theorem 2 of §3) for an infinite class of quadratic irrationalities, and show that it is in fact a consequence of well known properties of simple continued fractions. Theorem 1 is deduced from the case $k=1$ of Theorem 2 in §4. In §5 we shall show that Theorem 2 can also be deduced from a general result of Markoff on simple continued fractions. The author believes that the list of references at the end of this paper is the first reasonably complete bibliography on Beatty sequences and related topics.

Although our subject has roots in [6] and [48], it was the striking problem posed by the late Professor Samuel Beatty of Toronto in [5] that brought it to

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the attention of the mathematical community at large. The topic was then disseminated and pursued on a worldwide basis. As items [2, 3, 5, 18–21, 25, 34, 44, 46] indicate only inadequately, this was done nowhere more vigorously than in Beatty's native Canada.

Let $\theta=1+1/k+1/k+\cdots=1+\frac{1}{2}\{\sqrt{(k^2+4)}-k\}$. The Beatty sequence $a_n=[n\theta]$, where $1\le n<\infty$, is central to the theory of Wythoff's game and its generalizations [1, 3, 18, 19, 21, 22, 29, 40, 41, 43, 67, 68]. Theorem 2 of §3 asserts that the first difference of this sequence can be generated very rapidly from the $k+1$ tuple $([\theta], [2\theta], \ldots, [(k+1)\theta])$ by means of a certain sequence of shift operations. In fact, let $f(n)=[(n+1)\theta]-[n\theta]$ and let

$$F=(f(1), f(2), f(3), \ldots).$$

Then F is a fixed point of a certain transformation T such that (i) $T(F)$ depends only on the first k elements of F, and (ii) T is the limit of a sequence of shift operators, defined in §2, corresponding to elements of the sequence defined by $t_0=1$, $t_1=k$, and $t_n=kt_{n-1}+t_{n-2}$ for $n\ge 2$.

Our proof hinges on the following well known result.

LEMMA. *Let $n\ge 1$. Let p_n/q_n be the n-th convergent to the irrational number ϕ. If a and b are integers with $b>0$ and $|\phi b-a|<|\phi q_n-p_n|$, then $b\ge q_{n+1}$.*

A proof can be found, for example, in [53, p. 163].

The standard way of generating a_n is as follows. Let $a_1=1$ and $b_1=k+1$. After a_i and b_i have been generated for $i<n$, let a_n be the smallest positive integer which is distinct from all these a_i and b_i, with $i<n$, and let $b_n=a_n+nk$. Since $\theta^{-1}+(\theta-k)^{-1}=1$ we have [see, e.g., 5, 21, or 52 pp. 34–35] that $a_n=[n\theta]$ and $b_n=[n(\theta+k)]$. Thus $\{a_n\}$ and $\{b_n\}$ are complementary Beatty sequences.

2. **Certain shift operators.** Let $C=(c_1, c_2, c_3, \ldots)$ be a finite or infinite sequence of integers. Let $e_i(C)=c_i$ be the ith element of the sequence C. If $C_j=(c_{1j}, c_{2j}, \ldots)$ we say that $\lim_{j\to\infty} C_j=C$ if $e_i(C_j)=c_i$ for j sufficiently large.

If $C=(c_1, \ldots, c_n)$ and $D=(d_1, \ldots, d_m)$ are *finite* sequences, we denote by CD their juxtaposition; i.e.,

$$CD=(c_1, \ldots, c_n, d_1, \ldots, d_m).$$

We allow infinite juxtaposition; thus D^∞ is the infinite sequence of period m whose first m members are $d_1, \ldots, d_m$. We denote by (C, t) the finite sequence whose first t elements are $c_1, \ldots, c_t$; i.e.,

$$(C, t)=(c_1, \ldots, c_t).$$

Now let $T=\{t_n\}_{n=1}^\infty$ be a sequence of strictly increasing positive integers. Define $T_n(C)$ inductively by

$$T_1(C)=(C, t_1)^\infty \tag{2.1}$$

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and

(2.2) $$T_n(C) = (T_{n-1}(C), t_n)^\infty.$$

Set

$$T(C) = \lim_{n\to\infty} T_n(C);$$

the limit obviously exists. Note that if the sequences C and C' have the same initial t_1 members, then $T_1(C) = T_1(C')$ and hence $T(C) = T(C')$.

For example, let $C = (1, 0)^\infty$ and let $T = (2, 3, 5, 8, 13, \ldots)$ be the sequence of Fibonacci numbers that exceed 1. Then $T_1(C) = (1, 0)^\infty$, $T_2(C) = (1, 0, 1)^\infty$, $T_3(C) = (1, 0, 1, 1, 0)^\infty$, and $T_4(C) = (1, 0, 1, 1, 0, 1, 0, 1)^\infty$.

3. **The main result.** Recall that $\theta = 1 + 1/k + 1/k + \cdots$ and that $F = (f(1), f(2), f(3), \ldots)$ where $f(n) = [(n+1)\theta] - [n\theta]$.

THEOREM 2. *Let $k \geq 2$ be a positive integer and let $T = \{t_n\}_{n=1}^\infty$ where $t_0 = 1$, $t_1 = k$, and $t_n = kt_{n-1} + t_{n-2}$ for $n \geq 2$. Then*

$$T(F) = F.$$

If $k = 1$ the result is valid for $T = \{t_n\}_{n=2}^\infty$.

Before giving a proof, we require some notation and several lemmas. Given an irrational continued fraction $x = a_0 + 1/a_1 + 1/a_2 + \cdots$ define (as usual) sequences $\{p_n\}$ and $\{q_n\}$ inductively as follows:

$$p_{-2} = 0 \qquad p_{-1} = 1 \qquad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for} \quad n \geq 0,$$
$$q_{-2} = 1 \qquad q_{-1} = 0 \qquad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for} \quad n \geq 0.$$

Let $r_j = p_j/q_j$.

LEMMA 3.1. *For $\varphi = \theta + k - 1$ we have $a_i = k$ for all k, so $p_j = t_{j+1}$ and $q_j = t_j$.*

LEMMA 3.2. *We have $t_{2j}\varphi - t_{2j+1} > 0$ and $t_{2j+2} - \varphi t_{2j+1} > 0$ for $j \geq 0$.*

Proof. For $j \geq 0$, it is well known that $r_{2j} < x < r_{2j+1}$ [53, *p.* 156], so this follows from Lemma 3.1.

LEMMA 3.3

(i) *If $t_{2j+2} > m$, then*

$$1 > m\varphi - [m\varphi] \geq t_{2j+2} - \varphi t_{2j+1} > 0. \qquad (j \geq 0)$$

(ii) *If $t_{2j+1} > m$, then*

$$1 + [m\varphi] - m\varphi \geq t_{2j}\varphi - t_{2j+1} > 0. \qquad (j \geq 1).$$

Proof. This is an immediate consequence of Lemmas 3.1, 3.2, and the contrapositive of the Lemma quoted in the introduction; simply let $b = m$ and take $a = [m\varphi]$ for (i) and $a = 1 + [m\varphi]$ for (ii).

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LEMMA 3.4

(i) *If* $t_{2j+2} > m$, *then*

$$t_{2j+2} + [m\varphi] = [(t_{2j+1} + m)\varphi]. \qquad (j \ge 0)$$

(ii) *If* $t_{2j+1} > m$, *then*

$$t_{2j+1} + [m\varphi] = [(t_{2j} + m)\varphi]. \qquad (j \ge 1).$$

Proof. Let

$$r = t_{2j+1}\varphi - t_{2j+2} + m\varphi - [m\varphi]$$

and

$$s = m\varphi - [m\varphi] + t_{2j}\varphi - t_{2j+1}.$$

If $t_{2j+2} > m$, Lemma 3.3(i) shows that $[r] = 0$. But upon adding $[r]$ to the left side of (i), we get the right side. If $t_{2j+1} > m$, Lemma 3.3(ii) shows that $[s] = 0$. But upon adding $[s]$ to the left side of (ii), we get the right side. Thus the lemma is established.

Thus we have proved

COROLLARY 3.1. *If* $j \ge 1$ *and* $m < t_{j+1}$, *then*

$$[(t_j + m)\varphi] = t_{j+1} + [m\varphi].$$

We now establish

COROLLARY 3.2. *If* $j \ge 1$ *and* $n + 1 < t_{j+1}$, *then*

$$f(n + t_j) = f(n).$$

Proof. Recall that $\varphi = \theta + k - 1$. Thus

$$f(n + t_j) = [(n + 1 + t_j)\theta] - [(n + t_j)\theta]$$
$$= [(n + 1 + t_j)\varphi] - (k - 1)(n + 1 + t_j) - [(n + t_j)\varphi] + (n + t_j)(k - 1).$$

By Corollary 3.1 the right side of the above is

$$t_{j+1} + [(n + 1)\varphi] - (t_{j+1} + [n\varphi]) - (k - 1)$$
$$= [(n + 1)\varphi] - [n\varphi] - (k - 1) = [(n + 1)\theta] - [n\theta] = f(n).$$

This proves Corollary 3.2. We comment that it is not always valid for $n + 1 = t_{j+1}$.

We can now prove the theorem. Say $k \ge 2$, and note that $n + 1 < t_{j+1}$ means $n \le t_{j+1} - 2$. The sequence F certainly agrees with itself in the first k places. Since $t_1 = k$, the sequence $T_1(F)$, by Corollary 3.2, will agree with F in the first $t_1 + t_2 - 2$ places. In general, say $T_j(F)$ agrees with F in the first $t_{j+1} + t_j - 2$ places. Since $t_{j+1} \le t_{j+1} + t_j - 2$, these places completely determine $T_{j+1}(F)$. By Corollary 3.2, the sequence $T_{j+1}(F)$ will agree with F in the first $t_{j+2} + t_{j+1} - 2$ places. Thus $T(F) = F$. The proof for $k = 1$ is the same, aside from the initialization.

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4. **The "golden mean" Beatty sequence.** We shall now prove Theorem 1. For $h \geq 1$ let $a_h = [h\alpha]$ and $b_h = a_h + h$ where $\alpha = \frac{1}{2}(1+\sqrt{5})$. Thus $\{a_h\} = (1, 3, 4, 6, 8, 9, \ldots)$.

LEMMA 4.1. *The inequality $a_h \leq n$ holds if and only if $b_h \leq a_{n+1}$.*

Proof. If $a_h \leq n$ then $h\alpha < n+1$, so

$$b_h = [h(\alpha+1)] = [h\alpha^2] \leq [(n+1)\alpha] = a_{n+1}. \tag{4.1}$$

On the other hand, if $b_h \leq a_{n+1}$ then

$$b_h = [h(\alpha+1)] < (n+1)\alpha. \tag{4.2}$$

We now claim that $h\alpha < n+1$. If not, then $h\alpha + h \geq n+1+h$ and $b_h = [h\alpha + h] \geq n+1+h$. From this and (4.2), we find

$$b_h\alpha > b_h + h\alpha \geq b_h + n + 1$$

and

$$b_h\alpha^2 > b_h\alpha + b_h,$$

a contradiction to the fact that $\alpha^2 = \alpha + 1$. Hence $h\alpha < n+1$, so $a_h \leq n$ and the proof is complete.

Now let $A(n)$ be the number of a_h such that $a_h \leq n$.

LEMMA 4.2. *The sequence a_k can be defined inductively by $a_1 = 1$ and $a_{n+1} = n + 1 + A(n)$.*

Proof. It is clear that $1 \leq a_{n+1} - a_n \leq 2$ and that

$$a_{n+1} = 1 + \sum_{h=1}^{n} (a_{h+1} - a_h).$$

Each term of the above sum is either 1 or 2, so $a_{n+1} = 1 + n + K(n)$ where $K(n)$ is the number of twos. It is clear from the standard way of generating the a_k (see §1) that $K(n)$ is the number of b_i such that $b_i \leq a_{n+1}$. But by Lemma 4.1, we have that $K(n) = A(n)$. This completes the proof.

We can now prove Theorem 1. By Lemma 4.2 we have $a_{n+1} - a_n = 1 + [A(n) - A(n-1)]$. The expression in brackets is 1 if $n = [h\alpha]$ for some integer h and 0 otherwise. Thus the sequence of differences is, aside from an additive constant of 1 (which doesn't really matter), the characteristic function of the sequence $[h\alpha]$. But we can apply the Theorem of §3 to the sequence of differences. This completes the proof.

5. **A proof by Markoff's theorem.** Let ξ be a real number such that $0 < \xi < 1$, and set $f(m) = [(m+1)\xi] - [m\xi]$. Let

$$\xi = \frac{1}{k+}\frac{1}{k_1+}\frac{1}{k_2+}\cdots \tag{5.1}$$

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be the expansion of ξ into a simple contained fraction. Let $S(c, d)$ be the free semigroup on the letters c and d.

DEFINITION 5.1. For $m \geq 1$, let c_m and d_m be the sequences of elements from $S(c, d)$ defined inductively by

(5.2) $$c_1 = c^{k-1}d, \qquad d_1 = c^k d$$

and

(5.3) $$c_{m+1} = c_m^{k_m - 1} d_m, \qquad d_{m+1} = c_m^{k_m} d_m$$

for $m \geq 1$. Let

(5.4) $$x = c_1 c_2 c_3 \cdots = \prod_{i=1}^{\infty} c_i.$$

Now write

(5.5) $$x = \prod_{i=1}^{\infty} g_i$$

where each g_i is either c or d.

THEOREM (MARKOFF). *For $m \geq 1$ we have $f(m) = 0$ if $g_m = c$ and $f(m) = 1$ if $g_m = d$.*

This result was originally obtained by A. A. Markoff [48] in his investigation of a problem of the astronomer J. Bernoulli [6]. The above formulation is from Venkov [65, pp. 65–68], who also gives a proof. We shall apply this to the case

(5.6) $$\xi = \theta - 1 = \frac{1}{k+}\frac{1}{k+}\frac{1}{k+}\cdots$$

to obtain an alternate proof of our own result.

DEFINITION 5.2. For $w \in S(c, d)$, let $\bar{w}$ be the word obtained from w by replacing each c by $c^{k-1}d$ and each d by $c^{k-1}dc$. For example, $\overline{cd} = c^{k-1}dc^{k-1}dc$.

It is clear that $\overline{w_1 w_2} = \overline{w_1} \cdot \overline{w_2}$ for words w_1 and w_2 in $S(c, d)$. It is also clear that $\bar{w}$ makes sense for infinite words $w = a_1 a_2 a_3 \cdots$ where each a_i is either c or d.

REMARK. We have from (5.2), (5.3), and (5.6) that

(5.7) $$c_1 = c^{k-1}d, \qquad c_2 = (c^{k-1}d)^{k-1}c^k d$$

and

(5.8) $$c_{n+2} = c_{n+1}^{k-1} c_n c_{n+1} \quad \text{for} \quad n \geq 1.$$

For the next lemma, consider $S(c, d)$ as imbedded in $G(c, d)$, the free group on c and d.

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LEMMA 5.1. *For* $r \geq 1$ *we have*

(5.9) $$c_1 c_{r+1} c_1^{-1} = \overline{c_r}.$$

Proof. This is true for $r = 1$, since

$$c_1 c_2 c_1^{-1} = (c^{k-1}d)^k c^k d(c^{k-1}d)^{-1} = (c^{k-1}d)^k c$$
$$= (c^{k-1}d)^{k-1} c^{k-1} dc = \bar{c}^{k-1}\bar{d} = \overline{c_1}.$$

Say it is true for $r \leq n$. Then

$$c_1 c_{n+2} c_1^{-1} = (c_1 c_{n+1} c_1^{-1})^{k-1}(c_1 c_n c_1^{-1})(c_1 c_{n+1} c_1^{-1})$$
$$= \overline{(c_n^{k-1} c_{n-1} c_n)} = \bar{c}_{n+1}$$

and the lemma is proved.

LEMMA 5.2. *For* $r \geq 1$, *we have*

(5.10) $$\overline{(c_1 \cdots c_r)} c_1 = c_1 c_2 \cdots c_{r+1}.$$

Proof. Multiply both sides by c_1^{-1} on the right, and apply the previous lemma.

DEFINITION 5.3. Let $w_1 = c_1$ and $w_{n+1} = \overline{w_n}$. Let $\ell(w_n)$ denote the length of w_n; that is, the number of letters in w_n. For example, $\ell(c^3 d) = 4$.

LEMMA 5.3. *There is a unique infinite word* w *such that* (i) w *has* c_1 *as an initial segment, and* (ii) $\bar{w} = w$. *In fact,* $w = x$.

Proof. It is clear that the lengths of the words w_n are strictly increasing. Since each w_n must be an initial segment of w, the infinite word w is unique (if it exists). On the other hand, the infinite word x defined in (5.4) begins with c_1, and $\bar{x} = x$ follows from Lemma 5.2. Take $w = x$. This completes the proof.

We thus have that the sequence $\{w_n\}$ converges to w (with essentially the same notion of convergence as in §2).

LEMMA 5.4. *For* $n \geq 2$, *we have* $w_{n+1} = w_n^k w_{n-1}$.

Proof. Define $w_0 = c$. Then $w_{m+1} = \bar{w}_m$ for $m \geq 0$. Since $w_2 = (c^{k-1}d)^k c = w_1^k w_0$, the lemma is valid for $n = 1$. Say it is true for $n \leq m$. Then

$$w_{m+2} = \bar{w}_{m+1} = \overline{w_m^k w_{m-1}} = w_{m+1}^k w_m$$

and the lemma is proved.

Since $w_{n-1} = w_n(w_{n-1}^{k-1} w_{n-2})^{-1}$, we have

(5.11) $$w_{n+1} = w_n^{k+1}(w_{n-1}^{k-1} w_{n-2})^{-1}.$$

This shows that w_{n+1} is an initial segment of the infinite word formed by repeated juxtaposition of w_n. Thus we once again have our result on shift operators.

6. **Bibliography.** Let $\alpha > 0$ and β be real numbers, and let $[x]$ denote the greatest integer in x. This bibliography attempts to list all papers concerning the sequence $[\alpha n + \beta]$, $n = 1, 2, 3, \ldots$, which can be considered as "descendants" of Beatty [5], Markoff [48], and Wythoff [67]. I believe it is complete at least up to 1972, but perhaps this is wishful thinking. I would greatly appreciate any further references. A fairly complete bibliography of this area should be useful, since it is notorious for the frequency with which known facts are rediscovered and republished.

A quick overview of the subject is given in Coxeter [21]. For more detailed introductions to Beatty sequences and Wythoff's game, see Niven [52, pp. 34–45] and Connell [18] respectively. Most (if not all) of the known facts about Bernoulli sequences can be found in Venkov [65, pp. 65–68] and Uspensky [16, 17]. For research areas which are presently active, see Fraenkel [25–30] and Cohn [14–16].

The following question seems untouched: for an irrational $\alpha > 0$, what can be said about the second difference of $[\alpha n^2 + \beta]$?

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References

1. W. Ahrens, *Mathematische Unterhaltungen und Spiele*, vol. 1, 2nd ed., Teubner, Leipzig, 1910 (pp. 84–88).

2. M. Angel, *Partitions of the natural numbers*, Canad. Math. Bull. **7** (1964), 219–236. MR 28 #5030.

3. W. W. R. Ball, *Mathematical Recreations and Essays*, revised by H. S. M. Coxeter, MacMillan, New York, 1939 (pp. 36–40).

4. Th. Bang, *On the sequence* $[n\alpha]$, $n = 1, 2, \ldots$, Math. Scand. **5** (1957), 69–76. MR 19 (1958), p. 1159.

5. S. Beatty, *Problem* 3173, Amer. Math. Monthly **33** (1926), 159. Solutions, ibid., **34** (1927), 159.

6. J. Bernoulli III, *Sur une nouvelle espèce de calcul, Recueil pour les astronomes*, vols. 1, 2, Berlin, 1772.

7. R. Bricard, *Sur un problème relatif aux nombres incommensurables*, Nouv. Ann. de Math. (6) **1** (1926), 100–103.

8. L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville, *Fibonacci representations*, Fib. Quart. **10** (1972), 1–28.

9. ——, ——, and ——, *Lucas representations*, Fib. Quart. **10** (1972), 29–42, 70, 112.

10. ——, ——, and ——, *Fibonacci representations of higher order*, Fib. Quart. **10** (1972), 71–80.

11. ——, ——, and ——, *Fibonacci representations of higher order II*, Fib. Quart. **10** (1972), 71–80.

12. ——, ——, and ——, *Pellian representations*, Fib. Quart. **10** (1972), 449–488; Addendum, ibid., 527–530. MR 47 #8420.

13. L. Carlitz, R. Scoville, and T. Vaughan, *Some arithmetic functions related to Fibonacci numbers*, Fib. Quart. **11** (1973), 337–386. MR 48 #10969.

14. H. Cohn, *Representation of Markoff's binary quadratic forms by geodesics on a perforated torus*, Acta Arith. **18** (1971), 125–136. MR 44 #5277.

15. ——, *Markoff forms and primitive words*, Math. Ann. **196** (1972), 8–22. MR 45 #6899.

16. ——, *Some direct limits of primitive homotopy words and of Markoff geodesics*, esp. pp. 97–98,

in Discontinuous Groups and Riemann Surfaces, L. Greenberg ed., Ann. of Math. Studies **79,** Princeton University Press, Princeton, 1974.

17. ——, *Problem E2544*, Amer. Math. Monthly **82** (1975), 660. Solution, ibid., 83 (1976), 745–747.

18. I. G. Connell, *A generalization of Wythoff's game*, Canad. Math. Bull. **2** (1959), 181–190.

19. ——, *Some properties of Beatty sequences I*, Canad. Math. Bull. **2** (1959), 190–197. MR #7805.

20. ——, *Some properties of Beatty sequences II*, Canad. Math. Bull. **3** (1960), 17–22. MR 22 #1558.

21. H. S. M. Coxeter, *The golden section, phyllotaxis and Wythoff's game*, Scripta Math. **19** (1953), 135–143. MR 15 (1954), p. 246.

22. A. P. Domoryad, *Mathematical Games and Diversions* (Russian), Gosudarstv. Izdat. Fiz-Mat. Lit., Moscow, 1961 (pp. 59–67). MR 23 #A761.

23. K. Fan, *Problem* 4399, Amer. Math. Monthly **57** (1950), 343. Solutions, ibid., **59** (1952), 48–49.

24. Faucheux, *Sur une question concernant des suites de nombres incommensurables*, Nouv. Ann. de Math. (6) **1** (1926), 237–239.

25. A. S. Feaenkel, *The bracket function and complementary sets of integers*, Can. J. Math. **21** (1969), 6–27. MR 38 #3214.

26. ——, J. Levitt, and M. Shimshoni, *Characterization of the set of values* $f(n)=[n\alpha]$, $n=1,2,\ldots$, Discrete Math. **2** (1972), 335–345. MR 46 #1743.

27. ——, *Complementing and exactly covering sequences*, J. Comb. Theory Ser. **A14** (1973), 8–20. MR 46#8875.

28. ——, *A characterization of exactly covering congruences*, Discrete Math. **4** (1973), 359–366. MR 47 #4906.

29. —— and I. Borosh, *A generalization of Wythoff's game*, J. Comb. Theory **15** (1973), 175–191.

30. A. Fraenkel, *Further characterizations and properties of exactly covering congruences*, Discrete Math. **12** (1975), 93–100.

31. ——, *Complementary systems of integers*, in preparation.

32. E. N. Gilbert, *Functions which represent all integers*, Amer. Math. Monthly **70** (1963), 736–738; Acknowledgement, 1082.

33. H. W. Gould, *Problem H*-1, Fib. Quart. **1** (1963), 46.

34. ——, *Generalization of a bracket formula of L. Moser*, Canad. Math. Bull. **6** (1963), 275–277; Editor's comment, 277–278.

35. ——, *Non-Fibonacci numbers*, Fib. Quart. **3** (1965), 177–183. MR 35 #118.

36. R. L. Graham, *On a theorem of Uspensky*, Amer. Math. Monthly **70** (1963), 407–409. MR 26 #6062.

37. —— and H. O. Pollak, *Note on a nonlinear recurrence related to* $\sqrt{2}$, Math. Mag. **43** (1970), 143–145. MR 42 #180.

38. ——, *Covering the positive integers by disjoint sets of the form* $\{[n\alpha+\beta]: n=1,2,\ldots\}$, J. Comb. Theory Ser. **A15** (1973), 354–358.

39. H. D. Grossman, *A set containing all integers*, Amer. Math. Monthly **69** (1962), 532–533.

40. H. Hijikata, *On a game of Wythoff* (Japanese), Sûgaku **11** (1959/60), 220–222. MR 25 #42.

41. J. C. Holladay, *Cartesian products of termination games*, Contributions to the theory of games, Vol. 3, Princeton University Press, New Jersey, 1957. MR 20 #2236.

42. ——, *Some convergent recursive sequences, homeomorphic identities, and inductively defined complementary sequences*, Fib. Quart. **4** (1966), 1–36. MR 36 #5063.

43. ——, *Some generalizations of Wythoff's game and other related games*, Math. Mag. **41** (1968), 7–13.

44. R. Honsberger, *Ingenuity in Mathematics*, Random House, New York, 1970 (pp. 93–110).

45. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley and Sons, New York, 1974 (pp. 305–318).

46. J. Lambek and L. Moser, *Inverse and complementary sequences of natural numbers*, Amer. Math. Monthly **61** (1954), 454–458. MR 16 (1955), p. 17.

47. W. J. Leveque, editor, *Reviews in Number Theory* vol. **1**, American Mathematical Society, Providence, Rhode Island, 1974 (pp. 331–333). MR #2040.

48. A. A. Markoff, *Sur une question de Jean Bernoulli*, Math. Ann. **19** (1882), 27–36.

49. ——, *Collected Works* (Russian), Izdat. Akad. Nauk. SSSR, Leningrad, 1951 (pp. 13, 15–16).

50. J. C. P. Miller, *The sum of the integral parts in an arithmetical progression*, Math. Gazette **36** (1952), 234–243. MR 14 (1953), p. 536.

51. D. J. Newman, *Problem* 5252, Amer. Math. Monthly **71** (1964), 1138. Solutions, ibid., **72** (1965), 1144–1145.

52. I. Niven, *Diophantine Approximations*, Interscience Tracts in Pure and Applied Mathematics #**14**, Interscience Publishers, John Wiley and Sons, New York, c. 1963 (pp. 34–45). MR 26 #6120.

53. —— and H. Zuckerman, *An Introduction to the Theory of Numbers*, John Wiley and Sons, New York, 2nd ed., 1966. MR 22#5605.

54. T. H. O'Beirne, *Puzzles and Paradoxes*, Oxford University Press, London, 1965 (pp. 130–138).

55. Th. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. **5** (1957), 57–68. MR 19 (1958), p. 1159.

56. ——, *Über einige Eigenschaften der Zahlenmengen* $[\alpha n+\beta]$ *bei irrationalem* α *mit einleitenden Bemerkungen über einige kombinatorische Probleme*, Norske Vid. Selsk. Forh. (Trondheim) **30** (1957), 118–125. MR 19 (1958), pp. 1159–1160.

57. ——, *Some remarks on the triple systems of Steiner*, Math. Scand. **6** (1958), 273–280. MR 19 (1958), p. 1159.

58. N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York, 1973 (pp. 29–30).

59. B. M. Stewart, *Theory of Numbers*, 2nd ed., MacMillan, New York, 1964 (pp. 76–77). MR 37 #6232.

60. R. Sprague, *Ein Satz über Teilfolgen der Reihe der natürlichen Zahlen*, Math. Ann. **115** (1938), 153–156.

61. J. V. Uspensky, *On a problem arising out of the theory of a certain game*, Amer. Math. Monthly **34** (1927), 516–521.

62. —— and M. A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York, 1939 (p. 98). MR 1 (1940), p. 38.

63. ——, *On a problem of John Bernoulli. I, II, III*, Revista Union Mat. Argentina **11** (1946), 141–154; 164–183; 239–255. MR 8 (1947), pp. 5, 6, 443.

64. ——, *On a problem of John Bernoulli, IV*, Revista Union Mat. Argentina **12** (1946), 10–19. MR 8 (1947), p. 443.

65. B. A. Venkov, *Elementary Number Theory*, Translated and edited by Helen Alderson, Wolters-Noordhoff, Groningen, 1970 (pp. 65–68). MR 42 #178.

66. I. M. Vinogradov, *Elements of Number Theory*, translated from the fifth revised edition by S. Kravetz, Dover, 1954 (p. 29). MR 15 (1954), p. 933.

67. W. A. Wythoff, *A modification of the game of Nim*, Nieuw. Archief voor Viskunde (2), **7** (1907), 199–202.

68. A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, translated by J. McCawley, Jr., revised and edited by B. Gordon, vol. 2, Holden-Day, San Francisco, 1967 (pp. 20, 105–112). MR 29 #2193.

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