

Simple Decompositions of the Exceptional Jordan Algebra

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Abstract. This paper presents some results on the simple exceptional Jordan algebra over an algebraically closed field Φ of characteristic not 2. Namely an example of simple decomposition of $H(O_3)$ into the sum of two subalgebras of the type $H(Q_3)$ is produced, and it is shown that this decomposition is the only one possible in terms of simple subalgebras.

1 Introduction

Let Φ be an algebraically closed field of characteristic not 2. Then all composition algebras of dimension ≥ 2 are split in the sense of having zero divisors. Therefore, they are uniquely determined by their dimensions. In particular, there exists only one octonion algebra over Φ , denoted O . The algebra of split octonions can be represented as a direct vector sum of Q and νQ , where Q is the split quaternion algebra and $\nu^2 = 1$, $O = Q \oplus \nu Q$. Multiplication in O is defined by

$$(q + rv)(s + tv) = (qs + \bar{t}r) + (tq + r\bar{s})\nu$$

for q, r, s, t in Q . The involution in O is $x = q + rv \rightarrow \bar{x} = \bar{q} - rv$. Therefore, O is a vector space of dimension 8 and degree 2. Next we consider the set $H_3(O)$ of Hermitian 3×3 matrices over Φ . An arbitrary element A from this set has the form

$$A = \begin{pmatrix} \alpha & x & y \\ \bar{x} & \beta & z \\ \bar{y} & \bar{z} & \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \Phi$, $x, y, z \in Q$. Obviously, $H_3(O)$ is a vector space of dimension 27. If $x[ij]$ denotes $xE_{ij} + \bar{x}E_{ji}$ where $x \in O$, E_{ij} a standard matrix unit, then $A = \alpha E_{11} + \beta E_{22} + \gamma E_{33} + x[12] + y[13] + z[23]$.

Let O be an octonion algebra with canonical involution “ $\bar{}$ ”. Let $a, b, c, d, x, z \in O$, and recall [2]:

- (i) $x^2 - t(x)x + n(x) = 0$ where $n(x) = x\bar{x} = \bar{x}x \in \Phi$;
- (ii) $t(x) = t(x, 1)$, where $t(a, b) := n(a+b) - n(a) - n(b) = a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a = t(\bar{a}, b)$;
- (iii) $\bar{a}(ab) = n(a)b = (ba)\bar{a}$.

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Van der Blij and Springer [1] obtained the standard basis $\{x_i, y_i \mid 1 \leq i \leq 4\}$ of the split octonions, which satisfies the following identities:

$$\begin{aligned}x_1 x_i &= x_i, & y_1 y_i &= y_i, & 1 \leq i \leq 4, \\x_i y_i &= -x_1, & x_i x_{i+1} &= y_{i+2}, & y_i y_{i+1} &= x_{i+2}, & 2 \leq i \leq 4,\end{aligned}$$

where the indices are taken modulo 3 and all other products are zero or obtained by applying the canonical involution to one of the above.

2 Example of a Simple Decomposition of $H(O_3)$

To exhibit an example of a simple decomposition of $H(O_3)$ we need the following lemma.

Lemma 1 *The set \mathcal{S} consisting of all hermitian matrices of the form*

$$(1) \quad \begin{pmatrix} \alpha & xv & yv \\ -xv & \beta & z \\ -yv & \bar{z} & \gamma \end{pmatrix},$$

where $x, y, z \in Q$ and Q is a simple algebra of the type $H(Q_3)$.

Proof First, it is easy to verify that \mathcal{S} is closed with respect to the product $2X \circ Y = (X \cdot Y + Y \cdot X)$, where $X \cdot Y$ is the ordinary matrix product in O_3 . Therefore, \mathcal{S} is a Jordan subalgebra of $H(O_3)$.

Next we are going to show that \mathcal{S} has no proper non-zero ideals. Assume the contrary, that is, there exists an ideal I such that $I \neq \{0\}$, $I \neq \mathcal{S}$. Hence we can select a non-zero a_0 from I of the form $a_0 = \alpha_0 E_{11} + \beta_0 E_{22} + \gamma_0 E_{33} + x_0 v[12] + y_0 v[13] + z_0[23]$. Let one of the off-diagonal elements be non-zero. For clarity, $x_0 \neq 0$. Then multiplying a_0 by E_{11} and E_{22} (in the Jordan sense), we obtain $a_1 = x_0 v[12]$, which is also an element of I . First we assume that $n(x_0) = x_0 \bar{x}_0 = 0$. Then for any $b = \alpha E_{11} + \beta E_{22} + \gamma E_{33} + x[12] + y[13] + z[23]$ and $c = \alpha' E_{11} + \beta' E_{22} + \gamma' E_{33} + x'[12] + y'[13] + z'[23]$,

$$(2) \quad b \circ c = \alpha'' E_{11} + \beta'' E_{22} + \gamma'' E_{33} + b_{12}[12] + b_{13}[13] + b_{23}[23],$$

where

$$\begin{aligned}b_{12} &= (\alpha x' + \beta' x + y z' + \alpha' x + \beta x' + y' z) v, \\b_{13} &= (\alpha y' + x \bar{z}' + \gamma' y + \alpha' y + x' \bar{z} + \gamma y') v, \\b_{23} &= \bar{y}' x + \beta z' + \gamma' z + \bar{y} x' + \beta' z + z' \gamma, \\ \alpha'' &= 2\alpha \alpha' + t(x, x') + t(y, y'), \\ \beta'' &= 2\beta \beta' + t(x, x') + t(z, z'), \\ \gamma'' &= 2\gamma \gamma' + t(z', z) + t(y, y').\end{aligned}$$

Set $c = a_1 = x_0v[12]$. If we choose b such that $\alpha = 0$ and the other coefficients of b are non-zero, then $b \circ c$ has the form (2) where $b_{12} = \beta x_0v, b_{13} = x_0\bar{z}v, b_{23} = \bar{y}x_0, \alpha'' = \beta'' = t(x_0, x), \gamma'' = 0$. Again multiplying $b \circ c$ by $d = x''[12] + z''[13], x'', z'' \neq 0$, we obtain

$$(b \circ c) \circ d = \alpha'''E_{11} + \beta'''E_{22} + \gamma'''E_{33} + c_{12}[12] + c_{13}[13] + c_{23}[23]$$

such that $c_{12} = (\alpha''x'' + b_{13}z'' + \beta''x'')v = (2t(x_0, x'')x'' + x_0\bar{z}z'')v$. Next we are going to show that we can choose x'' in such a way that $n(2t(x_0, x'')x'' + x_0\bar{z}z'') \neq 0$. Since z'' is an arbitrary element, we can set $z'' = 0$. Consider $r = 2t(x_0, x'')x''$. Since $t(x_0, x'')$ is a non-singular bilinear form and $x_0 \neq 0$, there exists $q \in Q$ such that $t(x_0, q) \neq 0$. If $n(q) \neq 0$, then $n(2t(x_0, q)q) = 4t(x_0, q)^2n(q) \neq 0$. Therefore, $x'' = q$ is a required element. If $n(q) = 0$, then we consider $q + \delta$ where $\delta \in \Phi$. Notice that $n(q + \delta) = n(q) + \delta t(q) + \delta^2 = \delta(t(q) + \delta) \neq 0$ if $\delta \neq 0, -t(q)$, and $t(x_0, q + \delta) = t(x_0, q) + \delta t(x_0)$. If $t(x_0) = 0$, then $t(x_0, q + \delta) \neq 0$ for any $\delta \in \Phi$. If $t(x_0) \neq 0$, then $t(x_0, q + \delta) \neq 0$ for any $\delta \neq t(x_0, q)/t(x_0)$. Hence, if $\delta \neq 0, -t(q), t(x_0, q)/t(x_0)$, then $n(2t(x_0, q + \delta), q + \delta) \neq 0$. Therefore, $x'' = q + \delta$ is a required element. As shown above we can always choose x'' such that $n(c_{12}) \neq 0$. Then performing multiplication of $(b \circ c) \circ d$ by E_{11} and E_{22} , we come to $\bar{a}_0 = c_{12}v[12]$, where $n(c_{12}) \neq 0$.

These considerations allow us to assume that $n(x_0) \neq 0$, therefore we have $a_0^2 = -n(x_0)(E_{11} + E_{22})$, with $n(x_0) \in \Phi, n(x_0) \neq 0$. Hence $E_{11} + E_{22} \in I$. It follows that for any $\alpha, \beta \in \Phi, x \in Q$ we have $\alpha E_{11} + \beta E_{22} + xv[12] \in I$. Then it is easily seen that $I = \mathcal{S}$, a contradiction.

As a result, \mathcal{A} is a simple special Jordan algebra of degree 3 and dimension 15. Hence, $\mathcal{S} \cong H(\mathcal{Q}_3)$, and this proves the lemma. ■

Before we continue our discussion we need a few facts concerning $\text{Aut } H(O_3)$. Let $\{e_1, e_2, e_3\}$ be a reducing set of idempotents of $H(O_3)$, and let $H(O_3) = \sum_{ij} J_{ij}$ be the corresponding Pierce decomposition. Set $\mathcal{C} = \Phi(e_1 + e_2) \oplus J_{12}, \dim \mathcal{C} = 9$. Then \mathcal{C} is an algebra of the bilinear form $Q(x, y)$ where $Q(x) = \frac{1}{2}t(x^2)$. From [2], if $v_1, \dots, v_{2r} \in \mathcal{C}, \Pi_i Q(v_i) = 1$, then $\eta = U_{\theta(v_1)}U_{\theta(v_2)} \dots U_{\theta(v_{2r})}$, where $\theta(x) = e_3 + x$ is an automorphism of $H(O_3)$ that leaves e_3 fixed.

Let $e_3 = E_{33}$, and

$$(3) \quad a = \begin{pmatrix} 1/\sqrt{3} & (1+iv)/\sqrt{3} & 0 \\ (1-iv)/\sqrt{3} & -1/\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(4) \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Obviously, $a, b \in \mathcal{C}$, and $Q(a) = 1, Q(b) = 1$, therefore $\varphi = U_{\theta(a)}U_{\theta(b)}$ is an automorphism of $H(O_3)$.

Example 1 There exists a simple decomposition of $H(O_3)$ of the form $H(O_3) = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \cong H(\mathcal{Q}_3)$.

Proof First we notice that by [2] any two subalgebras of the type $H(Q_3)$ are conjugate under automorphism of $H(O_3)$. Therefore, we can assume that the first subalgebra has the canonical form

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & x & y \\ \bar{x} & \beta & z \\ \bar{y} & \bar{z} & \gamma \end{pmatrix} \right\},$$

where $\alpha, \beta, \gamma \in \Phi$, $x, y, z \in Q$. Next we choose a of the form (3) and b of the form (4). As shown above, $\varphi = U_{\theta(a)}U_{\theta(b)}$ is an automorphism of $H(O_3)$. Recall $xU_{\theta(y)} = 2(x \circ \theta(y)) \circ \theta(y) - x \circ \theta(y)^2$. Notice that $(a + E_{33})^2 = (b + E_{33})^2 = E_{11} + E_{22} + E_{33}$. Let $\mathcal{B} = \varphi(\mathcal{S})$ where \mathcal{S} is an algebra of the form (1). Any $b \in \mathcal{B}$ has the form $b = \beta_1 E_{11} + \beta_2 E_{22} + \beta_3 E_{33} + \sum_{ij} \bar{b}_{ij} [ij]$, where

$$\begin{aligned} \bar{b}_{12} &= xv - \alpha(1 - iv), \quad \alpha \in \Phi, \\ \bar{b}_{13} &= 1/\sqrt{3}\bar{z} - \left(1 + \frac{2}{\sqrt{3}}\right)y + 1/\sqrt{3}\bar{z}i)v, \\ \bar{b}_{23} &= -\left(1 - \frac{2}{\sqrt{3}}\right)\bar{z} + \frac{2}{\sqrt{3}}yv(1 + iv). \end{aligned}$$

It is easy to verify that $H(O_3) = \mathcal{A} + \mathcal{B}$. ■

Remark The exceptional Jordan algebra can be represented as the sum of two special algebras.

Racine [4] has determined the maximal (unital) subalgebras of finite-dimensional special simple linear Jordan algebras. In [5] he completes the determination of the maximal subalgebras of finite-dimensional simple Jordan algebras by considering the exceptional algebras. The main result in [5] is the following classification theorem.

Theorem 2 (Racine) *A subalgebra \mathcal{B} of a finite-dimensional simple exceptional central quadratic Jordan algebra \mathcal{J} is maximal if and only if it is isomorphic to an algebra of one of the following types:*

- (i) $\mathcal{A}^{(+)}$, \mathcal{A} an associative division algebra of degree 3 over its center Φ ;
- (ii) $H(\mathcal{A}, *)$, \mathcal{A} an associative division algebra of degree 3 over its center E , an involution $*$ of the second kind with $\Phi = H(E, *)$;
- (iii) $H(\mathcal{L}_3, J)$, \mathcal{L} is a division quaternion algebra;
- (iv) $\mathcal{J}_1(e) \oplus \mathcal{J}_0(e) = \Phi e \oplus \mathcal{J}_0(e)$, e a primitive idempotent;
- (v) $H(\mathcal{S}_3)$, $\mathcal{S} = \Phi x_0 \oplus \Phi x_1 \oplus \Phi x_2 \oplus \Phi y_0 \oplus \Phi y_1 \oplus \Phi y_3$;
- (vi) $\mathcal{J}(\Phi z)$, the idealizer of Φz .

Note that their dimensions are 9, 9, 15, 11, 21 and 18, respectively. The first three are special simple, while the fourth is semisimple and special. The last two are exceptional. Notice the Wedderburn factor of $H(\mathcal{S}_3)$ is $H(\mathcal{L}_3)$ where \mathcal{L} is a quaternion algebra, and the Wedderburn factor of $\mathcal{J}(\Phi z)$ is $S = \Phi e_1 \oplus \Phi e_2 \oplus (\Phi x_1 \oplus \Phi x_2 \oplus \Phi x_3 \oplus \Phi y_1 \oplus \Phi y_2 \oplus \Phi y_3)$ [12] (we use here the basis of [1]).

Theorem 3 *Decomposition of the form $H(O_3) = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \cong H(Q_3)$ is the only possible simple decomposition in $H(O_3)$ in terms of simple subalgebras.*

Proof Assume the contrary, that is, there exists a simple decomposition different from the one in Example 2. Let $H(O_3) = \mathcal{A} + \mathcal{B}$, where \mathcal{A}, \mathcal{B} are simple subalgebras, at least one of them not of the type $H(Q_3)$. Recall that $\dim H(F_3) = 6$, $\dim F_3^{(+)} = 9$, $\dim H(Q_3) = 15$. Then by the dimension arguments, one of the above subalgebras, for example \mathcal{B} , should have the type $J(f, 1)$ (the algebra of a bilinear form f with the identity 1).

According to Racine's classification of maximal subalgebras, \mathcal{B} cannot be maximal in $H(O_3)$. Therefore, we can cover \mathcal{B} with some maximal subalgebra \mathcal{M} of one of the above types. If \mathcal{M} has type (1) or (2), then $\dim \mathcal{M} = 9$. Hence, $\dim \mathcal{B} \leq 9$. Likewise if \mathcal{M} has the type (4), then $\dim \mathcal{M} \leq 11$. Therefore, $\dim \mathcal{B} \leq 11$.

Now if $\mathcal{M} \cong H(Q_3) \cong H(F_6, j)$, then $\dim \mathcal{B} \leq 4$ (see [5]).

For the last two possibilities for \mathcal{M} we recall that $H(S_3)$ and $\mathcal{J}(Fz)$ are both exceptional Jordan algebras with the non-zero radicals:

$$H(S_3) = H(Q_3) \oplus R,$$

$$\mathcal{J}(Fz) = S \oplus R,$$

where $S = \Phi e_1 \oplus \Phi e_2 \oplus (\Phi x_1 \oplus \Phi x_2 \oplus \Phi x_3 \oplus \Phi y_1 \oplus \Phi y_2 \oplus \Phi y_3)$ [12], $\dim S \leq 8$. By Malcev's theorem (see [3]) if $\mathcal{B} \subseteq \mathcal{M} = S \oplus R$, there exists an automorphism $\varphi: \mathcal{M} \rightarrow \mathcal{M}$, such that $\varphi(\mathcal{B}) \subseteq S$. Hence, in the case of $\mathcal{M} = H(S_3)$, $\dim \mathcal{B} \leq 4$, and in the case of $\mathcal{M} = \mathcal{J}(\Phi z)$, $\dim \mathcal{B} \leq 8$.

As a result, if \mathcal{B} of the type $J(f, 1)$ is a subalgebra of $H(O_3)$, then $\dim \mathcal{B} \leq 11$. Further, \mathcal{A} has either the type $J(f, 1)$ or $H(Q_3)$. Hence, $\dim \mathcal{A} \leq 15$. Thus, by the dimension arguments, the decomposition does not hold, and the theorem is proved. ■

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