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Selection principles and proofs from the Book

Boaz Tsaban

To Adina

Abstract. I provide simplified proofs for each of the following fundamental theorems regarding selection principles:

(1) The Quasinormal Convergence Theorem, due to the author and Zdomskyy, asserting that a certain, important property of the space of continuous functions on a space is actually preserved by Borel images of that space.
(2) The Scheepers Diagram Last Theorem, due to Peng, completing all provable implications in the diagram.
(3) The Menger Game Theorem, due to Telgársky, determining when Bob has a winning strategy in the game version of Menger's covering property.
(4) A lower bound on the additivity of Rothberger's covering property, due to Carlson.

The simplified proofs lead to several new results.

1 Introduction

The study of *selection principles* unifies notions and studies originating from dimension theory (Menger and Hurewicz), measure theory (Borel), convergence properties (Császár–Laczkovicz), and function spaces (Gerlits–Nagy and Arhangel'skiĭ), notions analyzed and developed in numerous studies of later mathematicians, especially since the 1996 paper of Just, Miller, Scheepers, and Szeptycki [8]. The unified notions include, among others, many classic types of special sets of real numbers, local properties in function spaces, and more recent types of convergence properties.

Selective topological covering properties form the kernel of selection principles. These covering properties are related via the Scheepers Diagram (Figure 1). This is a diagram of covering properties and implications among them. The properties in this diagram are obtained as follows:

For families A and B of sets, let $\mathsf{S}_1(\mathrm{A},\mathrm{B})$ be the statement: For each sequence of elements of the family A, we can pick one element from each sequence member, and obtain an element of the family B. When $\mathrm{A} = \mathrm{B} = \mathrm{O}(X)$, the family of open covers of a topological space X, we obtain *Rothberger's property* (1941), the topological version

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$$\begin{tikzcd}
 & & \underset{\text{Hurewicz}}{\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)} \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega) \arrow[r] & \underset{\text{Menger}}{\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})} \\
 & & \mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega) \arrow[ur] & & \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[r] \arrow[uurr] & \mathsf{S}_1(\Gamma,\Omega) \arrow[rr] \arrow[ur] & & \mathsf{S}_1(\Gamma,\mathrm{O}) \arrow[uur] & \\
 & & \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega) \arrow[uu] & & \\
\underset{\text{Gerlits–Nagy}}{\mathsf{S}_1(\Omega,\Gamma)} \arrow[r] \arrow[uu] & \mathsf{S}_1(\Omega,\Omega) \arrow[rr] \arrow[uu] \arrow[ur] & & \underset{\text{Rothberger}}{\mathsf{S}_1(\mathrm{O},\mathrm{O})} \arrow[uu] &
\end{tikzcd}$$

Figure 1: The Scheepers Diagram.

of Borel's strong measure zero. We say that a space X satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ if the assertion $\mathsf{S}_1(\mathrm{O}(X),\mathrm{O}(X))$ holds, and similarly for the other selective properties.

The hypothesis $\mathsf{S}_{\mathrm{fin}}(\mathrm{A},\mathrm{B})$ is obtained by replacing *one* by *finitely many* in the above definition. The property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ is, by an observation of Hurewicz, equivalent to Menger's basis property, a dimension-type property. The property $\mathsf{U}_{\mathrm{fin}}(\mathrm{A},\mathrm{B})$ is obtained by further allowing us to take the *unions* of the selected finite subsets – this matters for some types of covers. For technical reasons, this property does not consider all covers of type A, but only those that have no finite subcover.

A cover of a space is an *ω-cover* if no member of the cover covers the entire space, but every finite subset of the space is covered by some member of the cover. For a space X, $\Omega(X)$ is the family of open ω-covers of the space. A *point-cofinite cover* is an infinite cover where every point of the space belongs to all but finitely many members of the cover. $\Gamma(X)$ is the family of open point-cofinite covers of the space X.

Applying the mentioned selection principles to the cover types O, Ω, and Γ, we obtain additional important properties, such as Hurewicz's property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ [7]. We also obtain the Gerlits–Nagy γ-property $\mathsf{S}_1(\Omega,\Gamma)$ [6], characterizing the Fréchet–Urysohn property of the function space $\mathrm{C_p}(X)$ of continuous real-valued functions, with the topology of pointwise convergence: A topological space is *Fréchet–Urysohn* if every point in the closure of a set is actually a limit of a sequence in the set. This duality between the spaces X and $\mathrm{C_p}(X)$ also translates various tightness and convergence properties of the space $\mathrm{C_p}(X)$ – discovered earlier by Arhangel'skiiĭ, Bukovský, Sakai, and others – to the selective covering properties $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$, $\mathsf{S}_1(\Gamma,\Gamma)$, and $\mathsf{S}_1(\Omega,\Omega)$. In Section 2, we provide a surprisingly simple proof of one of the most important results of this type. While the result itself does not involve selective covering properties explicitly, its proof does that extensively.

A topological space is *Lindelöf* if every open cover has a countable subcover. For example, all sets of real numbers are Lindelöf. Since all selection principles concern countable sequences, the theory mainly deals with Lindelöf spaces. For Lindelöf spaces, the Scheepers Diagram is the result of a classification of all properties thus introduced; each property is equivalent to one in the diagram [8]. It was long open whether any additional implication could be established among the properties in the diagram. In Section 3, we deal with the recent, surprising solution of this problem.

Menger's covering property $S_{fin}(O, O)$ is the oldest, most general, and most applied property in the Scheepers Diagram. Initially, Menger conjectured his property to coincide with σ-compactness. While this turned out false [19], the game version of this property does provide a characterization of σ-compactness. A very transparent proof of this deep result is presented in Section 4.

In Section 5, we consider a connection to combinatorial set theory. We show that a nontrivial lower bound on the additivity of Rothberger's property follows easily from basic knowledge on selection principles.

The Book is a popular myth by Paul Erdős: A transfinite book containing the most simple proofs for all theorems. I would like to believe that the proofs presented here are similar to ones from the Book ...or from some of its preliminary drafts, at any rate.

2 The Quasinormal Convergence Theorem

By *set of real numbers*, or *real set* in short, we mean a topological space where every open set is a countable union of clopen sets. Such are, for example, totally disconnected subsets of the real line and, in particular, subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$. In general, every perfectly normal space with any of the properties considered in this section is a real set.

Let X be a real set. A sequence of real-valued functions $f_1, f_2, \dots$ on X converges *quasinormally* to a real-valued function f if there are positive real numbers $\varepsilon_1, \varepsilon_2, \dots$ converging to 0 such that for each point $x \in X$, we have

$$|f_n(x) - f(x)| \le \varepsilon_n$$

for all but finitely many n. Quasinormal convergence generalizes uniform convergence.

A real set X is a *QN space* if every sequence of continuous real-valued functions on X that converges to 0 pointwise, converges to 0 quasinormally. Equivalently, convergence in the space $C_p(X)$ is quasinormal.

QN spaces were studied intensively, e.g., by Bukovský, Recław, Repický, Scheepers, Nowik, Sakai, and Haleš ([20] and the references therein). This and other properties of similar type are preserved by continuous images, and all experience prior to the paper of the author and Zdomskyy [20] suggested that they are not preserved by Borel images. Thus, the following theorem [20, Theorem 9] came as a surprise.

The Baire space $\mathbb{N}^{\mathbb{N}}$ is quasiordered by the relation $\le^*$ of eventual dominance: $f \le^* g$ if $f(n) \le g(n)$ for all but finitely many n. A subset Y of the Baire space $\mathbb{N}^{\mathbb{N}}$ is *bounded* if it is bounded with respect to eventual dominance.

Theorem 2.1 (Quasinormal Convergence Theorem) *The following assertions are equivalent for real sets X:*

(1) *The set X is a QN space.*
(2) *Every Borel image of the set X in the Baire space $\mathbb{N}^{\mathbb{N}}$ is bounded.*

The second property in the theorem is well known and straightforward to apply: The most natural transformations needed in proofs regarding these notions are always easily seen to be Borel. Consequently, the theorem had a dramatic impact on the study of QN spaces: First, many of the previous sophisticated arguments could be replaced by straightforward ones. Second, many properties that were hitherto considered separately turned out provably equivalent. Consequently, this theorem settled all problems concerning these properties [20].

The original proof of the Quasinormal Convergence Theorem is long and involved, and some of its parts are difficult to follow. A more natural proof was later published by Bukovský and Šupina [4, Section 4]. Inspired by a paper of Gerlits and Nagy [6], I have discovered the following surprisingly simple proof. All needed proof ingredients were already available at the time the Quasinormal Convergence Theorem was established. The following lemma provides the key to the proof.

For a space X, let $\mathrm{PF}(X)$ be the collection of countably infinite point-finite families of open sets in X.

Lemma 2.2 *Let X be a topological space. The following assertions are equivalent:*

(1) *Every Borel image of the space X in the Baire space* $\mathbb{N}^{\mathbb{N}}$ *is bounded.*
(2) *The space X satisfies* $\mathsf{S}_1(\mathrm{PF}, \mathrm{PF})$.

Proof. Let $\mathrm{F}(X)$ (respectively, $\mathrm{B}(X)$) be the family of countable closed (respectively, Borel) covers of the set X, and let $\mathrm{F}_\Gamma(X)$ (respectively, $\mathrm{B}_\Gamma(X)$) be the family of infinite closed (respectively, Borel) point-cofinite covers of the set X. The properties (1), $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$, and $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$ are equivalent [16, Theorem 1]. For a family $\mathcal{U}$ of open sets, we have $\mathcal{U} \in \mathrm{PF}(X)$ if and only if

$$\{\, U^{\mathrm{c}} : U \in \mathcal{U} \,\} \in \mathrm{F}_\Gamma(X).$$

It follows that $\mathsf{S}_1(\mathrm{PF}, \mathrm{PF}) = \mathsf{S}_1(\mathrm{F}_\Gamma, \mathrm{F}_\Gamma)$.

$(1) \Rightarrow (2)$: Clearly, $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$ implies $\mathsf{S}_1(\mathrm{F}_\Gamma, \mathrm{F}_\Gamma)$.

$(2) \Rightarrow (1)$: A theorem of Bukovský–Recław–Repický [3, Corollary 5.3] asserts that

$$\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma) = \mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma).$$

The usual argument [15, Proposition 11] shows that $\mathsf{S}_1(\mathrm{F}_\Gamma, \mathrm{F}_\Gamma)$ implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma)$: If $\{\, C_n : n \in \mathbb{N}\} \in \mathrm{F}(X)$ and there is no finite subcover, then $\{\, \bigcup_{k=1}^{n} C_k : n \in \mathbb{N}\} \in \mathrm{F}_\Gamma(X)$. ∎

A topological space Y has Arhangel'skiĭ's property α_1 if for every sequence $s_1, s_2, \dots$ of sequences converging to the same point, there is a sequence s such that the sets $\mathrm{im}(s_n) \setminus \mathrm{im}(s)$ are finite for all natural numbers n. This property is defined by properties of sets (images of sequences) rather than sequences. Fix a bijection $\varphi\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. For sequences $s_1, s_2, \dots$, with $s_n = (s_{(n,1)}, s_{(n,2)}, \dots)$ for each n, define

$$\bigsqcup_{n=1}^{\infty} s_n := (s_{\varphi(1)}, s_{\varphi(2)}, \dots).$$

Since convergence of a sequence does not depend on the order of its elements, it does not matter, for our purposes, which bijection φ is used. A sequence $\bigsqcup_{n=1}^{\infty} s_n$ converges to a point p if and only if each sequence s_n converges to p, and for each neighborhood U of p, we have $\mathrm{im}(s_n) \subseteq U$ for all but finitely many n.

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Lemma 2.3 *Let Y be an α_1 space. For every sequence $s_1, s_2, \ldots$ of sequences in the space Y converging to the same point p, there are tails t_n of s_n, for $n \in \mathbb{N}$, such that the sequence $\bigsqcup_{n=1}^{\infty} t_n$ converges to p.*

Proof. There is a sequence s such that the sets $\operatorname{im}(s_n) \setminus \operatorname{im}(s)$ are finite for all natural numbers n. By moving to a subsequence, we may assume that $\operatorname{im}(s) \subseteq \bigcup_{n=1}^{\infty} \operatorname{im}(s_n)$. Suppose that $s = (a_1, a_2, \ldots)$. For each natural number n, since the sequence s_n converges to the point p, every element other than p may appear in the sequence s_n only finitely often. Thus, there is a tail t_n of the sequence s_n such that

$$\operatorname{im}(t_n) \subseteq \{\, a_k : k \geq n \,\} \cup \{p\}.$$

Let U be a neighborhood of p. There is a natural number N such that

$$\operatorname{im}(t_n) \subseteq \{\, a_k : k \geq n \,\} \cup \{p\} \subseteq U$$

for all natural numbers $n \geq N$. Thus, the direct sum $t := \bigsqcup_{n=1}^{\infty} t_n$ converges to the point p. ∎

Sakai [12, Theorem 3.7] and Bukovský–Haleš [2, Theorem 11] proved that a real set X is a QN space if, and only if, the space $C_p(X)$ is an α_1 space. Thus, the Quasinormal Convergence Theorem can be stated, and proved, as follows.

Theorem 2.4 *The following assertions are equivalent for real sets X:*

(1) *The space $C_p(X)$ is an α_1 space.*
(2) *Every Borel image of the set X in the Baire space $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. (2) ⇒ (1): This is the straightforward implication. For completeness, we reproduce its proof [20, Theorem 9].

Let $s_1, s_2, \ldots$ be sequences in the space $C_p(X)$ that converge to a function $f \in C_p(X)$. For each natural number n, suppose that

$$s_n = (f_1^n, f_2^n, f_3^n, \ldots).$$

Define a Borel function $\Psi\colon X \to \mathbb{N}^{\mathbb{N}}$ by

$$\Psi(x)(n) := \min\left\{ k : (\forall m \geq k)\ |f_m^n(x) - f(x)| \leq \frac{1}{n} \right\}.$$

Let $g \in \mathbb{N}^{\mathbb{N}}$ be a $\leq^*$-bound for the image $\Psi[X]$. Then the sequence

$$\bigsqcup_{n=1}^{\infty} (f_{g(n)}^n, f_{g(n)+1}^n, f_{g(n)+2}^n, \ldots)$$

converges to the function f.

(1) ⇒ (2): This is the main implication. By Lemma 2.2, it suffices to prove that the set X satisfies $S_1(\mathrm{PF}, \mathrm{PF})$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{PF}(X)$. By thinning out the point-finite covers, we may assume that they are pairwise disjoint [15, Lemma 4]. For each set $U \in \bigcup_{n=1}^{\infty} \mathcal{U}_n$, let $\mathcal{C}_U$ be a countable family of clopen sets with $\bigcup \mathcal{C}_U = U$. For each

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natural number n, let

$$\mathcal{V}_n := \bigcup_{U \in \mathcal{U}_n} \mathcal{C}_U.$$

Every set $C \in \mathcal{V}_n$ is contained in at most finitely many sets $U \in \mathcal{U}_n$. Thus, the family $\mathcal{V}_n$ is infinite and point-finite. Let s_n be a bijective enumeration of the family

$$\{ \chi_V : V \in \mathcal{V}_n \}.$$

The sequence s_n is in $\mathrm{C_p}(X)$, and it converges to the constant function 0.

As the space $\mathrm{C_p}(X)$ is α_1, there is for each n a tail t_n of the sequence s_n such that the sequence $s := \bigsqcup_{n=1}^{\infty} t_n$ converges to 0. For each natural number n, pick a set $U_n \in \mathcal{U}_n$ with $\mathcal{C}_{U_n} \subseteq \mathrm{im}(t_n)$. Then the family $\{ U_n : n \in \mathbb{N}\}$ is infinite and point-finite. ■

For a set $X \subseteq \{0,1\}^{\mathbb{N}}$, Gerlits and Nagy (and, independently, Nyikoš) define a space $\mathrm{T}(X)$ as follows: Let $\{0,1\}^*$ denote the set of finite sequences of elements of the set $\{0,1\}$. Let $X \subseteq \{0,1\}^{\mathbb{N}}$. For each point $x \in X$, let $A_x \subseteq \{0,1\}^*$ be the set of initial segments of the point x. Let $X \cup \{0,1\}^*$ be the topological space where the points of the set $\{0,1\}^*$ are isolated, and for each point $x \in X$, a neighborhood base of x is given by the sets $\{x\} \cup B$, where B is a cofinite subset of the set A_x. Let $\mathrm{T}(X)$ be the one-point compactification of this space, and let ∞ be the compactifying point.

Gerlits and Nagy prove that if a set $X \subseteq \{0,1\}^{\mathbb{N}}$ is a Siepiński set, then the space $\mathrm{T}(X)$ is α_1, and that if the space $\mathrm{T}(X)$ is α_1, then the set X is a σ-set [6, Theorem 4]. The following theorem unifies these results and improves upon them. Indeed, every Borel image of a Siepiński set in the Baire space is bounded, and every set with bounded Borel images in the Baire space is a σ-set ([16] and the references therein).

Theorem 2.5 *Let $X \subseteq \{0,1\}^{\mathbb{N}}$. The following assertions are equivalent:*

(1) *The space $\mathrm{T}(X)$ is an α_1 space.*
(2) *Every Borel image of the set X in the Baire space $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. For a finite sequence $s \in \{0,1\}^*$, let $[s]$ be the basic clopen subset of the Cantor space $\{0,1\}^{\mathbb{N}}$ consisting of all functions extending s. Every open set in the space $\{0,1\}^{\mathbb{N}}$ is a disjoint union of basic clopen sets. A sequence $a_1, a_2, \dots$ in the set $\{0,1\}^*$ converges to ∞ in the space $\mathrm{T}(X)$ if, and only if, the set $\{ [a_n] : n \in \mathbb{N}\}$ is point-finite in the space X [6]. The argument in the proof of Theorem 2.4 applies. ■

3 The Scheepers Diagram Last Theorem

The implications in the Scheepers Diagram 1 were all rather straightforward to establish, and almost all other potential implications were ruled out by counterexamples [8]. Only two problems remained open: Does $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$ imply $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Omega)$? And if not, does $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ imply $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Omega)$? [8, Problems 1 and 2]. For nearly three decades, it was expected that the remaining two potential implications were refutable. Only when Peng came up with an entirely new method for refuting implications among selective covering properties [11], these problems were resolved. But not in the expected way: Having proved that $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$ does not imply $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Omega)$, Peng tried to

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$$\begin{tikzcd}
 & \underset{\text{Hurewicz}}{\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)} \arrow[r] & \mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega) \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega) \arrow[r] & \underset{\text{Menger}}{\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})} \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[ur] \arrow[r] & \mathsf{S}_1(\Gamma,\Omega) \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\Gamma,\mathrm{O}) \arrow[ur] & \\
 & & \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega) \arrow[uu] & & \\
\underset{\text{Gerlits–Nagy}}{\mathsf{S}_1(\Omega,\Gamma)} \arrow[uu] \arrow[r] & \mathsf{S}_1(\Omega,\Omega) \arrow[uu] \arrow[ur] \arrow[rr] & & \underset{\text{Rothberger}}{\mathsf{S}_1(\mathrm{O},\mathrm{O})} \arrow[uu] &
\end{tikzcd}$$

Figure 2: The Final Scheepers Diagram.

refute the last remaining potential implication. And, he failed. His close examination of the failure suggested a path for *proving* the last potential implication [11, Theorem 23]. Peng's results establish the final form of the Scheepers Diagram (Figure 2).

Peng's proof of the last implication is somewhat involved. The proof given below identifies the heart of Peng's argument, and replaces the other parts with simple, quotable observations about selective covering properties.

Let k be a natural number. A cover of a space is a *k-cover* if no member of the cover covers the entire space, but every k-element subset of the space is covered by some member of the cover. Thus, a cover is an ω-cover if and only if it is a k-cover for all natural numbers k. For a space X and a natural number k, let $\mathrm{O}_k(X)$ be the family of open k-covers of the space.

Lemma 3.1 *Let Π be a selection principle, and let* A *be a type of open covers. The following assertions are equivalent:*

(1) $\Pi(\mathrm{A},\Omega)$.
(2) *For each natural number k, we have* $\Pi(\mathrm{A},\mathrm{O}_k)$.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): Let $\mathcal{U}_1,\mathcal{U}_2,\dots$ be a sequence in A. Split the sequence into infinitely many disjoint sequences. For each natural number k, apply $\Pi(\mathrm{A},\mathrm{O}_k)$ to the kth sequence, to obtain a k-cover $\mathcal{V}_k$. Then $\bigcup_{k=1}^{\infty}\mathcal{V}_k$ is an ω-cover, in accordance with the required property $\Pi(\mathrm{A},\Omega)$. ■

Theorem 3.2 (Peng [11, Theorem 23]) *The Hurewicz property* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ *implies* $\mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega)$.

Proof. Let X be a Hurewicz space. By Lemma 3.1, it suffices to prove that $\mathsf{S}_{\mathrm{fin}}(\Gamma,\mathrm{O}_k)$ holds for all natural numbers k. Fix a natural number k.

Let $\mathcal{U}_1,\mathcal{U}_2,\dots$ be a sequence in $\Gamma(X)$. By moving to countably infinite subcovers, we may enumerate

$$\mathcal{U}_n = \{\, U^n_m : m \in \mathbb{N} \,\}$$

for each n. For each n and m, we may replace the set U^n_m with the smaller set

$$U^1_m \cap U^2_m \cap \cdots \cap U^n_m,$$

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so that we may assume that

$$U^1_m \supseteq U^2_m \supseteq U^3_m \supseteq \cdots$$

for all natural numbers m. The refined covers $\mathcal{U}_n$ remain in $\Gamma(X)$.

Let $g_0(m) := m$ for all m. We define, by induction, increasing functions $g_1, \dots, g_k \in \mathbb{N}^{\mathbb{N}}$. Let $l < k$ and assume that the function g_l is defined. For natural numbers n, m, and i, let

$$\begin{aligned} V^{l,n}_i &:= \bigcap_{m=i}^{g_l(i)} U^n_m, \\ W^{l,n}_m &:= \bigcup \{\, V^{l,n}_i : n \leq i, g_l(i) \leq m \,\}. \end{aligned}$$

For each l and n, the sets $W^{l,n}_m$ are increasing with m, and cover the space X. By the Hurewicz property, there is an increasing function $g_{l+1} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{\, W^{l,n}_{g_{l+1}(n)} : n \in \mathbb{N} \} \in \Gamma(X).$$

This completes the inductive construction.

We will show that

$$\{\, U^n_m : n \in \mathbb{N}, m \leq g_k(n) \,\} \in \mathrm{O}_k(X).$$

Let $x_1, \dots, x_k \in X$. Since $\{\, W^{l,n}_{g_{l+1}(n)} : n \in \mathbb{N} \} \in \Gamma(X)$ for all $l = 0, \dots, k-1$, there is a natural number N with

$$x_1, \dots, x_k \in W^{l,n}_{g_{l+1}(n)}$$

for all $l = 0, \dots, k-1$ and all $n \geq N$. Fix a number $n_0 \geq N$.

Since $x_1 \in W^{k-1,n_0}_{g_k(n_0)}$, there is n_1 with $n_0 \leq n_1, g_{k-1}(n_1) \leq g_k(n_0)$ and

$$x_1 \in V^{k-1,n_0}_{n_1} = \bigcap_{m=n_1}^{g_{k-1}(n_1)} U^{n_0}_m.$$

Since $x_3 \in W^{k-3,n_2}_{g_{k-2}(n_2)}$, there is n_3 with $n_2 \leq n_3, g_{k-3}(n_3) \leq g_{k-2}(n_2)$ and

$$x_3 \in V^{k-3,n_2}_{n_3} = \bigcap_{m=n_3}^{g_{k-3}(n_3)} U^{n_2}_m \subseteq \bigcap_{m=n_3}^{g_{k-3}(n_3)} U^{n_0}_m.$$

Since $x_k \in W^{0,n_{k-1}}_{g_1(n_{k-1})}$, there is n_k with $n_{k-1} \leq n_k = g_0(n_k) \leq g_1(n_{k-1})$ and

$$x_k \in V^{0,n_{k-1}}_{n_k} = U^{n_{k-1}}_{n_k} \subseteq U^{n_0}_{n_k}.$$

It follows that $x_1, \dots, x_k \in U^{n_0}_{n_k}$, and $n_k \leq g_k(n_0)$. ■

The proof of Theorem 3.2 establishes a stronger result. To this end, we need the following definitions and lemma. An infinite cover of a space X is *ω-groupable* [9] (respectively, *k-groupable*, for a natural number k) if there is a partition of the cover into finite parts such that for each finite (respectively, k-element) set $F \subseteq X$ and all but finitely many parts $\mathcal{P}$ of the partition, there is a set $U \in \mathcal{P}$ with $F \subseteq U$. Let $\Omega^{\mathrm{gp}}(X)$

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(respectively, $\mathrm{O}_k^{\mathrm{gp}}(X)$) be the family of open ω-groupable (respectively, k-groupable) covers of the space X.

Lemma 3.3 *Let Π be a selection principle, and let* A *be a type of open covers. The following assertions are equivalent:*

(1) $\Pi(\mathrm{A}, \Omega^{\mathrm{gp}})$;
(2) *For each natural number k, we have* $\Pi(\mathrm{A}, \mathrm{O}_k^{\mathrm{gp}})$.

Proof. The proof is similar to that of Lemma 3.1, once we observe that if $\{ U_n : n \in \mathbb{N}\}$ is a k-groupable cover for all k, then it is ω-groupable. This follows easily from the fact that for each countable family $\{ \mathcal{P}_k : k \in \mathbb{N} \}$ of partitions of $\mathbb{N}$ into finite sets, there is a partition $\mathcal{P}$ of $\mathbb{N}$ into finite sets that is eventually coarser than all of the given partitions, that is, such that for each k, all but finitely many members of the partition $\mathcal{P}$ contain a member of the partition $\mathcal{P}_k$. ■

Kočinac and Scheepers proved that if all finite powers of a space X are Hurewicz, then every open ω-cover of the space is ω-groupable. Together with Peng's Theorem 3.2, if *all finite powers* of a space X are Hurewicz, then the space satisfies $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Omega^{\mathrm{gp}})$. The following theorem shows that the assumption on the finite powers is not needed.

In the proof, we also mention the property $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Lambda^{\mathrm{gp}})$. An open cover is *large* if each point is in infinitely many members of the cover. Let $\Lambda(X)$ be the family of large open covers of the space X. An open cover $\mathcal{U}$ is in $\Lambda^{\mathrm{gp}}(X)$ [9] (also denoted $\gimel(\Gamma)$, depending on the context [13]) if there is a partition of the cover into finite parts such that for each point $x \in X$ and all but finitely many parts $\mathcal{P}$ of the partition, we have $x \in \bigcup \mathcal{P}$.

Theorem 3.4 *The following assertions are equivalent:*

(1) $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.
(2) $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Omega^{\mathrm{gp}})$.

Proof. (1) $\Rightarrow$ (2): The proof of Peng's Theorem 3.2, as written above, shows, for a prescribed number k, that for each k-element set F, there is a natural number N such that for each $n \geq N$, there is a member of the finite set $\mathcal{F}_n = \{ U_m^n : m \leq g_k(n) \}$ that contains the set F. By thinning out the point-cofinite covers, we may assume that they are pairwise disjoint [15, Lemma 4], and consequently so are the finite sets $\mathcal{F}_n$. Thus, $\bigcup_{n=1}^\infty \mathcal{F}_n \in \mathrm{O}_k^{\mathrm{gp}}$. This proves $\mathsf{S}_{\mathrm{fin}}(\Gamma, \mathrm{O}_k^{\mathrm{gp}})$ for all k. Apply Lemma 3.3.

(2) $\Rightarrow$ (1): $\Omega^{\mathrm{gp}} \subseteq \Lambda^{\mathrm{gp}}$. It is known that $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Lambda^{\mathrm{gp}}) = \mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ [13, Theorem 6]. For completeness, we provide a proof that $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Lambda^{\mathrm{gp}})$ implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

Assume that the space X satisfies $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Lambda^{\mathrm{gp}})$. It suffices to prove that it satisfies $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$. Given a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ in $\Gamma(X)$, we may (as in the proof of Theorem 3.2) assume that the covers get finer with n. Apply $\mathsf{S}_{\mathrm{fin}}(\Gamma, \Lambda^{\mathrm{gp}})$ to obtain a cover $\mathcal{U} \in \Lambda^{\mathrm{gp}}$, with parts $\mathcal{P}_n$ (for $n \in \mathbb{N}$) witnessing that.

Let $\mathcal{F}_1 \subseteq \mathcal{U}_1$ be a finite set refined by $\mathcal{P}_1$, that is, such that for each set $U \in \mathcal{P}_1$, there is a set $V \in \mathcal{F}_1$ with $U \subseteq V$. The set $\mathcal{F}_1$ exists since the covers $\mathcal{U}_n$ get finer as n increases. Let n_2 be minimal with $\mathcal{P}_{n_2} \subseteq \bigcup_{n=2}^\infty \mathcal{U}_n$, and let $\mathcal{F}_2 \subseteq \mathcal{U}_2$ be a finite set refined by $\mathcal{P}_{n_2}$.

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Let n_3 be minimal with $\mathcal{P}_{n_3} \subseteq \bigcup_{n=3}^{\infty} \mathcal{U}_n$, and let $\mathcal{F}_3 \subseteq \mathcal{U}_3$ be a finite set refined by $\mathcal{P}_{n_3}$. Continuing in this manner, we obtain finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$ for $n \in \mathbb{N}$, with $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \Gamma(X)$. ■

As mentioned in the proof, we have $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma) = \mathsf{S}_{\text{fin}}(\Gamma, \Lambda^{\text{gp}})$. Kočinac and Scheepers proved that $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma) = \mathsf{S}_{\text{fin}}(\Omega, \Lambda^{\text{gp}}) = \mathsf{S}_{\text{fin}}(\Lambda, \Lambda^{\text{gp}})$ [9, Theorem 14]. However, $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma) \neq \mathsf{S}_{\text{fin}}(\Omega, \Omega^{\text{gp}})$: The latter property is equivalent to satisfying $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$ in all finite powers [9, Theorem 16], a property strictly stronger than $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$ [8, Theorem 2.12].

4 When Bob has a winning strategy in the Menger game

Menger [10] conjectured that his property $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$ implies σ-compactness. While his conjecture turned out false ([19] and the references therein), a closely related assertion is true. The *Menger game* [7], $\mathsf{G}_{\text{fin}}(\mathrm{O}, \mathrm{O})$, is the game associated with Menger's property $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$. It is played on a topological space X, and has an inning per each natural number n. In each inning, Alice picks an open cover $\mathcal{U}_n$ of the space, and Bob chooses a finite set $\mathcal{F}_n \subseteq \mathcal{U}_n$. Bob wins if $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a cover of the space, and otherwise Alice wins. Telgársky [17] proved that if Bob has a winning strategy in the Menger game played on a metric space, then the space is σ-compact.

Scheepers [14, Theorem 1] provided a direct proof of Telgársky's theorem, using the notion of H-closed sets. We will eliminate the notion of H-closed sets and the closure operations from Scheepers's proof, and obtain a more transparent proof.

A subset K of a topological space X is *relatively compact* if every open cover $\mathcal{U}$ of the entire space X has a finite subcover of the set K. A subset K of a regular space is relatively compact if and only if its closure is compact.

Lemma 4.1 *Let κ be a cardinal number. If a regular space X is a union of at most κ relatively compact sets, then it is the union of at most κ compact sets.*

Proof. If $X = \bigcup_{\alpha<\kappa} K_\alpha$, then $X = \bigcup_{\alpha<\kappa} \overline{K_\alpha}$. ■

For a basis $\mathcal{B}$ for the topology of a space X, let $\mathrm{O}_{\mathcal{B}}(X)$ be the family of subsets of $\mathcal{B}$ that cover the space X.

Lemma 4.2 *Let X be a topological space with a basis $\mathcal{B}$, and let σ be a function on the family $\mathrm{O}_{\mathcal{B}}(X)$ such that for each cover $\mathcal{U} \in \mathrm{O}_{\mathcal{B}}(X)$, $\sigma(\mathcal{U})$ is a finite subset of $\mathcal{U}$. Then the set*

$$K := \bigcap_{\mathcal{U} \in \mathrm{O}_{\mathcal{B}}(X)} \bigcup \sigma(\mathcal{U})$$

is relatively compact.

Proof. Every open cover is refined by a cover $\mathcal{U} \in \mathrm{O}_{\mathcal{B}}(X)$ which, in turn, has the finite subcover $\sigma(\mathcal{U})$. ■

If Bob has a winning strategy in the Menger game played on X, then the space X is Menger and, in particular, Lindelöf. If X is, in addition, metric, then the space is

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regular and second countable. On the other hand, Urysohn's Metrization Theorem asserts that every second countable regular space is metrizable.

Theorem 4.3 (Telgársky) *Let X be a metric space. If Bob has a winning strategy in the Menger game* $\mathsf{G}_{\text{fin}}(\mathrm{O},\mathrm{O})$ *played on X, then the space X is σ-compact.*

Proof. We follow the steps of Scheepers's proof [14, Theorem 1], removing what is not necessary.

The space X is regular and second countable. Let σ be a winning strategy for Bob. Fix a countable base $\mathcal{B}$ for the topology of the space X. Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers. We consider all possible games where Alice chooses her covers from the family $\mathrm{O}_{\mathcal{B}}(X)$.

Since the base $\mathcal{B}$ is countable, the family $\{\,\sigma(\mathcal{U}) : \mathcal{U} \in \mathrm{O}_{\mathcal{B}}\,\}$ (the possible first responds of Bob) is countable, too. Choose elements $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{O}_{\mathcal{B}}$ with

$$\{\,\sigma(\mathcal{U}_n) : n \in \mathbb{N}\} = \{\,\sigma(\mathcal{U}) : \mathcal{U} \in \mathrm{O}_{\mathcal{B}}\,\}.$$

By induction, for a given natural number n and each sequence $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, the family

$$\{\,\sigma(\mathcal{U}_{s_1}, \mathcal{U}_{s_1,s_2}, \ldots, \mathcal{U}_{s_1,\ldots,s_n}, \mathcal{U}) : \mathcal{U} \in \mathrm{O}_{\mathcal{B}}\,\}$$

is countable. Choose elements $\mathcal{U}_{s_1,\ldots,s_n,1}, \mathcal{U}_{s_1,\ldots,s_n,2}, \ldots \in \mathrm{O}_{\mathcal{B}}$ with

$$\{\,\sigma(\mathcal{U}_{s_1}, \ldots, \mathcal{U}_{s_1,\ldots,s_n}, \mathcal{U}_{s_1,\ldots,s_n,m}) : m \in \mathbb{N}\,\} = \{\,\sigma(\mathcal{U}_{s_1}, \ldots, \mathcal{U}_{s_1,\ldots,s_n}, \mathcal{U}) : \mathcal{U} \in \mathrm{O}_{\mathcal{B}}\,\}.$$

This completes our inductive construction.

By Lemma 4.2, for each sequence $s = (s_1, \ldots, s_n) \in \mathbb{N}^*$, the set

$$K_s := \bigcap_{m=1}^{\infty} \bigcup \sigma(\mathcal{U}_{s_1}, \ldots, \mathcal{U}_{s_1,\ldots,s_n}, \mathcal{U}_{s_1,\ldots,s_n,m})$$

is relatively compact. By Lemma 4.1, it remains to see that $X = \bigcup_{s\in\mathbb{N}^*} K_s$.

Assume that some element $x \in X$ is not in $\bigcup_{s\in\mathbb{N}^*} K_s$.

(1) Since $x \notin K_{()}$, there is m_1 with $x \notin \bigcup \sigma(\mathcal{U}_{m_1})$.
(2) Since $x \notin K_{m_1}$, there is m_2 with $x \notin \bigcup \sigma(\mathcal{U}_{m_1}, \mathcal{U}_{m_1,m_2})$.
(3) Since $x \notin K_{m_1,m_2}$, there is m_3 with $x \notin \bigcup \sigma(\mathcal{U}_{m_1}, \mathcal{U}_{m_1,m_2}, \mathcal{U}_{m_1,m_2,m_3})$.
(4) Etc.

Then the play

$$\mathcal{U}_{m_1}, \sigma(\mathcal{U}_{m_1}), \mathcal{U}_{m_1,m_2}, \sigma(\mathcal{U}_{m_1}, \mathcal{U}_{m_1,m_2}), \mathcal{U}_{m_1,m_2,m_3}, \sigma(\mathcal{U}_{m_1}, \mathcal{U}_{m_1,m_2}, \mathcal{U}_{m_1,m_2,m_3}), \ldots$$

is lost by Bob; a contradiction. ∎

Let α be an ordinal number. The transfinite Menger game $\mathsf{G}^{\alpha}_{\text{fin}}(\mathrm{O},\mathrm{O})$ is defined as the ordinary Menger game, with the only difference that now there is an inning per each ordinal number $\beta < \alpha$. Clearly, if $\alpha_1 < \alpha_2$ and Bob has a winning strategy in the α_1-Menger game, then Bob has a winning strategy in the α_2-Menger game: He can use a winning strategy in the first α_1 innings, and then play arbitrarily. Thus, the following theorems are more general than Theorem 4.3.

The *weight* of a topological space is the minimal cardinality of a base for its topology.

Theorem 4.4 *Suppose that X is a regular topological space of weight κ, and $\lambda \leq \kappa$. If Bob has a winning strategy in the game $\mathsf{G}^{\lambda}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ played on X, then the space X is a union of at most $\kappa^{<\lambda}$ compact sets.*

Proof. The proof is identical to that of Theorem 4.3, only that here we begin with a base of cardinality κ. Here, the sequences s range over the set of sequences in κ of length smaller than λ. ∎

Corollary 4.5 *Let X be a regular topological space of weight κ. If Bob has a winning strategy in the Menger game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, then the space X is a union of at most κ compact sets.*

For spaces of uncountable weight κ, the converse of Corollary 4.5 is false: The discrete space of cardinality κ has weight κ, and it is a union of κ compact sets (singletons). This space is not Lindelöf, and thus not Menger, so Bob has no winning strategy in the Menger game played on this space.

5 The additivity of Rothberger's property

Let $\mathrm{add}(\mathcal{N})$ be the minimal cardinality of a family $F \subseteq \mathbb{N}^{\mathbb{N}}$ such that there is no function $S\colon \mathbb{N} \to [\mathbb{N}]^{<\infty}$ with $|S(n)| \leq n$ for all n, such that for each function $f \in F$, we have $f(n) \in S(n)$ for all but finitely many n.

The notation $\mathrm{add}(\mathcal{N})$ is explained by a result of Bartoszyński and Judah [1, Theorem 2.11]: The cardinal number $\mathrm{add}(\mathcal{N})$ is the minimal cardinality of a family of Lebesgue null sets of real numbers whose union is not Lebesgue null. In general, the *additivity* of a property is the minimum cardinality of a family of sets satisfying the property, whose union does not. The following theorem is attributed to Carlson by Bartoszyński and Judah [1, Theorem 2.9].

Theorem 5.1 (Carlson) *Let $\kappa < \mathrm{add}(\mathcal{N})$. If a Lindelöf space is a union of at most κ spaces satisfying $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$, then the space X satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. That is, for Lindelöf spaces, $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathsf{S}_1(\mathrm{O}, \mathrm{O}))$.*

This theorem is an easy consequence of a simple, basic fact concerning selection principles. We need the following lemmata.

Let A and B be types of open covers. A topological space X satisfies $\mathsf{S}_n(\mathrm{A}, \mathrm{B})$ if for all $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{A}(X)$, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $|\mathcal{F}_n| \leq n$ for all n, and $\bigcup_{n=1}^{\infty} \mathcal{F}_n \in \mathrm{B}(X)$.

Garcia-Ferreira and Tamariz-Mascarua [5, Lemma 3.12] established the following observation in the case $\mathrm{A} = \mathrm{O}$.

Lemma 5.2 [19, Theorem A.1] *Let A be a type of countable covers such that every pair of covers of type A has a joint refinement of type A. Then $\mathsf{S}_n(\mathrm{A}, \mathrm{O}) = \mathsf{S}_1(\mathrm{A}, \mathrm{O})$.*

Lemma 5.3 (Folklore) *If a space* X *satisfies* $\mathsf{S}_1(\mathrm{A},\mathrm{O})$*, then for each sequence* $\mathcal{U}_1,\mathcal{U}_2,\dots$ *in* $\mathrm{O}(X)$*, there are elements* $U_1\in\mathcal{U}_1, U_2\in\mathcal{U}_2,\dots$ *such that for each point* $x\in X$*, we have* $x\in U_n$ *for infinitely many* n*.*

Proof. As usual, we split the sequence of open covers to infinitely many disjoint sequences, and apply the property $\mathsf{S}_1(\mathrm{A},\mathrm{O})$ to each subsequence separately. ∎

Theorem 5.4 *Let* A *be a type of countable covers such that every pair of covers of type* A *has a joint refinement of type* A*. Then* $\operatorname{add}(\mathcal{N})\leq\operatorname{add}(\mathsf{S}_1(\mathrm{A},\mathrm{O}))$*.*

Proof. Let $\kappa<\operatorname{add}(\mathcal{N})$ and $X=\bigcup_{\alpha<\kappa}X_\alpha$, where each space X_α satisfies $\mathsf{S}_1(\mathrm{A},\mathrm{O})$. By Lemma 5.2, it suffices to show that the space X satisfies $\mathsf{S}_n(\mathrm{A},\mathrm{O})$.

Let $\mathcal{U}_n=\{U^n_m : m\in\mathbb{N}\}\in\mathrm{A}(X)$, for $n\in\mathbb{N}$. For each ordinal number $\alpha<\kappa$, as the space X_α satisfies $\mathsf{S}_1(\mathrm{A},\mathrm{O})$, there is a function $f_\alpha\in\mathbb{N}^\mathbb{N}$ such that for each point $x\in X_\alpha$, we have

$$x\in U^n_{f_\alpha(n)}$$

for infinitely many n.

There is a function $S\colon\mathbb{N}\to[\mathbb{N}]^{<\infty}$ with $|S(n)|\leq n$ for all n, such that for each $\alpha<\kappa$, we have

$$f_\alpha(n)\in S(n)$$

for all but finitely many n. Then $\bigcup_{n=1}^\infty\{\,U^n_m : m\in S(n)\,\}\in\mathrm{O}(X)$. ∎

Judging by an extensive survey on the topic [18], the result in the second item below seems to be new.

Corollary 5.5 (1) *For Lindelöf spaces,* $\operatorname{add}(\mathcal{N})\leq\operatorname{add}(\mathsf{S}_1(\mathrm{O},\mathrm{O}))$.
(2) $\operatorname{add}(\mathcal{N})\leq\operatorname{add}(\mathsf{S}_1(\Gamma,\mathrm{O}))$.

Proof. (1) Since the spaces are Lindelöf, we may restrict attention to countable covers, and the assumptions of Theorem 5.4 hold.

(2) A countably infinite subset of a point-cofinite cover is also a point-cofinite cover. Thus, we may restrict attention to countable point-cofinite covers. It is well known that every pair of point-cofinite covers has a joint refinement that is a point-cofinite cover. Indeed, let $\mathcal{U}$ and $\mathcal{V}$ be countable point-cofinite covers. Enumerate them $\mathcal{U}=\{\,U_n : n\in\mathbb{N}\}$ and $\mathcal{V}=\{\,V_n : n\in\mathbb{N}\}$. Then $\{\,U_n\cap V_n : n\in\mathbb{N}\}\in\Gamma(X)$. Theorem 5.4 applies. ∎

We can extract additional information from this proof method. A topological space X satisfies $\mathsf{U}_n(\Gamma,\Gamma)$ [19] if for all $\mathcal{U}_1,\mathcal{U}_2,\dots\in\Gamma(X)$, there are finite sets $\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\dots$ such that $|\mathcal{F}_n|\leq n$ for all n, and $\{\bigcup\mathcal{F}_n : n\in\mathbb{N}\}\in\Gamma(X)$. This property is strictly inbetween $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ [19, Theorems 3.3 and 3.8].

Theorem 5.6 $\operatorname{add}(\mathcal{N})\leq\operatorname{add}(\mathsf{U}_n(\Gamma,\Gamma))$.

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Proof. Let $\kappa < \mathrm{add}(\mathcal{N})$ and $X = \bigcup_{\alpha<\kappa} X_\alpha$, where each space X_α satisfies $\mathsf{U}_n(\Gamma, \Gamma)$. It suffices to show that the space X satisfies $\mathsf{U}_{n^2}(\Gamma, \Gamma)$, where the cardinality of the nth selected finite set is at most n^2 [19, Lemma 3.2].

Let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\} \in \Gamma(X)$, for $n \in \mathbb{N}$. For each $\alpha < \kappa$, as the space X_α satisfies $\mathsf{U}_n(\Gamma, \Gamma)$, there is a function $S_\alpha : \mathbb{N} \to \prod_n [\mathbb{N}]^{\le n}$ such that for each point $x \in X_\alpha$, we have

$$x \in \bigcup_{m \in S_\alpha(n)} U^n_m$$

for infinitely many n.

There is a function $S : \mathbb{N} \to \prod_n [\mathbb{N}]^{\le n}$ with $|S(n)| \le n$ for all n, such that for each $\alpha < \kappa$, we have

$$S_\alpha(n) \in S(n)$$

for all but finitely many n. For each natural number n, let $F_n := \bigcup S(n)$. Then $|F_n| \le n^2$ for all n, and $\{ \bigcup_{m \in F_n} U^n_m : n \in \mathbb{N}\} \in \Gamma(X)$. ∎

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Department of Mathematics, Bar-Ilan University, Ramat Gan, Israel
e-mail: tsaban@math.biu.ac.il

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