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A REPRESENTATION FOR THE INVERSE GENERALISED FOURIER–FEYNMAN TRANSFORM VIA CONVOLUTION PRODUCT ON FUNCTION SPACE

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Abstract

We study a representation for the inverse transform of the generalised Fourier–Feynman transform on the function space $C_{a,b}[0,T]$ which is induced by a generalised Brownian motion process. To do this, we define a transform via the concept of the convolution product of functionals on $C_{a,b}[0,T]$. We establish that the composition of these transforms acts like an inverse generalised Fourier–Feynman transform and that the transforms are vector space automorphisms of a vector space $\mathcal{E}(C_{a,b}[0,T])$. The vector space $\mathcal{E}(C_{a,b}[0,T])$ consists of exponential-type functionals on $C_{a,b}[0,T]$.

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1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space. The concept of the analytic Fourier–Feynman transform (FFT) of functionals on Wiener space $C_0[0,T]$ was introduced by Brue [1] and developed in [2, 8, 11]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean space. In [8–10], Huffman *et al.* defined a convolution product (CP) for functionals on $C_0[0,T]$ and obtained various results involving the analytic FFT and the CP. For an introduction to the analytic FFT and further results, see [12] and the references therein.

In [5], Chang and Skoug defined a generalised analytic Fourier–Feynman transform (GFFT) for functionals on the very general function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$, induced by a generalised Brownian motion process (GBMP), was introduced by Yeh [13, 14] and used extensively in [3–7].

The representation for an inverse transform of the 'analytic' GFFT has been studied [4, 6, 7], but the inverse transform of the GFFT investigated in [4, 6, 7] is

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not an analytic transform. In this paper, we study other representations for the inverse transform of the analytic GFFT on the function space $C_{a,b}[0,T]$. To do this, we define a transform via the concept of the CP of functionals on $C_{a,b}[0,T]$. We next establish that the composition of the transforms studied in this paper acts like an inverse GFFT and that these transforms are vector space automorphisms of a vector space $\mathcal{E}(C_{a,b}[0,T])$. The vector space $\mathcal{E}(C_{a,b}[0,T])$ consists of exponential-type functionals on $C_{a,b}[0,T]$.

The Wiener process used in [1, 2, 8–11] is free of drift and stationary in time, while the stochastic process used in this paper, as well as in [3–7], is nonstationary in time and is subject to a drift.

2. Preliminaries

In this section we present a brief background and some well-known results about the function space $C_{a,b}[0,T]$ induced by a GBMP.

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$ and $a'(t)\in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. The GBMP Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. For more details, see [3, 5, 7, 13, 14]. By [14, Theorem 14.2], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence, the function space induced by Y is $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$, where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. We complete this function space to obtain the measure space $(C_{a,b}[0,T],\mathcal{W}(C_{a,b}[0,T]),\mu)$, where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

Remark 2.1. (i) The coordinate process defined by $e_t(x)=x(t)$ on $C_{a,b}[0,T]\times[0,T]$ is also the GBMP determined by $a(t)$ and $b(t)$.

(ii) The function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$, considered in [1, 2, 8–11], if and only if $a(t)\equiv 0$ and $b(t)=t$ for all $t\in[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided that ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho>0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided that $\mu(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided that F is defined on a scale-invariant measurable set and $F(\rho\,\cdot)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho>0$. If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., we write $F\approx G$. Note that the relation '$\approx$' is an equivalence relation.

Let $L^2_{a,b}[0,T]$ (see [3] and [5]) be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; that is,

$$L^2_{a,b}[0,T]=\left\{v:\int_0^T v^2(s)\,db(s)<+\infty \text{ and } \int_0^T v^2(s)\,d|a|(s)<+\infty\right\},$$

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where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b} = \int_0^T u(t)v(t)\,dm_{|a|,b}(t) \equiv \int_0^T u(t)v(t)\,d[b(t)+|a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$. In particular, note that $\|u\|_{a,b} \equiv \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$.

Let

$$C'_{a,b}[0,T] = \Big\{w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)\,db(s) \text{ for some } z \in L^2_{a,b}[0,T]\Big\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)\,db(s)$ for $t \in [0,T]$, we define the operator $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1,w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)\,db(t)$$

is a separable Hilbert space. The two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by (2.1). The inverse operator of D is given by

$$(D^{-1}z)(t) = \int_0^t z(s)\,db(s), \quad t \in [0,T].$$

In addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty. \tag{2.2}$$

The function $a : [0,T] \to \mathbb{R}$ satisfies (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w,a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)\,db(t) = \int_0^T z(t)\frac{a'(t)}{b'(t)}\,db(t) = \int_0^T z(t)\,da(t).$$

For more details, see [7].

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal set in $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ such that the De_n are of bounded variation on $[0,T]$. For $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we define the Paley–Wiener–Zygmund (PWZ) stochastic integral $(w,x)^\sim$ as follows:

$$(w,x)^\sim = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (w,e_j)_{C'_{a,b}} De_j(t)\,dx(t)$$

if the limit exists.

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Note that for $w_1, w_2 \in C'_{a,b}[0,T]$ with $Dw_j = z_j$, $j = 1, 2$,

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T z_1(t)z_2(t)\,db(t) \neq \int_0^T z_1(t)z_2(t)\,d[b(t) + |a|(t)] = (z_1, z_2)_{a,b}.$$

This fact tells us that the two Hilbert spaces $C'_{a,b}[0,T]$ and $L^2_{a,b}[0,T]$ are not isometric. Thus, in this sense, our definition of the PWZ stochastic integral is different from the definition given in [3–6]. But we emphasise the following fundamental facts. For each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w, x)^\sim$ exists for s-a.e. $x \in C_{a,b}[0,T]$. If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w, x)^\sim$ equals the Riemann–Stieltjes integral $\int_0^T z(t)\,dx(t)$. Furthermore, for each $w \in C'_{a,b}[0,T]$, $(w, x)^\sim$ is a Gaussian random variable with mean $(w, a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. Also, we note that for $w, x \in C'_{a,b}[0,T]$, $(w, x)^\sim = (w, x)_{C'_{a,b}}$.

We make use of the following integration formula:

$$\int_{\mathbb{R}} \exp\{-\alpha u^2 + \beta u\}\,du = \sqrt{\frac{\pi}{\alpha}} \exp\Big\{\frac{\beta^2}{4\alpha}\Big\} \tag{2.3}$$

for complex numbers α and β with $\mathrm{Re}(\alpha) > 0$.

3. Generalised Fourier–Feynman transform and convolution product

Denote the function space integral of a $\mathcal{W}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_{a,b}[0,T]} F(x)\,d\mu(x)$$

whenever the integral exists. Throughout this paper, $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively. For each $\lambda \in \widetilde{\mathbb{C}}$, $\lambda^{1/2}$ denotes the principal square root of λ; that is, $\lambda^{1/2}$ is always chosen to have nonnegative real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \widetilde{\mathbb{C}}_+$.

We are now ready to state the definitions of the generalised analytic Feynman integral and the L_1 analytic GFFT.

Definition 3.1. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional such that for each $\lambda > 0$, the function space integral

$$J(\lambda) = E_x[F(\lambda^{-1/2}x)] = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)\,d\mu(x)$$

exists and is finite. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ and, for $\lambda \in \mathbb{C}_+$, we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda).$$

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Let q be a nonzero real number and let F be a functional such that $E^{\mathrm{an}_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalised analytic Feynman integral of F with parameter q and we write

$$E^{\mathrm{anf}_q}[F] \equiv E_x^{\mathrm{anf}_q}[F(x)] = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} E_x^{\mathrm{an}_\lambda}[F].$$

Definition 3.2. Let F be a scale-invariant measurable functional on $C_{a,b}[0,T]$ and let q be a nonzero real number. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y+x)].$$

We define the L_1 analytic GFFT, $T_q^{(1)}(F)$, of F by the formula (if it exists)

$$T_q^{(1)}(F)(y) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda(F)(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

If $T_q^{(1)}(F)$ exists and if $F \approx G$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(G) \approx T_q^{(1)}(F)$. By the definitions of the generalised analytic Feynman integral and the L_1 analytic GFFT, it is easy to see that for a nonzero real number q,

$$T_q^{(1)}(F)(y) = E_x^{\mathrm{anf}_q}[F(y+x)]$$

if the integrals exist. In particular, if both integrals exist,

$$T_q^{(1)}(F)(0) = E_x^{\mathrm{anf}_q}[F(x)]. \tag{3.1}$$

Next we give the definition of the convolution product (CP) of functionals on the function space $C_{a,b}[0,T]$.

Definition 3.3. Let F and G be scale-invariant measurable functionals on $C_{a,b}[0,T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$, we define their CP, $(F*G)_\lambda$ (if it exists), by

$$(F*G)_\lambda(y) = \begin{cases} E_x^{\mathrm{an}_\lambda}[F((y+x)/\sqrt{2})\,G((y-x)/\sqrt{2})], & \lambda \in \mathbb{C}_+, \\ E_x^{\mathrm{anf}_q}[F((y+x)/\sqrt{2})\,G((y-x)/\sqrt{2})], & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases}$$

When $\lambda = -iq$, we denote $(F*G)_\lambda$ by $(F*G)_q$.

4. Exponential-type functionals

Let $\mathcal{E}$ be the class of all functionals having the form

$$\Psi_w(x) = \exp\{(w,x)^\sim\} \tag{4.1}$$

for each $w \in C'_{a,b}[0,T]$ and for μ-a.e. $x \in C_{a,b}[0,T]$. Given $q \in \mathbb{R} \setminus \{0\}$, let $\mathcal{E}_{q,a}$ be the class of all functionals having the form

$$\Psi_{w,q,a}(x) = K_{w,q,a}\Psi_w(x) \tag{4.2}$$

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for μ-a.e. $x \in C_{a,b}[0,T]$, where Ψ_w is given by (4.1) and $K_{w,q,a}$ is the complex number given by

$$K_{w,q,a} \equiv \exp\left\{\frac{i}{2q}\|w\|^2_{C'_{a,b}} + (-iq)^{-1/2}(w,a)_{C'_{a,b}}\right\}. \tag{4.3}$$

Also, given $q \in \mathbb{R} \setminus \{0\}$, let $\mathcal{E}_{q,-a}$ be the class of all functionals having the form

$$\Psi_{w,q,-a}(x) = K_{w,q,-a}\Psi_w(x) \tag{4.4}$$

for μ-a.e. $x \in C_{a,b}[0,T]$, where $K_{w,q,-a}$ is the complex number given by

$$K_{w,q,-a} \equiv \exp\left\{\frac{i}{2q}\|w\|^2_{C'_{a,b}} - (-iq)^{-1/2}(w,a)_{C'_{a,b}}\right\}. \tag{4.5}$$

The functionals given by (4.1) and linear combinations (with complex coefficients) of the Ψ_w are called the (partially) exponential-type functionals on $C_{a,b}[0,T]$. The functionals given by (4.2) and (4.4) are also partially exponential-type functionals.

Remark 4.1. The classes $\mathcal{E}$, $\mathcal{E}_{q,a}$ and $\mathcal{E}_{q,-a}$ are dense in $L_2(C_{a,b}[0,T])$.

We denote the set of all partially exponential-type functionals on $C_{a,b}[0,T]$ by $\mathcal{E}(C_{a,b}[0,T])$, that is,

$$\mathcal{E}(C_{a,b}[0,T]) = \operatorname{Span}\mathcal{E}.$$

For notational convenience, let $\Psi_{w,0,a}(x) = \Psi_{w,0,-a}(x) = \Psi_w(x)$ and $\mathcal{E}_{0,a} = \mathcal{E}_{0,-a} = \mathcal{E}$. Then we see that

$$\bigcup_{q\in\mathbb{R}}(\mathcal{E}_{q,a} \cup \mathcal{E}_{q,-a}) \subset \mathcal{E}(C_{a,b}[0,T]).$$

We also observe that $\mathcal{E}(C_{a,b}[0,T]) = \operatorname{Span}\mathcal{E}_{q,a} = \operatorname{Span}\mathcal{E}_{q,-a}$ for every $q \in \mathbb{R}$. The class $\mathcal{E}(C_{a,b}[0,T])$ of exponential-type functionals is a complex linear space and is dense in $L_2(C_{a,b}[0,T])$.

Remark 4.2. The class $\mathcal{E}(C_{a,b}[0,T])$ is a commutative (complex) algebra under the pointwise multiplication and with identity $\Psi_0 \equiv 1$.

Note that every exponential-type functional is scale-invariant measurable. Since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, $\mathcal{E}(C_{a,b}[0,T])$ can be regarded as the space of all s-equivalence classes of partially exponential-type functionals. Throughout this paper, we assume that (4.1) holds for s-a.e. $x \in C_{a,b}[0,T]$. More precisely, the quotient space $\mathcal{E}(C_{a,b}[0,T])/\approx$ is again denoted by the same symbol $\mathcal{E}(C_{a,b}[0,T])$ in the rest of this paper.

Theorem 4.3. *Let $\Psi_w \in \mathcal{E}(C_{a,b}[0,T])$ be an exponential-type functional of the form (4.1). Then, for all real $q \neq 0$, the L_1 analytic GFFT of Ψ_w, $T_q^{(1)}(\Psi_w)$, exists and is given by the formula*

$$T_q^{(1)}(\Psi_w)(y) = \Psi_{w,q,a}(y) \tag{4.6}$$

for s-a.e. $y \in C_{a,b}[0,T]$, where $\Psi_{w,q,a}$ is given by (4.2) above. Thus, $T_q^{(1)}(\Psi_w)$ is an element of $\mathcal{E}(C_{a,b}[0,T])$.

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PROOF. By the change of variable theorem and (2.3), for every $\rho > 0$,

$$E_x[\exp\{\rho(w, x)^\sim\}] = \exp\Big\{\frac{\rho^2}{2}\|w\|^2_{C'_{a,b}} + \rho(w, a)_{C'_{a,b}}\Big\}.$$

Thus we obtain, for all $\lambda > 0$ and s-a.e. $y \in C_{a,b}[0, T]$,

$$\begin{aligned} T_\lambda(\Psi_w)(y) &= \exp\{(w, y)^\sim\}E_x[\exp\{\lambda^{-1/2}(w, x)^\sim\}] \\ &= \exp\Big\{(w, y)^\sim + \frac{1}{2\lambda}\|w\|^2_{C'_{a,b}} + \lambda^{-1/2}(w, a)_{C'_{a,b}}\Big\}. \end{aligned}$$

The last expression is an analytic function of λ throughout $\mathbb{C}_+$ for all $y \in C_{a,b}[0, T]$. In view of Definition 3.2, $T^{(1)}_q(\Psi_w)$ exists and is given by (4.6) for all $q \in \mathbb{R} \setminus \{0\}$. □

REMARK 4.4. Let F be an element of $\mathcal{E}(C_{a,b}[0, T])$. Then F can be written as

$$F \approx \sum_{j=1}^{n} c_j\Psi_{w_j} \tag{4.7}$$

for a finite sequence $\{w_1, \ldots, w_n\}$ of functions in $C'_{a,b}[0, T]$ and a sequence $\{c_1, \ldots, c_n\}$ in $\mathbb{C}$. Since the analytic GFFT $T^{(1)}_q$ is linear, from (4.6), (4.2) and (4.3), it follows that $T^{(1)}_q : \mathcal{E}(C_{a,b}[0, T]) \to \mathcal{E}(C_{a,b}[0, T])$ is a vector space epimorphism.

The following corollary follows from (3.1), (4.6), (4.2) and (4.3).

COROLLARY 4.5. *Let Ψ_w be as in Theorem 4.3. Then, for all real $q \neq 0$, the generalised analytic Feynman integral $E^{\mathrm{anf}_q}[\Psi_w]$ of Ψ_w exists and is given by the right-hand side of (4.3). Thus, in view of (4.7), every exponential-type functional F is generalised analytic Feynman integrable.*

In our next theorem, we obtain the CP of functionals in $\mathcal{E}(C_{a,b}[0, T])$.

THEOREM 4.6. *Let Ψ_{w_1} and Ψ_{w_2} be exponential-type functionals of the form (4.1). Then the CP of Ψ_{w_1} and Ψ_{w_2}, $(\Psi_{w_1} * \Psi_{w_2})_q$, exists for all real $q \neq 0$ and is given by*

$$(\Psi_{w_1} * \Psi_{w_2})_q(y) = K_{(w_1-w_2)/\sqrt{2},q,a}\Psi_{(w_1+w_2)/\sqrt{2}}(y) \tag{4.8}$$

for s-a.e. $y \in C_{a,b}[0, T]$, where $K_{(w_1-w_2)/\sqrt{2},q,a}$ is the complex number given by (4.3) with w replaced by $(w_1 - w_2)/\sqrt{2}$. Furthermore, $(\Psi_{w_1} * \Psi_{w_2})_q$ is an element of $\mathcal{E}(C_{a,b}[0, T])$.

PROOF. Proceeding as in the proof of Theorem 4.3, for all $\lambda > 0$ and for s-a.e. $y \in C_{a,b}[0, T]$,

$$\begin{aligned} (\Psi_{w_1} * \Psi_{w_2})_\lambda(y) &= E_x\Big[\Psi_{w_1}\Big(\frac{y + \lambda^{-1/2}x}{\sqrt{2}}\Big)\Psi_{w_2}\Big(\frac{y - \lambda^{-1/2}x}{\sqrt{2}}\Big)\Big] \\ &= \exp\Big\{\frac{(w_1 + w_2, y)^\sim}{\sqrt{2}}\Big\}E_x\Big[\exp\Big\{\frac{1}{\sqrt{\lambda}}\Big(\frac{w_1 - w_2}{\sqrt{2}}, x\Big)^\sim\Big\}\Big] \\ &= \exp\Big\{\Big(\frac{w_1 + w_2}{\sqrt{2}}, y\Big)^\sim\Big\}\exp\Big\{\frac{1}{2\lambda}\Big\|\frac{w_1 - w_2}{\sqrt{2}}\Big\|^2_{C'_{a,b}} + \lambda^{-1/2}\Big(\frac{w_1 - w_2}{\sqrt{2}}, a\Big)_{C'_{a,b}}\Big\}. \end{aligned}$$

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But the last expression is an analytic function of λ throughout $\mathbb{C}_+$ for all $y \in C_{a,b}[0,T]$. Hence, in view of Definition 3.3, $(\Psi_{w_1} * \Psi_{w_2})_q$ exists and is given by

$$\begin{aligned}
&(\Psi_{w_1} * \Psi_{w_2})_q(y) \\
&\quad = \exp\Big\{\Big(\frac{w_1+w_2}{\sqrt{2}}, y\Big)^{\sim} + \frac{i}{2q}\Big\|\frac{w_1-w_2}{\sqrt{2}}\Big\|^2_{C'_{a,b}} + (-iq)^{-1/2}\Big(\frac{w_1-w_2}{\sqrt{2}}, a\Big)_{C'_{a,b}}\Big\} \qquad (4.9)\\
&\quad = K_{(w_1-w_2)/\sqrt{2},q,a}\Psi_{(w_1+w_2)/\sqrt{2}}(y)
\end{aligned}$$

for all $q \in \mathbb{R}\setminus\{0\}$ and s-a.e. $y \in C_{a,b}[0,T]$. Since $\mathcal{E}(C_{a,b}[0,T])$ is a complex algebra,

$$\begin{aligned}
(\Psi_{w_1} * \Psi_{w_2})_q &\approx K_{(w_1-w_2)/\sqrt{2},q,a}\Psi_{(w_1+w_2)/\sqrt{2}} \\
&\approx K_{(w_1-w_2)/\sqrt{2},q,a}\Psi_{w_1/\sqrt{2}}(y)\Psi_{w_2/\sqrt{2}}
\end{aligned} \qquad (4.10)$$

is an element of $\mathcal{E}(C_{a,b}[0,T])$. □

REMARK 4.7. The class $\mathcal{E}(C_{a,b}[0,T])$ of exponential-type functionals is a non-commutative algebra with the operation $*$.

The following corollary follows immediately from (4.10).

COROLLARY 4.8. *Let $\Psi_w \in \mathcal{E}(C_{a,b}[0,T])$ be given by (4.1). Then, for all real $q \neq 0$, $(\Psi_w * \Psi_0)_q$ and $(\Psi_0 * \Psi_w)_q$ are elements of $\mathcal{E}(C_{a,b}[0,T])$. Also, it follows that*

$$(\Psi_w * \Psi_0)_q(y) = K_{w/\sqrt{2},q,a}\Psi_{w/\sqrt{2}}(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$, where $K_{w/\sqrt{2},q,a}$ is the complex number given by (4.3) with w replaced by $w/\sqrt{2}$, and

$$(\Psi_0 * \Psi_w)_q(y) = K_{w/\sqrt{2},q,-a}\Psi_{w/\sqrt{2}}(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$, where $K_{w/\sqrt{2},q,-a}$ is the complex number given by (4.5) with w replaced by $w/\sqrt{2}$.

Applying simple modifications of the proofs of Theorems 4.3 and 4.6, and using the parallelogram equality

$$\|w_1+w_2\|^2_{C'_{a,b}} + \|w_1-w_2\|^2_{C'_{a,b}} = 2(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}}),$$

we can obtain the following corollary.

COROLLARY 4.9. *Let Ψ_{w_1} and Ψ_{w_2} be as in Theorem 4.6. Then, for all real $q \neq 0$, $T_q^{(1)}((\Psi_{w_1} * \Psi_{w_2})_q)$, $T_q^{(1)}((\Psi_{w_1} * \Psi_0)_q)$ and $T_q^{(1)}((\Psi_0 * \Psi_{w_2})_q)$ all exist and are elements of $\mathcal{E}(C_{a,b}[0,T])$. Furthermore, for all real $q \neq 0$,*

$$T_q^{(1)}((\Psi_{w_1} * \Psi_{w_2})_q)(y) = T_q^{(1)}((\Psi_{w_1} * \Psi_0)_q)(y)T_q^{(1)}((\Psi_0 * \Psi_{w_2})_q)(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

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Remark 4.10. Any functionals F and G in $\mathcal{E}(C_{a,b}[0,T])$ can be represented as linear combinations of exponential-type functionals. Thus, for all real $q \neq 0$, $(F * G)_q$, $(F * \Psi_0)_q$, $(\Psi_0 * G)_q$, $T_q^{(1)}((F * G)_q)$, $T_q^{(1)}((F * \Psi_0)_q)$ and $T_q^{(1)}((\Psi_0 * G)_q)$ all exist and are elements of $\mathcal{E}(C_{a,b}[0,T])$. Furthermore, for all real $q \neq 0$,

$$T_q^{(1)}((F * G)_q)(y) = T_q^{(1)}((F * \Psi_0)_q)(y) T_q^{(1)}((\Psi_0 * G)_q)(y) \tag{4.11}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

5. A representation for the inverse GFFT

In [1, 2, 8–11], the authors established the existence of the analytic FFT $T_q(F)$ for F in several large classes of functionals on the Wiener space $C_0[0,T]$. Specifically, they demonstrated that for all real $q \neq 0$,

$$T_{-q}(T_q(F)) \approx F,$$

where T_q denotes the analytic FFT on $C_0[0,T]$. That is, the analytic FFT 'T_q' acting on various classes of functionals on $C_0[0,T]$ has the inverse transform 'T_{-q}'. On the other hand, for almost every functional F on $C_{a,b}[0,T]$, $T_{-q}^{(1)}(T_q^{(1)}(F)) \neq F$. This raises the question of how to construct an inverse transform of the analytic GFFT.

There have been several recent attempts to represent the inverse GFFT [4, 6, 7]. In this section, we define another function space transform via the CP of functionals on $C_{a,b}[0,T]$. We then investigate fundamental relationships among the analytic GFFT, the new function space transform and the CP. We also construct an inverse transform of the analytic GFFT for the functionals in the algebra $\mathcal{E}(C_{a,b}[0,T])$. The following observations will be very useful in the development of our inverse GFFT.

In view of Definitions 3.2 and 3.3,

$$T_q^{(1)}(F)(y) = E_x^{\mathrm{anf}_q}[F(y+x)] = E_x^{\mathrm{anf}_q}\Big[F\Big(\sqrt{2}\Big(\frac{y+x}{\sqrt{2}}\Big)\Big)\Big] = (F(\sqrt{2}\,\cdot) * \Psi_0)_q(y).$$

However, by the effect of the drift function $a(t)$ of the GBMP introduced in Section 2,

$$E_x^{\mathrm{an}_\lambda}[F(x)] \neq E_x^{\mathrm{an}_\lambda}[F(-x)]$$

for almost every functional F on $C_{a,b}[0,T]$. This yields

$$T_q^{(1)}(F)(y) \neq E^{\mathrm{anf}_q}[F(y-x)]$$

and so the CP of functionals on $C_{a,b}[0,T]$ is not commutative.

The above discussion leads us to the following definition for the inverse GFFT on the function space $C_{a,b}[0,T]$. Given a functional F on $C_{a,b}[0,T]$, let

$$T_q^{+}(F)(y) \equiv (F(\sqrt{2}\,\cdot) * \Psi_0)_q(y) \tag{5.1}$$

and

$$T_q^{-}(F)(y) \equiv (\Psi_0 * F(\sqrt{2}\,\cdot))_q(y). \tag{5.2}$$

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We have the identities

$$T_q^+(F)(y) = E_x^{\mathrm{anf}_q}[F(y+x)] = T_q^{(1)}(F)(y), \tag{5.3}$$

$$T_q^-(F)(y) = E_x^{\mathrm{anf}_q}[F(y-x)],$$

$$(F * \Psi_0)_q(y) = T_q^+(F(\cdot\,/\sqrt{2}))(y), \tag{5.4}$$

$$(\Psi_0 * F)_q(y) = T_q^-(F(\cdot\,/\sqrt{2}))(y) \tag{5.5}$$

for s-a.e. $y \in C_{a,b}[0,T]$ if the transforms exist.

LEMMA 5.1. *Let $\Psi_w \in \mathcal{E}(C_{a,b}[0,T])$ be as in Theorem 4.3. Then:*

(i) *for all real $q \neq 0$ and s-a.e. $y \in C_{a,b}[0,T]$,*

$$T_q^+(\Psi_w)(y) = T_q^{(1)}(\Psi_w)(y) = \Psi_{w,q,a}(y), \tag{5.6}$$

where $\Psi_{w,q,a}$ is given by (4.2); and

(ii) *for all real $q \neq 0$ and s-a.e. $y \in C_{a,b}[0,T]$,*

$$T_q^-(\Psi_w)(y) = \Psi_{w,q,-a}(y), \tag{5.7}$$

where $\Psi_{w,q,-a}$ is given by (4.4).

PROOF. Equation (5.6) follows from (5.1) and (5.3) with F replaced by Ψ_w and (4.6); (5.7) follows from (5.2) with F replaced by Ψ_w. □

REMARK 5.2. (i) Using (4.7), the fact that T_q^- is linear and (5.7), one can see that $T_q^- : \mathcal{E}(C_{a,b}[0,T]) \to \mathcal{E}(C_{a,b}[0,T])$ is a vector space epimorphism.

(ii) By a close examination, for each exponential-type functional $F \in \mathcal{E}(C_{a,b}[0,T])$,

$$T_q^+(T_q^-(F)) \approx T_q^-(T_q^+(F)),$$

that is,

$$T_q^{(1)}(T_q^-(F)) \approx T_q^-(T_q^{(1)}(F)).$$

Our next result now follows easily from (4.7), (4.11), (5.4), (5.5), (5.6) and (5.7).

PROPOSITION 5.3. *Let F and G be elements of $\mathcal{E}(C_{a,b}[0,T])$. Then, for all real $q \neq 0$,*

$$T_q^{(1)}((F*G)_q)(y) = T_q^{(1)}\Big(T_q^{(1)}\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)\Big)(y)\,T_q^{(1)}\Big(T_q^-\Big(G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)\Big)(y)$$

and

$$T_q^-((F*G)_q)(y) = T_q^-\Big(T_q^{(1)}\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)\Big)(y)\,T_q^-\Big(T_q^-\Big(G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)\Big)(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$, respectively.

Theorem 5.4 below follows easily from (5.6) and (5.7) and the fact that $\mathcal{E}(C_{a,b}[0,T])$ is the linear span of the exponential-type functionals.

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THEOREM 5.4. *Let F be an element of $\mathcal{E}(C_{a,b}[0,T])$. Then, for all real $q \neq 0$,*

$$T_{-q}^{-}(T_{-q}^{(1)}(T_q^{-}(T_q^{(1)}(F)))) \approx F.$$

That is, in view of (4.7), (5.6) and (5.7), the L_1 analytic GFFT, $T_q^{(1)} : \mathcal{E}(C_{a,b}[0,T]) \to \mathcal{E}(C_{a,b}[0,T])$, is a vector space automorphism and the GFFT $T_q^{(1)}$ has the inverse transform

$$\{T_q^{(1)}\}^{-1} = T_{-q}^{-} \circ T_{-q}^{(1)} \circ T_q^{-}.$$

REMARK 5.5. We finally emphasise that the inverse transform of the L_1 analytic GFFT $T_q^{(1)}$ cannot be represented as a single analytic GFFT. However, we have the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{E}(C_{a,b}[0,T]) \arrow[rrr, "T_q^{(1)} \equiv T_q^{+}"] & & & \mathcal{E}(C_{a,b}[0,T]) \arrow[d, "T_q^{-}"] \arrow[lll, dotted, bend left, "\{T_q^{(1)}\}^{-1} \equiv T_{-q}^{-} \circ T_{-q}^{(1)} \circ T_q^{-}"] \\
\mathcal{E}(C_{a,b}[0,T]) \arrow[u, "T_{-q}^{-}"] & & & \mathcal{E}(C_{a,b}[0,T]) \arrow[lll, "T_{-q}^{(1)} \equiv T_{-q}^{+}"]
\end{tikzcd}$$

In Theorem 5.4, we see that the composition of transforms $T_{-q}^{-} \circ T_{-q}^{(1)} \circ T_q^{-}$ on $\mathcal{E}(C_{a,b}[0,T])$ is an inverse transform of $T_q^{(1)}$. Moreover, we have the six possibilities for the inverse transform of $T_q^{(1)}$.

$$\begin{aligned}
T_{-q}^{-} \circ T_{-q}^{(1)} \circ T_q^{-} &= T_{-q}^{-} \circ T_q^{-} \circ T_{-q}^{(1)} = T_{-q}^{(1)} \circ T_{-q}^{-} \circ T_q^{-} \\
&= T_{-q}^{(1)} \circ T_q^{-} \circ T_{-q}^{-} = T_q^{-} \circ T_{-q}^{-} \circ T_{-q}^{(1)} = T_q^{-} \circ T_{-q}^{(1)} \circ T_{-q}^{-}.
\end{aligned}$$

In a similar way to the discussion of the inverse transform of $T_q^{(1)}$, it also follows that

$$\{T_q^{-}\}^{-1} = T_{-q}^{(1)} \circ T_{-q}^{-} \circ T_q^{(1)}.$$

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