Combinatorics, Probability and Computing (2024), **33**, pp. 270–285
doi:10.1017/S0963548323000378

ARTICLE

Spanning trees in graphs without large bipartite holes

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(Received 8 February 2023; revised 6 October 2023; accepted 9 October 2023; first published online 14 November 2023)

Abstract
We show that for any $\varepsilon > 0$ and $\Delta \in \mathbb{N}$, there exists $\alpha > 0$ such that for sufficiently large n, every n-vertex graph G satisfying that $\delta(G) \geq \varepsilon n$ and $e(X, Y) > 0$ for every pair of disjoint vertex sets $X, Y \subseteq V(G)$ of size αn contains all spanning trees with maximum degree at most Δ. This strengthens a result of Böttcher, Han, Kohayakawa, Montgomery, Parczyk, and Person.

Keywords: Bipartite holes; spanning trees; absorbing method
2020 MSC Codes: Primary: 05C35; Secondary: 05C05

1. Introduction

Determining the minimum degree condition for the existence of spanning structures is a central problem in extremal graph theory. The first result of this direction is Dirac's theorem [7] in 1952 which states that for $n \geq 3$, every n-vertex graph G with $\delta(G) \geq \frac{n}{2}$ contains a Hamiltonian cycle. Komlós et al. [24] proved that an n-vertex graph G with $\delta(G) \geq (\frac{1}{2} + o(1))n$ contains a copy of every bounded-degree spanning tree, and in [25], the result is extended to trees with maximum degree $O(\frac{n}{\log n})$. Another notable extension is the bandwidth theorem of Böttcher et al. [4], which finds the (asymptotically) optimal minimum degree condition forcing spanning subgraphs with bounded chromatic number and sublinear bandwidth and resolves a conjecture of Bollobás and Komlós [23].

Note that the extremal graphs in the above results usually have large independent sets, which makes them far from being typical. Hence a natural project is to study how the degree conditions drop if we forbid large independent sets from the host graph. Balogh et al. [1] initiated this study by proving if G is an n-vertex graph with $\delta(G) \geq (\frac{1}{2} + o(1))n$ and $\alpha(G) = o(n)$, then G contains a K_3-factor. This result is asymptotically optimal and requires a weaker degree bound than $\delta(G) \geq \frac{2}{3}n$ from the Corrádi–Hajnal theorem [6]. Nenadov and Pehova [35] extended this result to the case of K_r-factor and asked for the best possible minimum degree condition on G with $\alpha(G) = o(n)$ that guarantees a K_r-factor. Knierim and Su [22] resolved this question for $r \geq 4$ by showing $\delta(G) \geq (\frac{r-2}{r} + o(1))n$ is asymptotically best possible. Nenadov and Pehova [35] also generalised $\alpha(G)$ into ℓ-independence number and this inspires several recent works [5, 14, 15].

Jie Hu: Supported by Natural Science Foundation of China (12131013, 12161141006). Guanghui Wang: Research supported by Natural Science Foundation of China (12231018) and Young Taishan Scholars probgram of Shandong Province (201909001). Donglei Yang: Supported by the China Post-doctoral Science Foundation (2021T140413), Natural Science Foundation of China (12101365) and Natural ScienceFoundation of Shandong Province (ZR2021QA029).

However, just excluding large independent sets is not enough to guarantee the existence of large *connected* subgraph. The union of two disjoint copies of $K_{\frac{n}{2}}$ has independence number two, but it does not contain any connected subgraphs with more than $\frac{n}{2}$ vertices. Hence it is necessary to impose stronger conditions to overcome this. The following notion of bipartite hole was introduced by McDiarmid and Yolov [33] in the study of Hamilton cycles. Given two disjoint vertex sets A and B in a graph G, we use $E_G(A, B)$ to denote the set of edges joining A and B, and let $e_G(A, B) = |E_G(A, B)|$. An (s, t)-*bipartite-hole* in G consists of two disjoint sets $S, T \subseteq V(G)$ with $|S| = s$ and $|T| = t$ such that $e_G(S, T) = 0$. The *bipartite-hole number* $\widetilde{\alpha}(G)$ refers to the maximum integer r such that G contains an (s, t)-bipartite-hole for every pair of nonnegative integers s and t with $s + t = r$. In this paper, we adopt a slightly different notion of bipartite-hole number due to Nenadov and Pehova [35].

Definition 1.1. *The* bipartite independence number $\alpha^*(G)$ *is the maximum integer t such that G contains a (t, t)-bipartite-hole.*

It is clear from the definition that $2\alpha^*(G) + 1 \geq \widetilde{\alpha}(G) \geq \alpha^*(G) + 1$. McDiarmid and Yolov [33] showed that $\delta(G) \geq \widetilde{\alpha}(G)$ is enough to force G to be Hamiltonian and the minimum degree condition is sharp. Moreover, this was recently strengthened by Draganić, Correia, and Sudakov to the pancyclicity result [8]. Also, Kim et al. [21] studied the decomposition of an almost regular graph G with $\alpha^*(G) = o(n)$ into almost spanning trees of bounded maximum degree.

The main result of this paper is the following. We denote by $\mathcal{T}(n, \Delta)$ the family of all trees on n vertices with maximum degree at most Δ.

Theorem 1.2. *For each $\varepsilon > 0$ and $\Delta \in \mathbb{N}$, there exists $\alpha = \alpha(\varepsilon, \Delta) > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Every n-vertex graph G with $\delta(G) \geq \varepsilon n$ and $\alpha^*(G) < \alpha n$ is $\mathcal{T}(n, \Delta)$-universal, that is, G contains every $T \in \mathcal{T}(n, \Delta)$ as a subgraph.*

That is, for a graph G with sublinear $\alpha^*(G)$, the minimum degree condition forcing bounded-degree spanning trees is almost sublinear. We remark that the maximum degree Δ of the tree in Theorem 1.2 cannot be larger than $C\sqrt{n}$ for some constant $C = C_\alpha > 0$ (in contrast to, for example, the result of [25], which holds for trees of maximum degree $O(\frac{n}{\log n})$) by the following construction. Given any $\alpha > 0$, choose positive integers n, k, Δ, d such that k, Δ are odd, $n = \Delta k - k + 2$, $\Delta > (k + 2)d$ and $\frac{1}{d} \ll \alpha$. It is easy to see that $\Delta > \sqrt{dn}$. Let T be a caterpillar which consists of a path $P = v_1v_2 \ldots v_k$ with $\Delta - 1$ leaves attached to v_i for each $i \in \{1, k\}$ and $\Delta - 2$ leaves attached to v_j for each $j \in [2, k - 1]$. Now we are going to construct a graph G with $\delta(G) = \frac{n}{2} + d - 1$ and $\alpha^*(G) \leq \alpha n$, but that does not contain T as a subgraph. Let $G_0 = (V, E)$ be an $(\frac{n}{2}, d, \lambda)$-regular graph such that $\lambda \leq \alpha d$, whose existence is guaranteed by a result of Friedman [11] on random d-regular graphs. Then we take two identical copies V_1 and V_2 of V, and join $u \in V_1$ and $v \in V_2$ if $uv \in E(G_0)$. The resulting bipartite d-regular graph is denoted by G^*. Note that by the well-known Expander Mixing Lemma (applied to G_0), we have that for any $A \subseteq V_1, B \subseteq V_2$ each of size αn,

$$e_{G^*}(A, B) = e_{G_0}(A, B) \geq \frac{d}{|G_0|}|A||B| - \lambda\sqrt{|A||B|} \geq \frac{2d}{n}(\alpha n)^2 - (\alpha d)\alpha n > 0.$$

We further add two disjoint copies of $K_{n/2}$ on V_1, V_2 and call the resulting graph G. Thus we have $\alpha^*(G) \leq \alpha n$. Suppose G contains a copy of T and V_1 has at least $\frac{k+1}{2}$ vertices of P since k is odd. Indeed, whenever we embed a branch vertex $v_i \in V(P)$ into V_1, we have to embed at least $\Delta - 2 - d$ leaves, which are attached to v_i, into V_1 as well. Thus the order of V_1 is at least

$$\frac{k+1}{2} + \frac{k+1}{2}(\Delta - 2 - d) = \frac{n + \Delta - (k+1)d - 3}{2} > \frac{n}{2}$$

as $\Delta > (k + 2)d$, a contradiction.

https://doi.org/10.1017/S0963548323000378 Published online by Cambridge University Press

Another motivation of our result is its connection to the randomly perturbed graphs introduced by Bohman et al. [2], where the host graph is obtained by adding random edges to a deterministic graph with minimum degree conditions. Krivelevich et al. [28] showed that for any $\varepsilon > 0$, $\Delta \in \mathbb{N}$ and $T \in \mathcal{T}(n, \Delta)$, if G is an n-vertex graph with $\delta(G) \geq \varepsilon n$, then the randomly perturbed graph $G \cup G(n, \frac{C}{n})$ *a.a.s.* contains T as a subgraph, where C depends only on ε and Δ. They suggested that such a result can be improved to a universality result, that is, $G \cup G(n, \frac{C}{n})$ *a.a.s.* contains all members of $\mathcal{T}(n, \Delta)$ simultaneously. This is confirmed by Böttcher et al. [3]. Indeed, in their technical result which we state below, they replaced $G(n, \frac{C}{n})$ by a deterministic sparse expander graph, and thus get the universality part for free.

Theorem 1.3 ([3], Theorem 2). *For any $\varepsilon > 0$ and integers $C \geq 2$ and $\Delta > 1$, there exist $\alpha > 0$, D_0 and n_0 such that the following holds for any $D \geq D_0$ and $n \geq n_0$. Let G be an n-vertex graph satisfying the following two conditions:*

1. $\Delta(G) \leq CD$,
2. $e(U, W) \geq \frac{D}{Cn}|U||W|$ *for all sets* $U, W \subseteq V(G)$ *with* $|U|, |W| \geq \alpha n$.

Suppose G_ε is an n-vertex graph on the same vertex set and $\delta(G_\varepsilon) \geq \varepsilon n$. Then $H := G_\varepsilon \cup G$ is $\mathcal{T}(n, \Delta)$-universal.

Note that the graph H in Theorem 1.3 satisfies $\alpha^*(H) \leq \alpha^*(G) < \alpha n$ and $\delta(H) \geq \delta(G_\varepsilon) \geq \varepsilon n$, so H is $\mathcal{T}(n, \Delta)$-universal by Theorem 1.2. Hence Theorem 1.2 slightly improves upon Theorem 1.3, where we do not need to distinguish two graphs (or equivalently the maximum degree condition on the sparse graph G is no longer needed).

2. Proof strategy and preliminaries

2.1. Notation

For a graph $G = (V, E)$, let $v(G) = |V|$ and $e(G) = |E|$. Given a collection of subgraphs $\mathcal{F} = \{F_i : i \in I\}$, let $V(\mathcal{F}) = \cup_{i \in I} V(F_i)$. Given two vertex-disjoint graphs G_1, G_2, let $G_1 \cup G_2$ be the union of G_1 and G_2. For $U \subseteq V(G)$, let $G[U]$ be the induced subgraph of G on U and let $G - U$ be the induced graph after removing U, that is $G - U := G[V \setminus U]$. Given a vertex $v \in V(G)$ and $X, Y \subseteq V(G)$, denoted by $N_X(v)$ the set of neighbours of v in X and let $d_X(v) := |N_X(v)|$. The neighbourhood of X in G is denoted by $N_G(X) = (\cup_{v \in X} N(v)) \setminus X$ and let $N_Y(X) = N_G(X) \cap Y$. We omit the index G if the graph is clear from the context.

For a graph F on $[k] := \{1, \ldots, k\}$, we say that B is the *n-blow-up* of F if there exists a partition $V_1, \ldots, V_k$ of $V(B)$ such that $|V_1| = \cdots = |V_k| = n$ and we have that $\{u, v\} \in E(B)$ if and only if $u \in V_i$ and $v \in V_j$ for some $\{i, j\} \in E(F)$. Given a spanning subgraph G of B, we call the sequence $V_1, \ldots, V_k$ the *parts* of G and we define $\bar{\delta}(G) := \min_{\{i,j\} \in E(F)} \delta(G[V_i, V_j])$ where $G[V_i, V_j]$ is the bipartite subgraph of G induced by the parts V_i and V_j. A subset R of $V(G)$ is *balanced* if $|R \cap V_1| = \cdots = |R \cap V_k|$. In particular, we say a subset of $V(G)$ or a subgraph of G *transversal* if it intersects each part in exactly one vertex.

For a path P, the *length* of P is the number of edges in P. Given two vertices x, y, an *x, y-path* is a path with ends x and y. Let T be a tree and T' be obtained from T by removing all leaves. A *pendant star* is a maximal star centred at a leaf of T', where the unique neighbour of the centre inside T' is called the *root* of the pendant star. A *bare path* in T is a path whose internal vertices have degree exactly two in T. A *caterpillar* in T consists of a bare path in T' as the *central path* with some (possibly empty) leaves attached to the internal vertices of the central path, where *branch* vertices are the internal vertices attached with at least one leaf. The *length* and *ends* of the caterpillar refer to the length and ends of its central path.

For two graphs H and G, an *embedding* φ of H in G is an injective map $\varphi : V(H) \to V(G)$ such that $\{v, w\} \in E(H)$ implies $\{\varphi(v), \varphi(w)\} \in E(G)$. For all integers a, b with $a \leq b$, let $[a, b] :=$

https://doi.org/10.1017/S0963548323000378 Published online by Cambridge University Press

$\{i\in\mathbb{Z}: a\le i\le b\}$ and $[a]:=\{1,2,\ldots,a\}$. When we write $\alpha\ll\beta\ll\gamma$, we always mean that α,β,γ are constants in $(0,1)$, and $\beta\ll\gamma$ means that there exists $\beta_0=\beta_0(\gamma)$ such that the subsequent arguments hold for all $0<\beta\le\beta_0$. Hierarchies of other lengths are defined analogously. For the sake of clarity of presentation, we will sometimes omit floor and ceiling signs when they are not crucial.

2.2. Graph expansion and trees

We will introduce some graph expansion properties to embed the trees.

Definition 2.1 ([18]). Let $n\in\mathbb{N}$ and $d>0$. A graph G is an (n,d)-*expander* if $|G|=n$ and G satisfies the following two conditions.

1. $|N_G(X)|\ge d|X|$ for all sets $X\subseteq V(G)$ with $1\le|X|<\lceil\frac{n}{2d}\rceil$.
2. $e_G(X,Y)>0$ for all disjoint $X,Y\subseteq V(G)$ with $|X|=|Y|=\lceil\frac{n}{2d}\rceil$.

In [27], Krivelevich considered trees differently according to whether they contain many leaves or many disjoint bare paths.

Lemma 2.2 ([27]). *For any integers $n,k>2$, a tree on n vertices either has at least $\frac{n}{4k}$ leaves or a collection of at least $\frac{n}{4k}$ vertex-disjoint bare paths of length k.*

We will use the following corollary to divide $\mathcal{T}(n,\Delta)$ into trees with many pendant stars and trees with many vertex-disjoint caterpillars.

Corollary 2.3. *For any integer $n,k>2$, a tree on n vertices with maximum degree Δ either has at least $\frac{n}{4k\Delta}$ pendant stars or a collection of at least $\frac{n}{4k\Delta}$ vertex-disjoint caterpillars each of length k.*

Suppose T is an n-vertex tree with maximum degree at most Δ and T' is the subtree obtained by removing all leaves. Then it holds that $|T'|+|T'|(\Delta-1)\ge n$. We can apply Lemma 2.2 on T' to obtain at least $\frac{|T'|}{4k}\ge\frac{n}{4k\Delta}$ pendant stars or a collection of at least $\frac{n}{4k\Delta}$ vertex-disjoint caterpillars of length k in T.

In [16], Haxell extended a result of Friedman and Pippenger [12] and showed that one can embed every almost spanning tree with bounded maximum degree in a graph with strong expansion property. We will use the following result by Johannsen et al. [18].

Theorem 2.4 ([18]). *Let $n,\Delta\in\mathbb{N}$, $d\in\mathbb{R}^+$ with $d\ge2\Delta$ and G be an (n,d)-expander. Given any $T\in\mathcal{T}(n-4\Delta\lceil\frac{n}{2d}\rceil,\Delta)$, we can find a copy of T in G.*

As shown by Johannsen et al. [18], the following lemma is useful for attaching leaves onto certain vertices.

Definition 2.5 ([18]). Given a bipartite graph $G=(A,B,E)$ with $|A|\le|B|$ and a function $f:A\to\mathbb{N}$ with $\sum_{u\in A}f(u)=|B|$, an *f-matching from A into B* is a collection of vertex-disjoint stars $\{S_u:u\in A\}$ in G such that S_u has u as the centre and exactly $f(u)$ leaves inside B.

Lemma 2.6 ([18]). *Let $d,m\in\mathbb{N}$ and let G be a graph. Suppose that two disjoint sets $U,W\subseteq V(G)$ satisfy the following three conditions:*

1. *$|N_G(X)\cap W|\ge d|X|$ for all sets $X\subseteq U$ with $1\le|X|\le m$,*
2. *$e_G(X,Y)>0$ for all $X\subseteq U$ and $Y\subseteq W$ with $|X|=|Y|\ge m$,*
3. *$d_U(w)\ge m$ for all $w\in W$.*

Then for every $f:U\to\{1,\ldots,d\}$ with $\sum_{u\in U}f(u)=|W|$, there exists an f-matching from U into W.

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2.3. Proof overview

We give a brief outline of our proof here. Similar to previous works [27, 28, 34], we classify the trees and deal with them separately. Our classification is given in Corollary 2.3, which refines Lemma 2.2. First, for the pendant star case, let T_1 be a subtree obtained by deleting the centres and leaves of pendant stars in T (but not the roots). We randomly partition $V(G)$ into V_1, V_2 and V_3 and embed T_1 into $G[V_1]$ by Theorem 2.4. Then we greedily embed most of the centres into V_2. The α^* property guarantees that we are left with a small portion of centres and we shall embed them by the degree condition into V_3. Finally we use Lemma 2.6 to find a desired star-matching and complete the embedding.

For the caterpillar case, we shall embed a suitable subset of branch vertices into a random set with "good" expansion properties. In this way, we can greedily finish the last step of embedding leaves of caterpillars by a star-matching. To embed this suitable branch vertex set, we control the length of the caterpillars and let T_2 be the subforest obtained by deleting these caterpillars except the ends. First, we randomly partition $V(G)$ into V_1, V_2, and V_3, and embed T_2 into V_1 as the above case. To embed central paths of these caterpillars, we randomly partition V_2 set into $k-1$ equal parts $X_1, \ldots, X_{k-1}$, where k is the length of the caterpillars. Then we define an auxiliary graph H with $V(H) = X'_1 \cup X'_2 \cup \cdots \cup X'_{k-2}$ which is a spanning subgraph of the blow-up of C_{k-2} and X'_i is roughly the same as X_i. In this way, we transform the problem of embedding vertex-disjoint central paths into finding a transversal cycle-factor in H, as Lemma 2.7 below. Once central paths have been embedded, we can greedily embed leaves of caterpillars by a star-matching as mentioned above and complete the embedding. Now we state Lemma 2.7. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k and let $\alpha_b^*(G)$ be the largest integer s such that $G = (V_1, \ldots, V_k, E)$ contains an (s, s)-bipartite-hole (S, T) where $S \subseteq V_i$, $T \subseteq V_{i+1}$ for some $i \in [k]$.

Lemma 2.7. *Given a positive integer $k \in 4\mathbb{N}$ and a constant δ with $\delta > \frac{2}{k}$, there exists $\alpha > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha_b^*(G) < \alpha n$. Then G has a transversal C_k-factor.*

Although the minimum degree bound in Lemma 2.7 is not best possible and it only works for $k \in 4\mathbb{N}$, but it is enough for our purpose (indeed, we only need it work for large k). We suspect that the (asymptotic) tight condition should be $(1 + o(1))\frac{n}{k}$, given by the so-called space barrier. Indeed, let G be a (complete) n-blow-up of C_k with parts labelled as $V_1, \ldots, V_k$. One can specify a set U_i of size $n/k - 1$ in each cluster V_i, $i \in [k]$ and remove all edges not touching $U := \bigcup_{i \in [k]} U_i$ from G. Then we add a k-partite Erdős graph (obtained from random graph) to $V(G) \setminus U$ so that the resulting graph G' satisfy $\alpha_b^*(G') = o(n)$ but $G' - U$ is C_k-free. Now every transversal copy of C_k in G' must contain a vertex in U and $\bar{\delta}(G') \geq n/k - 1$. Since $|U| < n$, G' does not have a transversal C_k-factor.

If we remove the α_b^* condition in Lemma 2.7, then the minimum degree threshold for transversal C_k-factor is asymptotically $\bar{\delta}(G) \geq (1 + \frac{1}{k})\frac{n}{2} + o(n)$, as determined recently by Ergemlidze and Molla in [9].

Our proof of Lemma 2.7 is based on the absorption method, which will be given in Section 4.4. Finally, there are some other minimum degree-type results in blown-up graphs [9, 19] and in multi-partite graphs [10, 20, 29, 31, 32], but not with any randomness condition.

3. Proof of the main theorem

Proof of Theorem 1.2. Given a positive integer Δ and a constant $\varepsilon > 0$, we set $k = 48\lceil \frac{1}{\varepsilon} \rceil$, $\gamma = \frac{1}{4k\Delta^2}$ and $\eta = \frac{1}{4k\Delta}$. Choose

$$\frac{1}{n} \ll \alpha \ll \frac{1}{d} \ll \varepsilon, \frac{1}{\Delta}$$

and let G be an n-vertex graph with $\delta(G) \geq \varepsilon n$ and $\alpha^*(G) < \alpha n$. Given any tree $T \in \mathcal{T}(n, \Delta)$, by Corollary 2.3, we proceed the proof by considering the following two cases.

Case 1. T has at least ηn pendant stars.

We can easily pick a collection of γn vertex-disjoint pendant stars in T and label it as $\mathcal{D} = \{D_1, \ldots, D_{\gamma n}\}$ for convenience. Write $A := \{a_1, \ldots, a_{\gamma n}\}$ and $B := \{b_1, \ldots, b_{\gamma n}\}$ where a_i and b_i are the root and centre of D_i, respectively. Let $\mathcal{S} = \{S_1, \ldots, S_{\gamma n}\}$ where each S_i is obtained from D_i by removing the root. Let T_1 be a subtree of T obtained by deleting vertices of $\mathcal{S}$. Note that $|T_1| \leq n - 2\gamma n$. Moreover we claim that $V(G)$ can be partitioned into V_1, V_2, V_3 of sizes n_1, n_2, n_3, respectively, such that

$$d_{V_i}(v) \geq \frac{\varepsilon}{2}|V_i| \text{ for all } v \in V(G),\ i \in [3] \tag{1}$$

where

$$n_1 = \frac{d|T_1| + 4\Delta d}{d - 2\Delta}, \quad n_2 = \gamma n, \quad n_3 = n - n_1 - n_2. \tag{2}$$

Choose a partition $\{V_1, V_2, V_3\}$ of $V(G)$ uniformly at random where $|V_i| = n_i$. For every $v \in V(G)$, let $f_v^i = d_{V_i}(v)$ and note that $\mu_v^i := \mathbb{E}[f_v^i] \geq \varepsilon n_i$. Let q be the probability that there exist $v \in V(G)$ and $i \in [3]$ which violates property (1). Then by the union bound and Chernoff's inequality (see e.g. [17], Theorem 2.1),

$$q \leq 3n \exp\left(\frac{-(\mu_v^i/2)^2}{2\mu_v^i}\right) \leq 3n \exp\left(-\frac{\varepsilon n_i}{8}\right) = o(1)$$

for sufficiently large n. Therefore, with positive probability the randomly chosen partition $\{V_1, V_2, V_3\}$ satisfies property (1). Then we have

$$\begin{aligned} n_3 &\geq n - \gamma n - \frac{d(n - 2\gamma n) + 4\Delta d}{d - 2\Delta} = \frac{(d + 2\Delta)\gamma n - 2\Delta n - 4\Delta d}{d - 2\Delta} \\ &\geq \frac{d\gamma n - 2\Delta n}{2d} \geq \frac{\gamma n}{4} \geq \frac{6\Delta\alpha n}{\varepsilon}, \end{aligned} \tag{3}$$

where in the penultimate inequality we use $d\gamma n - 2\Delta n \geq \frac{d\gamma n}{2}$ because $\frac{1}{d} \ll \varepsilon, \frac{1}{\Delta}$ and the last inequality follows since $\alpha \ll \varepsilon, \frac{1}{\Delta}$.

Claim 3.1. $G[V_1]$ is an (n_1, d)-expander.

Proof. Let $m_1 = \lceil \frac{|V_1|}{2d} \rceil$ and $m_2 = \lceil \frac{\varepsilon |V_1|}{2d+2} \rceil$. Since $\alpha^*(G) < \alpha n \leq m_1$, there is at least one edge between any two vertex-disjoint sets of size m_1 in $G[V_1]$. For $X \subseteq V_1$ with $1 \leq |X| \leq m_2$, by (1), we have

$$|N_{V_1}(X)| \geq \frac{\varepsilon}{2}|V_1| - |X| \geq d|X|. \tag{4}$$

For $X \subseteq V_1$ with $m_2 < |X| < m_1$, since $\alpha \ll \frac{1}{d}, \varepsilon$, we can arbitrarily pick $Z \subseteq X$ with $|Z| = \alpha n$. As there is no edge between Z and $V_1 \backslash (Z \cup N_{V_1}(Z))$, we have $|V_1 \backslash (Z \cup N_{V_1}(Z))| < \alpha n$, then $|N_{V_1}(Z)| > |V_1| - 2\alpha n$. Thus

$$|N_{V_1}(X)| \geq |N_{V_1}(Z)| - (|X| - |Z|) \geq |V_1| - |X| - \alpha n \geq d|X|. \tag{5}$$

Together with (4), $G[V_1]$ is an (n_1, d)-expander. □

Note that $|T_1| = n_1 - 4\Delta(\frac{n_1}{2d} + 1) \leq n_1 - 4\Delta \lceil \frac{n_1}{2d} \rceil$, where the first equality follows since $n_1 = \frac{d|T_1| + 4\Delta d}{d - 2\Delta}$. Then by Theorem 2.4, there exists an embedding $f_1 : V(T_1) \to V_1$. Let $L_0 = V_1 \backslash f_1(T_1)$

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and $L_1 = f_1(A)$. Next, we will embed the centres of pendant stars into $V_2 \cup V_3$ and we do it in two steps.

In the first step, we embed most of the vertices of B into V_2. Consider the bipartite graph H on vertex sets L_1 and V_2, where $|L_1| = |V_2| = \gamma n$. Let M be a maximum matching in H and we claim that $|E(M)| \geq \gamma n - \alpha n$. Otherwise, since $\alpha^*(G) < \alpha n$, there is at least one edge in $H - V(M)$, contrary to the maximality of M. Let $L_2 = V(M) \cap V_2$ and $L_3 = V_2 \setminus L_2$. Without loss of generality, suppose we have embedded $B_1 = \{b_1, \ldots, b_t\}$ into L_2, where $t \geq \gamma n - \alpha n$.

In the second step, we shall embed the vertices of $B \setminus B_1$ into V_3. For every $u \in V(G)$, by (1) and (3), we have $d_{V_3}(u) \geq \frac{\varepsilon}{2}|V_3| \geq \Delta\alpha n$. Hence we can greedily embed $S_{t+1}, \ldots, S_{\gamma n}$ into $G[V_3]$. Let $\mathcal{S}_1 = \{S_1, \ldots, S_t\}$. It follows that there exists an embedding $f_2 : V(B_1) \cup V(\mathcal{S} \setminus \mathcal{S}_1) \to V_2 \cup V_3$ such that $f_2(B_1) = L_2$ and $f_2(\mathcal{S} \setminus \mathcal{S}_1) \subseteq V_3$. Let $L_4 = V_3 \setminus f_2(\mathcal{S} \setminus \mathcal{S}_1)$ and we have $|L_4| \geq |V_3| - \Delta\alpha n$.

Now it remains to embed the leaves attached to the vertices of B_1. Consider the bipartite graph Q on vertex sets L_2 and L, where $L = L_0 \cup L_3 \cup L_4$. Let $m := 2\alpha n$ and $d := \Delta - 1$. Since $\alpha^*(G) < \alpha n < m$, there is at least one edge between any two disjoint vertex set of size m in Q. For all $X \subseteq L_2$ with $1 \leq |X| \leq m$, by (1) and (3), we have $|N_Q(X) \cap L| \geq |N_Q(X) \cap L_4| \geq \frac{\varepsilon}{2}|V_3| - \Delta\alpha n \geq d|X|$. Moreover for each $u \in L$, we have $d_{L_2}(u) \geq \frac{\varepsilon}{2}|V_2| - \alpha n \geq m$ due to $\alpha \ll \varepsilon, \frac{1}{\Delta}$. Therefore by applying Lemma 2.6 on Q, we obtain an embedding f_3 of $\mathcal{S}_1$ in Q such that $f_3(u) = f_2(u)$ for every $u \in B_1$. In conclusion, it is clear that the map $f : V(T) \to V(G)$ defined by

$$f(u) := \begin{cases} f_1(u) & \text{if } u \in V(T_1) \\ f_2(u) & \text{if } u \in V(\mathcal{S}) \setminus V(\mathcal{S}_1) \\ f_3(u) & \text{if } u \in V(\mathcal{S}_1) \end{cases}$$

is an embedding of T in G. The proof of Case 1 is complete.

Case 2. T has at least ηn vertex-disjoint caterpillars of length k.

A caterpillar in T consists of a bare path in T' as the central path with some (possibly empty) leaves attached to the internal vertices of the central path, where T' is the subtree obtained by deleting the leaves of T and we say that the internal vertices attached with leaves are branch vertices. Observe that T either has a family of at least $\frac{\eta n}{2}$ caterpillars of length k that have at least one leaf or a family of at least $\frac{\eta n}{2}$ bare paths of length k. Here, we will give a detailed proof for the first subcase and the second subcase can be derived by the same argument.

Let $n' = \frac{\eta n}{2}$ and k' be an integer from $\{\frac{k}{2}, \frac{k}{2} - 1, \frac{k}{2} - 2, \frac{k}{2} - 3\}$ such that $k' = 2 \pmod 4$. It is easy to pick a collection of n' vertex-disjoint caterpillars of length k' in T such that one end of each caterpillar is adjacent to a branch vertex in T. Let $\mathcal{F} = \{F_1, \ldots, F_{n'}\}$ be such a collection and $\mathcal{P} = \{P_1, \ldots, P_{n'}\}$, where each P_i is the central path of F_i. Write $S := \{s_1, \ldots, s_{n'}\}$ and $W := \{w_1, \ldots, w_{n'}\}$ where s_i and w_i are the ends of F_i and assume that the neighbour of s_i in P_i is a branch vertex, $i \in [n']$.

Let T_2 be a subforest of T obtained by deleting the vertices of $\mathcal{F}$ except the ends of every caterpillar and note that $|T_2| \leq n - n'k'$. In a similar way as Case 1, there exists a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that

$$d_{V_i}(u) \geq \frac{\varepsilon}{2}|V_i| \text{ for all } u \in V(G),\ i \in [3] \tag{6}$$

where

$$|V_1| = |T_2| + \frac{2\Delta|T_2| + 4\Delta d}{d - 2\Delta},\ |V_2| = n'(k' - 1) - \frac{2\Delta|T_2| + 4\Delta d}{d - 2\Delta},\ |V_3| = n - |V_1| - |V_2|. \tag{7}$$

Then we have

$$|V_3| \geq n - (n - n'k') - n'(k' - 1) = n' \geq \frac{2\Delta\alpha n}{\varepsilon}, \tag{8}$$

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where the first inequality follows since $|T_2| \leq n - n'k'$ and the last inequality follows since $\alpha \ll \varepsilon, \frac{1}{\Delta}$.

Note that $|T_2| = |V_1| - 4\Delta(\frac{|V_1|}{2d} + 1) \leq |V_1| - 4\Delta\lceil \frac{|V_1|}{2d} \rceil$, where the first equality follows since $|V_1| = |T_2| + \frac{2\Delta|T_2| + 4\Delta d}{d - 2\Delta}$. Then Theorem 2.4 implies that there exists an embedding g_1 of T_2 in $G[V_1]$. Let $L_1 = V_1 \backslash g_1(T_2)$ and $V_2' = V_2 \cup L_1$. It now remains to embed n' caterpillars in $\mathcal{F}$.

First, we shall embed the internal vertices of $\mathcal{P}$ into V_2'. Randomly partition V_2' into $X_1, \ldots, X_{k'-1}$ each of size n'. The union bound and Chernoff's inequality imply that, if n is sufficiently large, then there exists a partition such that for every $u \in V(G)$ and $i \in [k' - 1]$, we have $d_{X_i}(u) \geq \frac{\varepsilon n'}{4}$. Since $\alpha^*(G) < \alpha n \leq n'$, there exists a matching M_1 between $g_1(S)$ and X_1 such that $t_1 := |E(M_1)| > n' - \alpha n$. Similarly, there exists a matching M_2 between $g_1(W)$ and $X_{k'-1}$ such that $t_2 := |E(M_2)| > n' - \alpha n$. Let $S_1 = \{s_1, \ldots, s_{t_1}\} \subseteq S$ and $W_1 = \{w_1, \ldots, w_{t_2}\} \subseteq W$. Without loss of generality, we assume that $g_1(S_1) \subseteq V(M_1)$ and $g_1(W_1) \subseteq V(M_2)$. By the choice of $X_1, \ldots, X_{k'-1}$, for each $u \in g_1((S \cup W)G' - Us(S_1 \cup W_1)) \subseteq V(G)$, we have

$$d_{X_2}(u) \geq \frac{\varepsilon n'}{4} \geq 2\alpha n \geq (n' - t_1) + (n' - t_2).$$

Therefore we can greedily find a matching M_3 between $g_1(S \setminus S_1)$ and X_2 covering $g_1(S \setminus S_1)$, and a matching M_4 between $g_1(W \setminus W_1)$ and X_2 covering $g_1(W \setminus W_1)$, where $V(M_3) \cap V(M_4) = \emptyset$. Let $X_1' := (V(M_1) \cap X_1) \cup (V(M_3) \cap X_2), X_{k'-1}' := (V(M_2) \cap X_{k'-1}) \cup (V(M_4) \cap X_2), X_2' := (X_2 \setminus (V(M_3) \cup V(M_4))) \cup (X_1 \setminus V(M_1)) \cup (X_{k'-1} \setminus V(M_2))$ and let $X_i' := X_i$ for $i \in [3, k' - 2]$. In this way, we obtain a new partition $\left\{X_1', \ldots, X_{k'-1}'\right\}$ of V_2' such that there exist perfect matchings between X_1' and $g_1(S)$ and between $X_{k'-1}'$ and $g_1(W)$. Let $X_1' = \{x_1, \ldots, x_{n'}\}$ and $X_{k'-1}' = \left\{y_1, \ldots, y_{n'}\right\}$ where for each $i \in [n']$, $\{x_i, g_1(s_i)\}$ and $\{y_i, g_1(w_i)\}$ are edges of the above perfect matchings.

Let $X_0' = \{z_1, \ldots, z_{n'}\}$ be a new set of vertices disjoint from $V(G)$. Define an auxiliary graph H with vertex set $V(H) = X_0' \cup X_2' \cup \cdots \cup X_{k'-2}'$. For $2 \leq i \leq k' - 3$, the edges of H between X_i' and X_{i+1}' are identical to those of G. For $v \in X_2'$ and $z_j \in X_0'$, $\left\{v, z_j\right\}$ is an edge of H if and only if $\left\{v, x_j\right\}$ is an edge of G. Similarly, for $u \in X_{k'-2}'$ and $z_j \in X_0'$, $\left\{u, z_j\right\}$ is an edge of H if and only if $\left\{u, y_j\right\}$ is an edge of G. Observe that if H has a transversal $C_{k'-2}$-factor, then G has n' vertex-disjoint paths of length $k' - 2$ that connect x_i and y_i. Since we moved at most $2\alpha n$ vertices when we constructed the new partition, now we have that

$$\bar{\delta}(H) \geq \frac{\varepsilon n'}{4} - 2\alpha n \geq \frac{\varepsilon n'}{8} > \frac{2n'}{k' - 2}. \tag{9}$$

Then Lemma 2.7 implies that H contains a transversal $C_{k'-2}$-factor. Together with the perfect matchings, we find an embedding g_2 of $V(\mathcal{P})$ to V_2' that connects $g_1(s_i)$ and $g_1(w_i)$ for each $i \in [n']$. Now it suffices to embed the leaves of caterpillars into V_3.

Let $I \subseteq V_2'$ be the set of images of branch vertices in $V(\mathcal{F})$. Since every vertex in S is adjacent to a branch vertex in F_i, we have $X_1' \subseteq I$. Let $m := \alpha n$ and $d := \Delta$. For all $X \subseteq I$ with $1 \leq |X| \leq m$, by (6) and (8), we have $|N_G(X) \cap V_3| \geq \frac{\varepsilon}{2}|V_3| \geq d|X|$. Moreover for each $u \in V_3$, by (9), we have $d_I(u) \geq d_{X_1'}(u) \geq \frac{\varepsilon n'}{8} \geq m$. Together with the assumption that $\alpha^*(G) < m$, Lemma 2.6 implies that there exists an embedding g_3 of $V(\mathcal{F}) \backslash V(\mathcal{P})$ to V_3 that respects the edges between the branch vertices and the leaves of caterpillars. It follows that the map $g : V(T) \to V(G)$ defined by

$$g(u) := \begin{cases} g_1(u) & \text{if } u \in V(T_2) \\ g_2(u) & \text{if } u \in V(\mathcal{P}) \\ g_3(u) & \text{if } u \in V(\mathcal{F}) \backslash V(\mathcal{P}) \end{cases}$$

is an embedding of T in G. This concludes the proof of the first subcase.

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As for the second subcase, we adopt a similar argument and the main difference is that we split $V(G)$ into two parts because the caterpillars have no leaves and it suffices to find g_1 and g_2. □

4. Transversal C_k-factor

4.1. Proof of Lemma 2.7

Following the typical absorption method, the main tasks are to (i) establish an absorbing set R and (ii) find an almost perfect transversal C_k-tiling in $G - R$. For (i), we will introduce some related definitions.

Definition 4.1. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k and F be a k-vertex graph.

1. We say that a balanced subset $R \subseteq V(G)$ is a ξ-*absorbing set* for some $\xi > 0$ if for every balanced subset $U \subseteq V(G)\backslash R$ with $|U| \leq \xi n$, $G[R \cup U]$ contains an F-factor which consists of transversal copies.
2. Given a subset $S \subseteq V(G)$ of size k and an integer t, we say that a subset $A_S \subseteq V(G)\backslash S$ is an (F, t)-*absorber* of S if $|A_S| \leq kt$ and both $G[A_S]$ and $G[A_S \cup S]$ contain an F-factor.

Now we state the first crucial lemma, whose proof can be found in Section 4.4.2.

Lemma 4.2. (Absorbing Lemma). *Given $k \in \mathbb{N}$ with $k \geq 4$ and positive constants δ, γ with $\delta > \frac{2}{k}$ and $\gamma \leq \frac{\delta}{2}$, there exist $\alpha, \xi > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha^*_{\rm b}(G) < \alpha n$. Then there exists a ξ-absorbing set $R \subseteq V(G)$ of size at most γn.*

For (ii), Lemma 4.3 provides an almost transversal C_k-tiling, whose proof will be given in Section 4.3.

Lemma 4.3. (Almost perfect tiling). *Given a positive integer $k \in 4\mathbb{N}$ and constants δ, ζ with $\delta > \frac{2}{k}$, there exists $\alpha > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha^*_b(G) < \alpha n$. Then G contains a transversal C_k-tiling covering all but at most ζn vertices.*

Now we are ready to prove Lemma 2.7 using Lemmas 4.2 and 4.3.

Proof of Lemma 2.7. Given $k \in 4\mathbb{N}$ and a constant δ with $\delta > \frac{2}{k}$, we set $\eta := \delta - \frac{2}{k}$ and choose $\frac{1}{n} \ll \alpha \ll \zeta \ll \xi \ll \gamma \ll \eta, \delta$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq (\frac{2}{k} + \eta)n$ and $\alpha^*_{\rm b}(G) < \alpha n$.

By Lemma 4.2 and the choice that $\gamma \ll \eta, \delta$, there exists a ξ-absorbing set $R \subseteq V(G)$ of size at most γn for some $\xi > 0$. Let $G' := G - R$ and note that G' is an $(n - \frac{|R|}{k})$-blow-up of C_k. Then we have

$$\bar{\delta}(G') \geq \left(\frac{2}{k} + \eta\right) n - \frac{\gamma n}{k} \geq \left(\frac{2}{k} + \frac{\eta}{2}\right) n.$$

Therefore by applying Lemma 4.3 on G', we obtain a transversal C_k-tiling $\mathcal{M}$ that covers all but a set U of at most ζn vertices in G'. Since $\zeta \ll \xi$, the absorbing property of R implies that $G[R \cup U]$ contains a transversal C_k-factor, which together with $\mathcal{M}$ forms a transversal C_k-factor in G. □

4.2. Regularity

The proof of Lemma 4.3 is based on a standard application of the regularity method. We will introduce some basic definitions and properties. Given a graph G and a pair (X, Y) of vertex-disjoint subsets in $V(G)$, the *density* of (X, Y) is defined as

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$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

Given constants $\varepsilon, d > 0$, we say that (X, Y) is *(ε, d)-regular* if $d(X, Y) \geq d$ and for all $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have

$$|d(X', Y') - d(X, Y)| \leq \varepsilon.$$

The following fact results from the definition.

Fact 4.4. Let (X, Y) be an (ε, d)-regular pair and $B \subseteq Y$ with $|B| \geq \varepsilon|Y|$. Then all but at most $\varepsilon|X|$ vertices in X have at least $(d - \varepsilon)|B|$ neighbours in B.

We now state a degree form of the regularity lemma (see [26], Theorem 1.10).

Lemma 4.5. (Degree form of Regularity Lemma [26]). *For every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that the following holds for any real number $d \in (0, 1]$ and $n \in \mathbb{N}$. Let $G = (V, E)$ be an n-vertex graph. Then there exists a partition $\mathcal{P} = V_0 \cup V_1 \cup \cdots \cup V_k$ and a spanning subgraph $G' \subseteq G$ with the following properties:*

- *(a)* $\frac{1}{\varepsilon} \leq k \leq N$;
- *(b)* $|V_0| \leq \varepsilon n$ *and* $|V_1| = \cdots = |V_k| = m \leq \varepsilon n$;
- *(c)* $d_{G'}(v) \geq d_G(v) - (d + \varepsilon)n$ *for all* $v \in V(G)$;
- *(d) every* V_i *is an independent set in* G' *for* $i \in [k]$;
- *(e) every pair* (V_i, V_j), $1 \leq i < j \leq k$ *is* ε*-regular in* G' *with density* 0 *or at least* d.

A widely used auxiliary graph accompanied with the regular partition is the reduced graph. The *reduced graph* R_d of $\mathcal{P}$ is a graph defined on the vertex set $\{V_1, \ldots, V_k\}$ such that V_i is adjacent to V_j in R_d if (V_i, V_j) has density at least d in G'. We use $d_R(V_i)$ to denote the degree of V_i in R_d for each $i \in [k]$.

Fact 4.6. Given positive constants d, ε and δ, fix an n-vertex graph $G = (V, E)$ with $\delta(G) \geq \delta n$ and let G' and $\mathcal{P}$ be obtained by Lemma 4.5, and R_d be given as above. Then for every $V_i \in V(R_d)$, we have $d_{R_d}(V_i) \geq (\delta - 2\varepsilon - d)k$.

4.3. Almost perfect tilings

Here we shall make use of the following result which provides a sufficient condition for a transversal path among given sets.

Proposition 4.7. *Given an integer $k \geq 2$ and a positive constant $\alpha \leq \frac{1}{2}$, let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\alpha^*_{\mathrm{b}}(G) < \alpha n$. For any integers i, j with $1 \leq i < j \leq k$ and a collection of subsets $X_s \subseteq V_s$ with $s \in [i, j]$, if $|X_i|$, $|X_j| \geq \alpha n$ and $|X_\ell| \geq 2\alpha n$ for $\ell \in [i + 1, j - 1]$ (possibly empty), then there exists a transversal path $x_i x_{i+1} \ldots x_{j-1} x_j$ where $x_s \in X_s$ for $s \in [i, j]$.*

Proof of Proposition 4.7. Without loss of generality, we may take $i = 1$, $j = k$ for instance. Let $Z_1 := X_1$ and $Z_2 := N(Z_1) \cap X_2$. By the fact that $\alpha^*_{\mathrm{b}}(G) < \alpha n \leq |Z_1|, |X_2|$, it holds that $|Z_2| > |X_2| - \alpha n \geq \alpha n$. If $k = 2$, then there exists an edge $\{x_1, x_2\}$ between Z_1 and Z_2 and we are done. If $k > 2$, then for each $s \in [k - 1]$, there exist $Z_s \subseteq X_s$ of size larger than αn such that Z_s is the set of neighbours of Z_{s-1}. Since $|Z_{k-1}|, |X_k| \geq \alpha n$, there exists an edge $\{x_{k-1}, x_k\}$ between Z_{k-1} and X_k. Therefore, we can find a transversal path $x_1 x_2 \ldots x_{k-1} x_k$, where $x_s \in X_s$ for $s \in [k]$. □

Now we are ready to prove Lemma 4.3.

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Proof of Lemma 4.3. Given $k \in 4\mathbb{N}$ and δ, ζ with $\delta > \frac{2}{k}$, we set $\eta := \delta - \frac{2}{k}$ and choose $\frac{1}{n} \ll \alpha \ll \frac{1}{N_0} \ll \varepsilon \ll \zeta, \delta, \eta$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha_b^*(G) < \alpha n$. By applying Lemma 4.5 on G with $d := \frac{\eta}{4}$, we obtain a partition $\mathcal{P} = \{U_0\} \cup \left\{U_{i,j} \subseteq V_i : i \in [k], j \in [N_0]\right\}$ that refines the partition $\{V_1, V_2, \ldots, V_k\}$ of G and a spanning subgraph G' of G with properties $(a)-(e)$, where we write $m := |U_{i,j}|$ for all $i \in [k], j \in [N_0]$. Let R_d be the reduced graph defined on the vertex set $\left\{U_{i,j} : i \in [k], j \in [N_0]\right\}$. For each $i \in [k]$, let $\mathcal{V}_i = \left\{U_{i,j} : j \in [N_0]\right\}$ and note that $\{\mathcal{V}_i : i \in [k]\}$ is a partition of R_d. Then Fact 4.6 implies that $\bar{\delta}(R_d) \geq (\delta - \frac{\eta}{2})N_0 = (\frac{2}{k} + \frac{\eta}{2})N_0$.

To obtain an almost perfect transversal C_k-tiling in G, we define an auxiliary graph H with k vertices and two disjoint edges and then use it for embedding copies of transversal C_k (see Claim 4.9). Now we will show that there exist N_0 vertex-disjoint copies of specific H in R_d.

Claim 4.8. *For every $i \in [k-1]$, $R_d[\mathcal{V}_i, \mathcal{V}_{i+1}]$ has a matching of size* $\min\{N_0, 2\bar{\delta}(R_d)\}$.

Proof. Let $m = \min\{N_0, 2\bar{\delta}(R_d)\}$ and without loss of generality, we may take $i = 1$ for instance. Let M be a maximum matching in $R_d[\mathcal{V}_1, \mathcal{V}_2]$ and assume for the contrary that $|E(M)| \leq m - 1$. Let U be the minimum vertex cover of $R_d[\mathcal{V}_1, \mathcal{V}_2]$. Then by König's theorem ([30]), it holds that $|U| = |E(M)|$. We write $A = U \cap \mathcal{V}_1$ and $B = U \cap \mathcal{V}_2$. By the pigeonhole principle, we get $|A| \leq \lfloor \frac{m-1}{2} \rfloor$ or $|B| \leq \lfloor \frac{m-1}{2} \rfloor$. Suppose $|A| \leq \lfloor \frac{m-1}{2} \rfloor$. Since $m \leq N_0$, we have $|\mathcal{V}_2 \backslash B| \neq 0$. Arbitrarily choose $u \in \mathcal{V}_2 \backslash B$, then u has no neighbour in $\mathcal{V}_1 \backslash A$. Thus we have $d_{\mathcal{V}_1}(u) = d_A(u) \leq \lfloor \frac{m-1}{2} \rfloor \leq \lfloor \frac{2\bar{\delta}(R_d)-1}{2} \rfloor = \bar{\delta}(R_d) - 1$, a contradiction. □

By Claim 4.8, for each $i \in [\frac{k}{2}]$, there exists a matching M_i of size $\min\{N_0, 2\bar{\delta}(R_d)\}$ between $\mathcal{V}_{2i-1}$ and $\mathcal{V}_{2i}$. Let $A_j \subseteq \mathcal{V}_j$ be the vertices uncovered by matchings for $j \in [k]$. We pick a family $\mathcal{H}$ of N_0 vertex-disjoint copies of H such that each copy contains two disjoint edges e_1 and e_2 where $e_1 \in M_{2i-1}$ and $e_2 \in M_{2i}$ for some $i \in [\frac{k}{4}]$ and exactly one vertex inside each A_j for $j \in [k] \backslash \{4i-3, 4i-2, 4i-1, 4i\}$. Since

$$\sum_{i=1}^{\frac{k}{4}} |M_{2i-1}| = \frac{k}{4} \cdot \min\{N_0, 2\bar{\delta}(R_d)\} \geq N_0,$$

and similarly $\sum_{i=1}^{k/4} |M_{2i}| \geq N_0$, we can greedily find N_0 vertex-disjoint copies of H that together cover all vertices in R_d.

Claim 4.9. *For each copy of H in $\mathcal{H}$, we can find a transversal C_k-tiling covering all but at most $\frac{\zeta m}{2}$ vertices in the union of its clusters in G.*

Proof of Claim 4.9. Given a copy of H, without loss of generality, we may assume that $V(H) = \left\{U_{1,1}, U_{2,2}, \ldots, U_{k,k}\right\}$ and $\left\{U_{1,1}, U_{2,2}\right\}, \left\{U_{3,3}, U_{4,4}\right\} \in E(H)$. Therefore $(U_{1,1}, U_{2,2})$ and $(U_{3,3}, U_{4,4})$ are (ε, d)-regular in G'. Now it suffices to show that for any $Z_i \subseteq U_{i,i}$ with $i \in [k]$ each of size at least $\frac{\zeta m}{2k}$, there exists a copy of C_k with exactly one vertex inside each Z_i.

Since $|Z_i| \geq \frac{\zeta m}{2k} \geq \varepsilon m$ for $i \in [4]$, Fact 4.4 implies that there exists a subset $Z_2' \subseteq Z_2$ of size at least $|Z_2| - \varepsilon m$ such that every vertex in Z_2' has at least $(d-\varepsilon)|Z_1|$ neighbours inside Z_1 and a subset $Z_3' \subseteq Z_3$ of size at least $|Z_3| - \varepsilon m$ such that every vertex in Z_3' has at least $(d-\varepsilon)|Z_4|$ neighbours inside Z_4 respectively.

By the assumption $\alpha_b^*(G) < \alpha n$ and the fact that $|Z_2'|, |Z_3'| \geq \frac{\zeta m}{2k} - \varepsilon m \geq \alpha n$, there is at least one edge between Z_2' and Z_3'. Arbitrarily choose one edge $\{u, v\}$ with $u \in {Z_2}'$ and $v \in {Z_3}'$, and let $Q_1 = N(u) \cap Z_1$ and $Q_2 = N(v) \cap Z_4$, respectively. Then we have $|Q_1|, |Q_2| \geq (d-\varepsilon) \cdot \frac{\zeta m}{2k} \geq \alpha n$ and note that $|Z_i| \geq \frac{\zeta m}{2k} \geq 2\alpha n$ for every $i \in [k]$. By applying Proposition 4.7 with $X_i := Q_1$ and $X_j := Q_2$, we can obtain a transversal path of length $k-3$, where the ends in Q_1 and Q_2

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are denoted by u' and v', respectively. Together with three edges $\{u', u\}$, $\{u, v\}$, $\{v, v'\}$, we can construct a copy of transversal C_k in $\cup_{i=1}^{k} Z_i$. Thus we can obtain a transversal C_k-tiling covering all but at most $\frac{\zeta m}{2}$ vertices in $\cup_{i=1}^{k} U_{i,i}$ in G. □

This would finish the proof as the union of these C_k-tilings taken over all copies of H in $\mathcal{H}$ would leave at most

$$|U_0| + |\mathcal{H}| \cdot \frac{\zeta m}{2} \leq \varepsilon n + \frac{\zeta n}{2} \leq \zeta n$$

vertices uncovered. □

4.4. Building an absorbing set

A typical step in the absorption method for F-factor is to show that every $k := |V(F)|$-set has polynomially many absorbers (see [14]). However, it remains unclear whether this property holds in our setting. Instead, a new approach due to Nenadov and Pehova [35] guarantees an absorbing set provided that every k-set has linearly many vertex-disjoint absorbers. Since the host graph in Lemma 4.2 is k-partite, we aim to show that every transversal k-set has linearly many vertex-disjoint absorbers. For this, we shall make use of the *lattice-based absorbing method* developed by Han [13].

4.4.1. Finding absorbers

To illustrate the lattice-based absorbing method, we introduce some definitions. Let G, F be given as aforementioned and m, t be positive integers. Then we say that two vertices $u, v \in V(G)$ are (F, m, t)*-reachable* (in G) if for any set W of m vertices, there is a set $S \subseteq V(G) \setminus W$ of size at most $kt - 1$ such that both $G[\{u\} \cup S]$ and $G[\{v\} \cup S]$ have F-factors, where we call such S an F*-connector* for u, v. Moreover, a set $U \subseteq V(G)$ is (F, m, t)*-closed* if every two vertices u, v in U are (F, m, t)-reachable, where the corresponding F-connector for u, v may not be included in U.

The following result builds a sufficient condition to ensure that every transversal k-set has linearly many vertex-disjoint absorbers.

Lemma 4.10. *Given $k \in \mathbb{N}$ with $k \geq 4$ and a constant $\delta > \frac{2}{k}$, there exist $\alpha, \beta > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha_b^*(G) < \alpha n$. Then every transversal k-set in G has at least $\frac{\beta n - k}{4k^2}$ vertex-disjoint $(C_k, 2k)$-absorbers.*

4.4.2. Proof of Lemma 4.2

In order to prove the existence of an absorbing set, we introduce a notion of F-fan.

Definition 4.11. ([15]) For a vertex $v \in V(G)$ and a k-vertex graph F, an *F-fan* $\mathcal{F}_v$ at v in $V(G)$ is a collection of pairwise disjoint sets $S \subseteq V(G) \setminus \{v\}$ such that for each $S \in \mathcal{F}_v$ we have that $|S| = k - 1$ and $\{v\} \cup S$ spans a copy of F.

To build an absorbing structure, we shall make use of bipartite templates as follows, which was introduced by Montgomery [34].

Lemma 4.12. *Let $\beta > 0$. There exists m_0 such that the following holds for every $m \geq m_0$. There exists a bipartite graph B_m with vertex classes $X_m \cup Y_m$ and Z_m and maximum degree 40, such that $|X_m| = m + \beta m$, $|Y_m| = 2m$ and $|Z_m| = 3m$, and for every subset $X'_m \subseteq X_m$ of size $|X'_m| = m$, the induced graph $B[X'_m \cup Y_m, Z_m]$ contains a perfect matching.*

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Proof of Lemma 4.2. Given $k \in \mathbb{N}$ with $k \geq 4$ and positive constants $\delta > \frac{2}{k}$ and $\gamma \leq \frac{\delta}{2}$, we choose $\frac{1}{n} \ll \alpha \ll \xi \ll \beta \ll \delta, \gamma, \frac{1}{k}$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha_b^*(G) < \alpha n$. Lemma 4.10 implies that every transversal k-set in G has at least $\frac{\beta n - k}{4k^2}$ vertex-disjoint $(C_k, 2k)$-absorbers. Let $\tau := \frac{\beta}{8k^2}$. Then for every $v \in V(G)$, there is a C_k-fan $\mathcal{F}_v$ in $V(G)$ of size at least $\frac{\beta n - k}{4k^2} \geq \tau n$. Now it suffices to find a ξ-absorbing set R for some $\xi > 0$ such that $|R| \leq \tau n \leq \gamma n$.

Let $q = \frac{\tau}{1000k^3}$ and $\beta' = \frac{q^{k-1}\tau}{2k}$. For $i \in [k]$, let $X_i \subseteq V_i$ be a set of size qn chosen uniformly at random. For every $v \in V(G)$, let f_v denote the number of the sets from $\mathcal{F}_v$ that lie inside $\cup_{i=1}^k X_i$. Note that $\mu := \mathbb{E}[f_v] = q^{k-1}|\mathcal{F}_v| \geq q^{k-1}\tau n$. By the union bound and Chernoff's inequality, we have

$$\mathbb{P}\left[\text{ there is } v \in V(G) \text{ with } f_v < \frac{\mu}{2}\right] \leq kn \exp\left(\frac{-(\mu/2)^2}{2\mu}\right) \leq kn \exp\left(-\frac{q^{k-1}\tau}{8}n\right) = o(1).$$

Therefore, as n is sufficiently large, there exist $X_i \subseteq V_i$ with $|X_i| = qn$ such that for each $v \in V(G)$, there is a subfamily $\mathcal{F}'_v$ of at least $\frac{q^{k-1}\tau n}{2} = k\beta' n$ sets from $\mathcal{F}_v$ contained in $\cup_{i=1}^k X_i$.

Let $m = |X_i|/(1+\beta')$ and note that m is linear in n. Let $\{I_i\}_{i\in[k]}$ be a partition of $[3km]$ with each $|I_i| = 3m$. For $i \in [k]$, arbitrarily choose k vertex-disjoint subsets Y_i, $Z_{i,j}$ for $j \in [k] \setminus \{i\}$ in $V_i \setminus X_i$ with $|Y_i| = 2m$ and $|Z_{i,j}| = 3m$. Let $X = \cup_{i=1}^k X_i$, $Y = \cup_{i=1}^k Y_i$ and $Z = \cup Z_{i,j}$. Then we have $|X| = (1+\beta')km$, $|Y| = 2km$ and $|Z| = 3k(k-1)m$. For each $j \in [k]$, we partition $\cup_{i\in[k]\setminus\{j\}} Z_{i,j}$ into a family $\mathcal{Z}_j$ of $3m$ transversal $(k-1)$-sets and take an arbitrary bijection $\phi_j : \mathcal{Z}_j \to I_j$. Moreover, we define a function φ on $[3km]$ such that $\varphi(x) := \phi_j^{-1}(x)$ if $x \in I_j$. Let T_i be the bipartite graph obtained by Lemma 4.12 with vertex classes $X_i \cup Y_i$ and I_i, and let $T = \cup_{i=1}^k T_i$. Then T is a bipartite graph between $X \cup Y$ and $[3km]$ with $\Delta(T) \leq 40$.

We claim that there exists a family $\{A_e\}_{e\in E(T)}$ of pairwise vertex-disjoint subsets in $V(G) \setminus (X \cup Y \cup Z)$ such that for every $e = \{w_1, w_2\} \in E(T)$ with $w_1 \in X \cup Y$ and $w_2 \in [3km]$, the set A_e is a $(C_k, 2k)$-absorber for the transversal k-set $\{w_1\} \cup \varphi(w_2)$. Indeed otherwise, there exists an edge $e' \in E(T)$ without such a subset. Recall that $m = \frac{qn}{1+\beta'}$ and $\Delta(T) \leq 40$, then we have

$$|X| + |Y| + |Z| + \left|\bigcup_{e\in E(T)\setminus\{e'\}} A_e\right| \leq 4km + 3km(k-1) + 2k^2|E(T)| \leq 4k^2 m + 80k^2 \cdot 3km \leq \frac{\tau n}{2}.$$

Since every transversal k-set has at least τn vertex-disjoint $(C_k, 2k)$-absorbers in G, we can choose one in $V(G) \setminus (X \cup Y \cup Z \cup \bigcup_{e\in E(T)\setminus\{e'\}} A_e)$ as the subset $A_{e'}$, a contradiction.

Let $R = X \cup Y \cup Z \cup \bigcup_{e\in E(T)} A_e$. Then $|R| \leq \tau n$ and we claim that R is a ξ-absorbing set in G. Indeed, for an arbitrary balanced subset $U \subseteq V(G) \setminus R$ with $|U| \leq \xi n$, we shall verify that $G[R \cup U]$ admits a C_k-factor. Note that if there exist $Q_i \subseteq X_i$ with $|Q_i| = \beta' m$ for $i \in [k]$ and a transversal C_k-factor in $G[\bigcup_{i=1}^k Q_i \cup U]$, then $G[R \cup U]$ contains a transversal C_k-factor. In fact by setting $X'_i = X_i \setminus Q_i$, Lemma 4.12 implies that there is a perfect matching M in T between $\bigcup_{i=1}^k X'_i \cup Y$ and $[3km]$. For each edge $e = \{w_1, w_2\} \in M$ take a transversal C_k-factor in $G[\{w_1\} \cup \varphi(w_2) \cup A_e]$ and for each $e' \in E(T) \setminus M$ take a transversal C_k-factor in $G[A_{e'}]$, which forms a transversal C_k-factor of $G[R \setminus \cup_{i=1}^k Q_i]$. Thus together with the above assumption, we can obtain a transversal C_k-factor in $G[R \cup U]$.

Hence, it suffices to find the desired Q_i as above. Recall that every $v \in V(G)$ has a subfamily $\mathcal{F}'_v$ of at least $k\beta' n$ sets in $\cup_{i=1}^k X_i$. By the choice that $\xi \ll \beta, \frac{1}{k}$ and thus $|U| \leq \xi n \leq \beta' n$, one can greedily find a family $\mathcal{C}_1$ of vertex-disjoint copies of transversal C_k covering U with vertices in $\cup_{i=1}^k X_i$. Let $Q_{i1} := X_i \cap V(\mathcal{C}_1)$. Then we have $|Q_{i1}| = \frac{k-1}{k}|U|$ and $\mathcal{C}_1$ is a transversal

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C_k-factor in $G[\bigcup_{i=1}^{k} Q_{i1} \cup U]$. Moreover, one can greedily pick a family $\mathcal{C}_2$ of $\beta' m - \frac{k-1}{k}|U|$ vertex-disjoint copies of transversal C_k in $G[X \setminus V(\mathcal{C}_1)]$. This is possible since every vertex v has at least $k\beta' n - (k-1)|U| \geq k(\beta' m - \frac{(k-1)}{k}|U|)$ sets from $\mathcal{F}_v$ that are disjoint from $V(\mathcal{C}_1)$. Let $Q_{i2} := X_i \cap V(\mathcal{C}_2)$ and $Q_i := Q_{i1} \cup Q_{i2}$. Then Q_i is desired because $\mathcal{C}_1 \cup \mathcal{C}_2$ is indeed a transversal C_k-factor of $G[\bigcup_{i=1}^{k} Q_i \cup U]$. This completes the entire proof. □

4.4.3. Proof of Lemma 4.10

Proof of Lemma 4.10. Given $k \in \mathbb{N}$ with $k \geq 4$ and $\delta > \frac{2}{k}$, we set $\eta := \delta - \frac{2}{k}$ and choose $\frac{1}{n} \ll \alpha \ll \beta \ll \delta, \eta$. Let $G = (V_1, \ldots, V_k, E)$ be a spanning subgraph of the n-blow-up of C_k with $\bar{\delta}(G) \geq \delta n$ and $\alpha_b^*(G) < \alpha n$.

Claim 4.13. *For each $i \in [k]$, V_i is $(C_k, \beta n, 2)$-closed.*

Proof of Claim 4.13. Without loss of generality, we may assume that $i = 1$. For any two vertices $u, v \in V_1$, since $\bar{\delta}(G) \geq \delta n$, we can choose four vertex-disjoint sets $D_1, D_2 \subseteq V_2$ and $D_3, D_4 \subseteq V_k$ each of size at least $\frac{\delta n}{2}$ such that $D_1 \subseteq N_{V_2}(u)$, $D_2 \subseteq N_{V_2}(v)$, $D_3 \subseteq N_{V_k}(u)$, and $D_4 \subseteq N_{V_k}(v)$, respectively. Given any vertex set $W \subseteq V(G) \setminus \{u, v\}$ of size at most βn, let $V_i' = V_i \setminus W$ and $D_j' = D_j \setminus W$ for $i \in [k]$ and $j \in [4]$. Note that $|V_i'| \geq n - \beta n \geq 10\alpha n$ and $|D_j'| \geq \frac{\delta n}{2} - \beta n \geq \alpha n$.

Since $\alpha_b^*(G) < \alpha n \leq |D_j'|, |V_1'|$, there exist subsets $S_j \subseteq V_1'$ of size at least $|V_1'| - \alpha n$ such that every vertex in S_j has at least one neighbour inside D_j'. By the fact that $|S_j| \geq |V_1'| - \alpha n > \frac{3}{4}|V_1'|$, we have that $S := \bigcap_{j=1}^{4} S_j \neq \emptyset$. Arbitrarily choose $x \in S$, and therefore there exist vertices y_1, y_2, y_3, y_4 satisfying $y_i \in N_{D_i'}(x)$ for $i \in [k]$. Note that $|N_{V_3'}(y_1)|, |N_{V_{k-1}'}(y_3)| \geq \delta n - \beta n \geq \alpha n$ and $|V_i'| \geq 10\alpha n$ for $i \in [k]$. When $k > 4$, by applying Proposition 4.7 with $X_i := N_{V_3'}(y_1)$ and $X_j := N_{V_{k-1}'}(y_3)$, we can obtain a transversal cycle C_1 passing through u, y_1, y_3. When $k = 4$, since $|N_{V_3'}(y_1)|$, $|N_{V_3'}(y_3)| \geq \delta n - \beta n \geq (\frac{1}{2} + \eta - \beta)n > \frac{n}{2}$, we can easily find a common neighbour of y_1, y_3 and thus obtain a transversal cycle C_1 passing through u, y_1, y_3. Similarly, we can obtain a transversal cycle C_2 in $V(G) \setminus (W \cup V(C_1))$ that passes through v, y_2, y_4. In fact, the set $\{x\} \cup (V(C_1) \setminus \{u\}) \cup (V(C_2) \setminus \{v\})$ is a C_k-connector for u, v. Therefore by definition, u is $(C_k, \beta n, 2)$-reachable to v. □

For every transversal k-subset $S \subseteq V(G)$, we greedily find as many pairwise disjoint $(C_k, 2k)$-absorbers for S as possible. For convenience, we write $S = \{s_1, s_2, \ldots, s_k\}$ where $s_i \in V_i$ for $i \in [k]$. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_\ell\}$ be a maximal family of such absorbers. Suppose to the contrary that $\ell < \frac{\beta n - k}{4k^2}$. Since each A_j has size at most $2k^2$, we have $\left|\bigcup_{j=1}^{\ell} A_j\right| < \frac{\beta n - k}{2}$.

Since $\alpha \ll \beta \ll \delta$, we can easily find a copy T of transversal C_k in $V(G) \setminus (\bigcup_{j=1}^{\ell} A_j \cup S)$ and write $T = \{t_1, t_2, \ldots, t_k\}$ where $t_i \in V_i$ for $i \in [k]$. By the closedness of V_i, we can pick a collection $\{I_1, I_2, \ldots, I_k\}$ of vertex-disjoint subsets in $V(G) \setminus (\bigcup_{j=1}^{\ell} A_j \cup S \cup T)$ such that each I_i is a C_k-connector for s_i, t_i with $|I_i| \leq 2k - 1$. In fact, for any $1 \leq k' \leq k$, we have

$$\left|\bigcup_{j=1}^{\ell} A_j \cup \left(\bigcup_{i=1}^{k'} I_i\right) \cup S \cup T\right| \leq \frac{\beta n - k}{2} + k(2k-1) + 2k < \beta n.$$

Therefore, we can choose such I_i one by one because s_i and t_i are $(C_k, \beta n, 2)$-reachable. At this point, it is easy to verify that $\bigcup_{i=1}^{k} I_i \cup T$ is actually a $(C_k, 2k)$-absorber for S, contrary to the maximality of ℓ. □

https://doi.org/10.1017/S0963548323000378 Published online by Cambridge University Press

Acknowledgements

We would like to thank the anonymous referees for their thorough reading of the manuscript and many helpful suggestions.

References

[1] Balogh, J., Molla, T. and Sharifzadeh, M. (2016) Triangle factors of graphs without large independent sets and of weighted graphs. *Random Struct. Algorithms* **49**(4) 669–693.
[2] Bohman, T., Frieze, A. and Martin, R. (2003) How many random edges make a dense graph hamiltonian? *Random Struct. Algorithms* **22**(1) 33–42.
[3] Böttcher, J., Han, J., Kohayakawa, Y., Montgomery, R., Parczyk, O. and Person, Y. (2019) Universality for bounded degree spanning trees in randomly perturbed graphs. *Random Struct. Algorithms* **55**(4) 854–864.
[4] Böttcher, J., Schacht, M. and Taraz, A. (2009) Proof of the bandwidth conjecture of Bollobás and Komlós. *Math. Ann.* **343**(1) 175–20531.
[5] Chang, F., Han, J., Kim, J., Wang, G. and Yang, D. (2023) Embedding clique-factors in graphs with low ℓ-independence number. *J. Comb. Theory Ser. B* **161** 301–330.
[6] Corrádi, K. and Hajnal, A. (1963) On the maximal number of independent circuits in graph. *Acta Math. Acad. Sci. Hung.* **14**(3-4) 423–439.
[7] Dirac, G. A. (1952) Some theorems on abstract graphs. *Proc. Lond. Math. Soc.* **3**(1) 69–81.
[8] Draganić, N., Correia, D. M. and Sudakov, B. (2023) A generalization of bondy's pancyclicity theorem. arXiv preprint arXiv: 2302.12752.
[9] Ergemlidze, B. and Molla, T. (2022) Transversal C_k-factors in subgraphs of the balanced blow-up of C_k. *Comb. Probab. Comput.* **31**(6) 1031–1047.
[10] Fischer, E. (1999) Variants of the Hajnal–Szemerédi theorem. *J. Graph Theory* **31** 275–282.
[11] Friedman, J. (2008) A proof of Alon's second eigenvalue conjecture. *Mem. Am. Math. Soc.* **195**(910) viii+100.
[12] Friedman, J. and Pippenger, N. (1987) Expanding graphs contain all small trees. *Combinatorica* **7**(1) 71–76.
[13] Han, J. (2017) Decision problem for perfect matchings in dense k-uniform hypergraphs. *Trans. Am. Math. Soc.* **369**(7) 5197–5218.
[14] Han, J., Hu, P., Wang, G. and Yang, D. (2023) Clique-factors in graphs with sublinear ℓ-independence number. *Comb. Prob. Comput.* **32**(4) 665–681.
[15] Han, J., Morris, P., Wang, G. and Yang, D. (2023) A Ramsey–Turán theory for tilings in graphs. *Random Struct. Algorithms*, 1–31. doi:10.1002/rsa.21182.
[16] Haxell, P. E. (2001) Tree embeddings. *J. Graph Theory* **36**(3) 121–130.
[17] Janson, S., Łuczak, T. and Ruciński, A. (2000) *Random Graphs*, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, New York
[18] Johannsen, D., Krivelevich, M. and Samotij, W. (2013) Expanders are universal for the class of all spanning trees. *Comb. Prob. Comput.* **22**(2) 253–281.
[19] Johansson, R. (2000) Triangle-factors in a balanced blown-up triangle. *Discrete Math.* **211**(1-3) 249–254.
[20] Keevash, P. and Mycroft, R. (2015) A multipartite Hajnal–Szemerédi theorem. *J. Comb. Theory Ser. B* **114** 187–236.
[21] Kim, J., Kim, Y. and Liu, H. (2020) Tree decompositions of graphs without large bipartite holes. *Random Struct. Algorithms* **57**(1) 150–168.
[22] Knierim, C. and Su, P. (2021) K_r-factors in graphs with low independence number. *J. Comb. Theory Ser. B* **148** 60–83.
[23] Komlós, J. (1999) The blow-up lemma. *Comb. Prob. Comput.* **8**(1-2) 161–176.
[24] Komlós, J., Sárközy, G. N. and Szemerédi, E. (1995) Proof of a packing conjecture of Bollobás. *Comb. Prob. Comput.* **4**(3) 241–255.
[25] Komlós, J., Sárközy, G. N. and Szemerédi, E. (2001) Spanning trees in dense graphs. *Comb. Prob. Comput.* **10**(5) 397–416.
[26] Komlós, J. and Simonovits, M. (1996) Szemerédi's regularity lemma and its applications in graph theory. *Combinatorica* **2** 295–352.
[27] Krivelevich, M. (2010) Embedding spanning trees in random graphs. *SIAM J. Discrete Math.* **24**(4) 1495–1500.
[28] Krivelevich, M., Kwan, M. and Sudakov, B. (2017) Bounded-degree spanning trees in randomly perturbed graphs. *SIAM J. Discrete Math.* **31**(1) 155–171.
[29] Lo, A. and Markström, K. (2013) A multipartite version of the Hajnal–Szemerédi theorem for graphs and hypergraphs. *Comb. Prob. Comput.* **22**(1) 97–111.
[30] Lovász, L. and Plummer, M. D. (2009) *Matching Theory*. American Mathematical Society Press.
[31] Magyar, C. and Martin, R. (2002) Tripartite version of the Corrádi–Hajnal theorem. *Discrete Math.* **254**(1-3) 289–308.
[32] Martin, R. and Szemerédi, E. (2008) Quadripartite version of the Hajnal–Szemerédi theorem. *Discrete Math.* **308**(19) 4337–4360.

[33] Mcdiarmid, C. and Yolov, N. (2017) Hamilton cycles, minimum degree and bipartite holes. *J. Graph Theory* **86**(3) 277–285.
[34] Montgomery, R. (2019) Spanning trees in random graphs. *Adv. Math.* **356** 106793.
[35] Nenadov, R. and Pehova, Y. (2020) On a Ramsey–Turán variant of the Hajnal–Szemerédi theorem. *SIAM J. Discrete Math.* **34**(2) 1001–1010.

Cite this article: Han J, Hu J, Ping L, Wang G, Wang Y, and Yang D (2024). Spanning trees in graphs without large bipartite holes. *Combinatorics, Probability and Computing* **33**, 270–285. https://doi.org/10.1017/S0963548323000378