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NEW EFFECTIVE RESULTS IN THE THEORY OF THE RIEMANN ZETA-FUNCTION

ALEKSANDER SIMONIČ

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Results in this dissertation are divided into four groups and are mainly effective estimates for the Riemann zeta-function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \operatorname{Re} s>1,$$

and associated functions in the Selberg class under the assumption of the (generalised) Riemann hypothesis ((G)RH), that is, $\zeta(s)\neq 0$ for $\operatorname{Re} s>1/2$. The zero-free regions for $\zeta(s)$ are connected with the distribution of prime numbers, for example, the celebrated prime number theorem

$$\sum_{n\le x}\Lambda(n)\sim x$$

is equivalent to the statement that $\zeta(1+it)\neq 0$ (see [13]), and that RH is equivalent to

$$\sum_{n\le x}\Lambda(n)=x+O(\sqrt{x}\log^2 x).$$

Here, $\Lambda(n)$ is the von Mangoldt function, equal to $\log p$ for $n=p^k, k\in\mathbb{N}$, and 0 otherwise. The fundamental connection between $\zeta(s)$ and $\Lambda(n)$ is

$$-\frac{\zeta'}{\zeta}(s)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},\qquad \operatorname{Re} s>1, \tag{1}$$

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which is an immediate consequence of the Euler product for $\zeta(s)$. Selberg [12] found a remarkably simple but powerful method to replace the right-hand side of (1) by the corresponding Dirichlet sum, together with the other terms which emerge from the singularities of $(\zeta'/\zeta)(s)$. His equation, known as the Selberg moment formula, and its variants are primary for establishing conditional estimates for $\log|\zeta(s)|$ and $|(\zeta'/\zeta)(s)|$. Although not considered in the thesis, explicit versions of these bounds prove to be useful in connection with the error term in the prime number theorem [4], as well as with the distribution of prime numbers in intervals [3] and in arithmetic progressions [8]. A very brief description of the four groups that constitute the thesis is given below.

The first group of results consists of explicit and RH estimates for the moduli of $S(t)$, that is, the argument of $\zeta(1/2+it)$, its antiderivative

$$S_1(t) = \int_0^t S(u)\,du, \quad t \geq 0,$$

and $\zeta(1/2+it)$. More precisely, explicit estimates of the bounds

$$S(t) \ll \frac{\log t}{\log\log t}, \quad S_1(t) \ll \frac{\log t}{(\log\log t)^2}, \quad \left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left(O(1)\frac{\log t}{\log\log t}\right)$$

are provided. We follow techniques outlined by Selberg [12] and Fujii [5, 6], and in the last case, also by Soundararajan [17]. As a corollary, we establish explicit and conditional upper bounds on gaps between consecutive zeros, for example,

$$\gamma' - \gamma \leq \frac{12.05}{\log\log\gamma}$$

for $\gamma' \geq \gamma \geq 10^{2465}$, where γ and γ' are the ordinates of two consecutive nontrivial zeros. Results from this chapter are published in [14].

The second group of results consists of explicit and RH estimates for $\log 1/|\zeta(s)|$ and $\log|\zeta(s)|$ to the right of the critical line by following techniques developed by Titchmarsh [18] while using also results on $S(t)$ and $S_1(t)$ from the first group. Moreover, we use these bounds to obtain effective and conditional estimates for

$$M(x) = \sum_{n\leq x}\mu(n) \quad \text{and} \quad Q_k(x) = \sum_{n\leq x}\sum_{d^k|n}\mu(d),$$

where $\mu(n)$ is the Möbius function and $Q_k(x)$ counts the number of k-free numbers not exceeding $x \geq 1$. Note that the prime number theorem is equivalent to $M(x) = o(x)$ and that RH is equivalent to $M(x) \ll_\varepsilon x^{1/2+\varepsilon}$. Our estimates are of the same strength as those of Titchmarsh [18], and Montgomery and Vaughan [10], that is,

$$M(x) \ll \sqrt{x}\exp\left(\frac{O(1)\log x}{\log\log x}\right), \quad Q_k(x) = \frac{x}{\zeta(k)} + O_{k,\varepsilon}(x^{1/(k+1)+\varepsilon}),$$

respectively. Results from this chapter are published in [15]. It should be noted that better (nonexplicit) results on various bounds on $\zeta(s)$ exist (see [1]). Improvements and generalisations of some of these results are currently in progress, and should yield also an improvement over our explicit estimate for $\zeta(1/2+it)$ from the first group.

The third group of results consists of GRH estimates for $|\log \mathcal{L}(s)|$ and $|(\mathcal{L}'/\mathcal{L})(s)|$ for functions in the Selberg class with a polynomial Euler product, where $\sigma \geq 1/2 + 1/\log\log(c_{\mathcal{L}}|t|)$ and $|t|$ is sufficiently large. The shape of these bounds are as in Littlewood [9] namely

$$\log \zeta(s) \ll_{\varepsilon,\sigma_0} (\log t)^{2(1-\sigma)+\varepsilon}$$

for $\varepsilon > 0$, $1/2 < \sigma_0 \leq \sigma \leq 1$ and t large, and are thus not the sharpest known. However, GRH can be replaced with a weaker assumption of having no zeros $\rho = \beta + i\gamma$ with $\beta > 1/2$ and $t - \gamma \ll \log\log|t|$. We provide effective estimates for $\zeta(s)$, Dirichlet L-functions with primitive characters and Dedekind zeta-functions, together with an improvement over a particular estimate for $M(x)$ from the second group of results, for example, for $x \geq 1$,

$$|M(x)| \leq 555.71x^{0.99} + 1.94 \cdot 10^{14}x^{0.98}$$

under RH. We also discuss a connection between a particular estimate on the 1-line and several classes of functions. Results from this chapter are published in [16]. We should mention that our bounds on $\mathcal{L}(s)$ have been already improved in [11], see also the next paragraph.

The fourth group of results consists of effective and GRH estimates for $|(L'/L)(\sigma,\chi)|$ for Dirichlet L-functions with primitive characters χ modulo q, where

$$\frac{1}{2} + \frac{1}{\log\log q} \leq \sigma \leq 1 - \frac{1}{\log\log q}$$

and also $\sigma = 1$, by combining methods from Selberg [12] and from the theory of band-limited functions applied to the Guinand–Weil exact formula. One of the results is that under GRH,

$$\left|\frac{L'}{L}(1,\chi)\right| \leq 2\log\log q - 0.4989 + 5.91\frac{(\log\log q)^2}{\log q}$$

for $q \geq 10^{30}$, which improves [7, Corollary 3.3.2]. We provide a similar conditional estimate also for $|(\zeta'/\zeta)(1+it)|$. Results in this group were obtained in collaboration with A. Chirre and M. V. Hagen, and are published in [2]. These techniques are used in [11] to obtain GRH estimates for $|\log \mathcal{L}(s)|$ and $|(\mathcal{L}'/\mathcal{L})(s)|$ for functions in the Selberg class with a polynomial Euler product, where

$$\frac{1}{2} + \frac{1}{\log\log q_{\mathcal{L}}|t|^{d_{\mathcal{L}}}} \leq \sigma \leq 1 - \frac{1}{\log\log q_{\mathcal{L}}|t|^{d_{\mathcal{L}}}}$$

and $|t|$ is sufficiently large. Here, $q_{\mathcal{L}}$ and $d_{\mathcal{L}}$ are the conductor and the degree of $\mathcal{L}$, respectively. This improves several known results. Moreover, our results are fully explicit under the additional assumption of the strong λ-conjecture.

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ALEKSANDER SIMONIČ, School of Science,
University of New South Wales (Canberra), Campbell, ACT 2612, Australia
e-mail: a.simonic@adfa.edu.au