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A CHARACTERIZATION OF LEFT PERFECT RINGS

YIQIANG ZHOU

ABSTRACT. In this note, we show that a ring R is a left perfect ring if and only if every generating set of each left R-module contains a minimal generating set. This result gives a positive answer to a question on left perfect rings raised by Nashier and Nichols.

Introduction. Throughout all rings R are associative with identity, and all modules are unitary left R-modules. For a module M, a subset X of M is said to be a *generating set* of M if $M = \sum_{x\in X} Rx$; and a minimal generating set of M is any generating set Y of M such that no proper subset of Y can generate M. A module is called *quasi-cyclic* if each of its finitely generated submodules is contained in a cyclic submodule [3]. For a sequence $\{a_n, n = 1, 2, \ldots\}$ of elements of R, let F be the free R-module with basis $x_1, x_2, \ldots$, G the submodule of F generated by the set $\{x_n - a_n x_{n+1} : n = 1, 2, \ldots\}$, and $[F, \{a_n\}, G]$ the quotient module F/G. It is an easy observation that every $[F, \{a_n\}, G]$ is a quasi-cyclic module. In [2], Neggers conjectured that a ring R was left perfect if and only if every R-module had a minimal generating set. A counterexample to this conjecture was given by Nashier and Nichols in [3], where they provided an interesting characterization of left perfect rings which says that the ring R is left perfect if and only if every quasi-cyclic module is cyclic if and only if every $[F, \{a_n\}, G]$ is cyclic. By means of the characterization, they observed that if, for a given ring R, every generating set of any R-module contains a minimal generating set, then the ring R must be left perfect. It remains open whether the converse holds. This question stimulates the work of the present paper.

A characterization of left perfect rings. The main result of this paper can be stated as follows.

THEOREM. *The ring R is a left perfect ring if and only if every generating set of each R-module contains a minimal generating set.*

We need the following lemma for the proof of the theorem.

LEMMA. *If M is a semi-simple R-module, then every generating set of M contains a minimal generating set.*

PROOF. Let M be a semi-simple R-module with a generating set X. By the Maximum Principle, there is a non-empty subset $X_1 \subseteq X$ maximal with respect to the condition

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that $\{Rx : x \in X_1\}$ is independent. Clearly X_1 is a minimal generating set of $\sum_{x\in X_1} Rx$. Suppose that we have chosen subsets $X_\alpha \subseteq X$ for all $\alpha < \sigma$ such that X_α is a minimal generating set of $\sum_{x\in X_\alpha} Rx$, and for each $\alpha + 1 < \sigma$ we have $X_\alpha \subseteq X_{\alpha+1}$ and $X_\alpha \subset X_{\alpha+1}$ if X_α does not generate M.

(1) σ is a limit ordinal. We choose $X_\sigma = \bigcup_{\alpha<\sigma} X_\alpha$. Thus, X_σ is a minimal generating set of $\sum_{x\in X_\sigma} Rx$.

(2) σ is not a limit ordinal. If $X_{\sigma-1}$ generates M, then we let $X_\sigma = X_{\sigma-1}$. Suppose that $X_{\sigma-1}$ does not generate M. Since M is semi-simple, $M = (\sum_{x\in X_{\sigma-1}} Rx) \oplus N$ for some N. Let π be the projection of M onto N. Since $X_{\sigma-1}$ does not generate M, we have $Y = \{x \in X : \pi(x) \neq 0\}$ is not empty. Again, there is a non-empty subset $Z \subseteq Y$ maximal with respect to the condition that $\{R\pi(x) : x \in Z\}$ is independent. Let $X_\sigma = X_{\sigma-1} \cup Z$. Then $X_{\sigma-1} \subset X_\sigma$. It is straightforward to verify that X_σ is a minimal generating set of $\sum_{x\in X_\sigma} Rx$.

By the Transfinite Induction, we can construct a chain of subsets of X:

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\alpha \subseteq \cdots \subseteq X_\sigma \subseteq \cdots$$

such that X_α is a minimal generating set of $\sum_{x\in X_\alpha} Rx$, and $X_\alpha \subset X_{\alpha+1}$ if X_α does not generate M. Since X is a set, there is an ordinal σ such that $X_\sigma = X_{\sigma+1}$. It shows that X_σ is a minimal generating set of M. ■

PROOF OF THE THEOREM. One direction is the observation of Nashier and Nichols [3]. For the other direction, we let R be a left perfect ring and M an R-module with a generating set X. We denote the Jacobson radical of R by J. As a module over the semi-simple ring R/J, $M/(JM)$ is semi-simple, with a generating set $\{x+JM : x \in X\}$. By the lemma, there is a subset $Y \subseteq X$ such that $\{x+JM : x \in Y\}$ is a minimal generating set of the R/J-module $M/(JM)$. This implies that Y is a minimal generating set of the R-module $\sum_{x\in Y} Rx$. Note that $M = \sum_{x\in Y} Rx + JM$. It follows that $M/(\sum_{x\in Y} Rx) = J[M/(\sum_{x\in Y} Rx)]$. Since J is left T-nilpotent, we have, by [1, 28.3], that $M/(\sum_{x\in Y} Rx) = \bar{0}$, *i.e.*, $M = \sum_{x\in Y} Rx$. Therefore, Y is a minimal generating set of M. ■

An element $r \in R$ is said to be *left cancellable* if, for any $a \in R$, $ra = 0$ implies $a = 0$. A right cancellable element is defined analogously. It is known that for a left perfect ring R, every left cancellable element of R is invertible (see [5, Lemma 1.10, p. 54]). We have the following consequence.

COROLLARY. *Every right cancellable element of a left perfect ring R is invertible.*

PROOF. Let $r \in R$ be a right cancellable element. We claim that r is left invertible. Consider the module $[F, \{a_n\}, G]$, where $a_n = r$ for all n. Let H_i be the submodule of $[F, \{a_n\}, G]$ generated by $\{x_k + G : k \leq i\}$. Then

$$(0) \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_i \subseteq \cdots, \quad \text{and} \quad [F, \{a_n\}, G] = \bigcup_{i\geq 0} H_i.$$

Suppose that r is not left invertible. Since r is right cancellable, it is straightforward to verify that $x_i + G \in H_i$ but $x_i + G \notin H_{i-1}$. We show that no minimal generating set can be extracted from the generating set $\{x_i + G : i = 1, 2, \ldots\}$ of $[F, \{a_n\}, G]$ and then our claim will follow from the theorem. Suppose that $\{x_i + G : i \in \mathbf{L}\}$ is a minimal generating set of $[F, \{a_n\}, G]$, where $\mathbf{L}$ is a subset of the set of positive integers. Let n be the least integer in $\mathbf{L}$. From $x_{n+1} + G \notin H_n$, it follows that $\{x_n + G\}$ can not be a minimal generating set of $[F, \{a_n\}, G]$. Therefore, there exists an integer $m \in \mathbf{L}$ with $n < m$. Clearly, $x_n + G = r^{m-n}(x_m + G)$. This implies that $\{x_i + G : i \in \mathbf{L} \setminus \{n\}\}$ is a generating set of $[F, \{a_n\}, G]$, a contradiction. Therefore, r is left invertible, *i.e.*, $tr = 1$ for some $t \in R$. It follows that r is left cancellable, and hence is invertible by [5, Lemma 1.10, p. 54]. ∎

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Mathematics Department
University of British Columbia
Vancouver, British Columbia
V6T 1Z2

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