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Sharpness Results and Knapp's Homogeneity Argument

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Abstract. We prove that the L^2 restriction theorem, and $L^p \to L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, boundedness of the surface averages imply certain geometric restrictions on the underlying hypersurface. We deduce that these bounds imply that a certain number of principal curvatures do not vanish.

1 Introduction

Let S be a smooth compact hypersurface in $\mathbb{R}^n$. Let

$$F_S(\xi) = \int_S e^{i\langle x,\xi\rangle}\, d\sigma(x) \tag{1}$$

denote the Fourier transform of the surface measure carried by S.

Let $\mathcal{R}f = \hat{f}|_S$, the restriction operator. It is well known (see [6], [2], [4]) that if

$$|F_S(\xi)| \le C(1 + |\xi|)^{-r}, \quad r > 0, \tag{2}$$

then

$$\|\mathcal{R}f\|_2 \le C_p \|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad \text{for } p \le p_0 = \frac{2(r+1)}{r+2}, \tag{3}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the standard Schwartz class.

However, it is not in general known whether this result is sharp. More precisely, it is natural to ask the following.

Question A *Does the estimate (3) imply the estimate (2)?*

Let

$$Tf(x, x_n) = \int f\big(x - y, x_n - \Phi(y)\big)\psi(y)\, dy, \tag{4}$$

where $x, y \in \mathbb{R}^{n-1}$, ψ is a smooth cutoff function, Φ is smooth, $\Phi(0, \ldots, 0) = 0$, and $\bigtriangledown\Phi(0, \ldots, 0) = (0, \ldots, 0)$.

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It is well known (see [5]) that if the estimate (2) holds, then

$$\|Tf\|_{p'} \le C_p \|f\|_p, \quad \text{where} \quad \frac{1}{p} - \frac{1}{2} \le \frac{r}{2(r+1)}, \tag{5}$$

where p' denotes the conjugate exponent of p.

The key estimate here is

$$\|Tf\|_{2(r+1)} \le C\|f\|_{\frac{2(r+1)}{2r+1}}; \tag{6}$$

the rest follows by interpolation. It is then natural to ask the following.

Question B *Does the estimate (5) imply the estimate (2)?*

The purpose of this paper is to answer questions A and B affirmatively in the case of the optimal exponents. We shall employ a multiparameter version of Knapp's homogeneity argument. (See *e.g.* [1] for a similar argument).

More precisely, we will show that if the the estimate (3) holds with $p = \frac{2(n+1)}{n+3}$, then the hypersurface has everywhere non-vanishing Gaussian curvature. Similarly, we will show that if the estimate (5) holds with $p = \frac{n+1}{n}$, then the hypersurface has non-vanishing Gaussian curvature.

We remark here, on the other hand, that non-vanishing Gaussian curvature implies that the estimate (2) holds with $r = \frac{n-1}{2}$ (see *e.g.* [4]). Thus Question A is answered affirmatively in the case $r = \frac{n-1}{2}$. Since the estimate (2) with $r = \frac{n-1}{2}$ implies the estimate (3) with $p = \frac{2(n+1)}{n+3}$, Theorem 2 below shows that the optimal decay of the Fourier transform (*i.e.*, $r = \frac{n-1}{2}$) implies that the hypersurface has non-vanishing Gaussian curvature.

We will also see that if a hypersurface has $\le k$ non-vanishing principal curvatures at each point, then the exponent p in the estimate (3) can never exceed $\frac{2n+k-2}{6}$. Consequently, the estimate (3) with $p \ge \frac{2n+k-2}{6}$ implies that at least k principal curvatures are non-zero at each point. (See Theorem 3 below). Similarly, we will show that if the estimate (5) holds with $p \ge \frac{2n+k+4}{2n+k+1}$, then at least k principal curvatures are non-zero at each point.

The sharpness of the estimate (3) is known in some cases. For example, if the hypersurface has non-vanishing Gaussian curvature, Knapp's homogeneity argument can be used to show that the exponent $p = \frac{2(n+1)}{n+3}$ is the best possible. Indeed, non-vanishing Gaussian curvature implies that the hypersurface has contact of order two with its tangent plane at every point. Let $f_\delta(x) = g(\delta^{-1}x, \delta^{-2}x_n)$, where $x = (x_1, \dots, x_{n-1})$, and g is the characteristic function of the rectangle with sides $(1, \dots, 1, C)$, C large, with the long side normal to the hypersurface.

It is not hard to check that $\|f_\delta\|_p \approx \delta^{(1-\frac{1}{p})(n+1)}$, whereas $\|\mathcal{R} f_\delta\|_2 \approx \delta^{\frac{n-1}{2}}$. The comparison yields $p \le \frac{2(n+1)}{n+3}$.

It should be noted that the above example does not verify even a special case of question A. For example, the above argument does not prove that if the estimate (3) holds with $p = \frac{2(n+1)}{n+3}$, then the estimate (2) holds with $r = \frac{n-1}{2}$. We will show (see Theorem 2 below) that this is indeed the case.

The sharpness of the estimate (5) can also be verified in some cases. By testing T against a characteristic function of a small ball it is not hard to check that if T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $(\frac{1}{p}, \frac{1}{q})$ must be contained in the triangle with the endpoints $(0,0)$,

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$(1,1)$, and $(\frac{n}{n+1}, \frac{1}{n+1})$. However, as before this does not prove that if the estimate (5) holds with $p = \frac{n+1}{n}$, then the estimate (2) holds with $r = \frac{n-1}{2}$. We will show (see Theorem 5 below) that this is indeed the case.

2 Statement of Results

Theorem 1 *Let $S = \{(x, x_n) \in \mathbb{R}^n : x_n = \Phi(x)\}$, where $x = (x_1, \dots, x_{n-1})$, Φ is a smooth function which does not vanish on a set of positive measure, $\Phi(0, \dots, 0) = 0$, and $\bigtriangledown\Phi(0, \dots, 0) = (0, \dots, 0)$. Suppose that the estimate (3) holds. Let G be any continuous function which does not vanish on a set of positive measure satisfying $G(0, \dots, 0) = 0$. Then*

$$(7) \qquad \big(|G(\delta)|\big)^r \geq CR(\delta)^{r+1}|\delta_1\delta_2\cdots\delta_{n-1}|,$$

where $R(\delta) = |\{x \in [-1,1]^{n-1} : |\Phi(\delta_1 x_1, \dots, \delta_{n-1}x_{n-1})| \leq C|G(\delta)|\}|$.

Remark If $G(\delta)$ is chosen to be $\Phi(\delta)$, and Φ is increasing in each variable separately, Theorem 1 says that the estimate (3) implies that $\big(|\Phi(\delta)|\big)^r \geq C\delta_1\delta_2\cdots\delta_{n-1}$. The same estimate would be true, of course, if we just assume that $R(\delta)$ is bounded below, which is a much weaker assumption. To prove Theorem 2, Theorem 3, Theorem 5, and Theorem 6 below we shall use Theorem 1 with

$$(8) \qquad G(\delta) = \sup_{\{x\in[-1,1]^{n-1}\}} |\Phi(x_1\delta_1, \dots, x_{n-1}\delta_{n-1})|.$$

Theorem 2 *Suppose that the estimate (3) holds with $p = \frac{2(n+1)}{n+3}$. Then the hypersurface S has everywhere non-vanishing Gaussian curvature.*

Theorem 3 *Suppose that the estimate (3) holds with $p \geq \frac{2n+k-2}{6}$. Then the hypersurface S has at least k non-vanishing principal curvatures at each point.*

Remark The conclusion of Theorem 3 can be motivated as follows. If the hypersurface has exactly k non-vanishing principal curvatures at a point, then after perhaps applying a rotation we can write it as a graph of the function $x_1^2 + \cdots + x_k^2 + A(x)$, where A is a higher order remainder. It is not hard to believe that the best possible estimate (2) is obtained if $A(x) = |x''|^3$, where $x'' = (x_{k+1}, \dots, x_{n-1})$. This gives us the estimate (2) with $r = \frac{k}{2} + \frac{n-1-k}{3}$. The conclusion of Theorem 3 is the consequence of the fact that $\frac{2(r+1)}{r+2} = \frac{2n+k-2}{6}$.

Theorem 4 *Let $\delta y = (\delta_1 y_1, \dots, \delta_{n-1}y_{n-1})$ and $g_\delta(s) = \Big|\{y \in \operatorname{supp}(\psi) : \big|s - |\Phi(\delta y)/\Phi(\delta)|\big| \leq C\}\Big|$. Suppose that the estimate (5) holds. Then for $|\delta|$ sufficiently small,*

$$(9) \qquad \big(|\Phi(\delta)|\big)^r \geq CP_\delta\|g_\delta\|_{L^{p'}(ds)}.$$

Theorem 5 *Suppose that the estimate (5) holds with $r = \frac{n-1}{2}$. Let $S = \{(x, x_n) : x \in \operatorname{supp}(\psi), x_n = \Phi(x)\}$. Then S has everywhere non-vanishing Gaussian curvature.*

Theorem 6 *Suppose that the estimate (5) holds with* $p \geq \frac{2n+k+4}{2n+k+1}$. *Then the hypersurface has at least k non-vanishing principal curvatures at each point.*

(See the remark after Theorem 3 for the motivation of the conclusion of Theorem 6).

3 Proof of Theorem 1

Let $\delta x = (\delta_1 x_1, \dots, \delta_{n-1}x_{n-1})$, and $\delta^{-1}x = (\delta_1^{-1}x_1, \dots, \delta_{n-1}^{-1}x_{n-1})$. Let $\hat{f}_\delta(x, x_n) = g(\delta^{-1}x, \frac{x_n}{|G(\delta)|})$, where g is the characteristic function of a rectangle with sides of length $(1, 1, \dots, 1, C)$. Let $P_\delta = |\delta_1\delta_2 \cdots \delta_{n-1}|$. It is not hard to see that

$$\|f_\delta\|_p \approx (P_\delta|G(\delta)|)^{(1-1/p)}. \tag{10}$$

On the other hand,

$$\|\mathcal{R}f_\delta\|_2^2 = \int \left| g\left(\delta^{-1}x, \frac{\Phi(x)}{|G(\delta)|}\right)\right|^2 dx = P_\delta \int \left| g\left(x, \frac{\Phi(\delta x)}{|G(\delta)|}\right)\right|^2 dx \approx CP_\delta R(\delta), \tag{11}$$

where $R(\delta)$ is defined in the statement of the theorem.

Comparing the estimates (10) and (11) we see that (3) can hold only if

$$(|G(\delta)|)^r \geq CP_\delta R^{r+1}(\delta), \tag{12}$$

for $|\delta|$ sufficiently small. This completes the proof of Theorem 1.

4 Proof of Theorem 2

Let $G(\delta) = \sup_{\{x\in[-1,1]^{n-1}\}} |\Phi(\delta_1 x_1, \dots, \delta_{n-1}x_{n-1})|$. It follows that $R(\delta) \equiv 1$, and so

$$\big(G(\delta)\big)^r \geq CP_\delta, \tag{13}$$

where $r = \frac{n-1}{2}$ by assumption.

After perhaps applying a rotation, we can use Taylor's theorem to write

$$\Phi(x) = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2 + A(x), \tag{14}$$

where $A(x)$ is a higher order remainder term, and $k \leq n-1$. If $k = n-1$, then in a sufficiently small neighborhood of the origin the determinant of the Hessian matrix of Φ never vanishes, which would verify the claim of Theorem 2. We shall henceforth assume that $k < n-1$.

It is not hard to check that

$$\big(G(\delta)\big)^{\frac{n-1}{2}} \leq (a_1\delta_1^2 + \cdots + a_k\delta_k^2 + C|\delta|^3)^{\frac{n-1}{2}}, \tag{15}$$

$|\delta|$ small.

We must show that the estimate (13) cannot hold if $k < n-1$. It suffices to show that the right hand side of (15) is not bounded below by CP_δ. We may assume that $A(x)$ is not

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identically 0, and that $A(x)$ depends on x_{n-1}, for otherwise the contradiction is immediate. Let $\delta_j = \delta_{n-1}^{\frac{3}{2}}$. If the right hand side were bounded below by CP_δ, we could use the fact that $A(x)$ is a higher order remainder term to force an inequality

$$|\delta_{n-1}|^{\frac{3(n-1)}{2}} \geq C|\delta_{n-1}|^{\frac{3n-4}{2}}, \tag{16}$$

δ_{n-1} small, which is not true. This shows that the estimate (8) cannot hold unless $k = n-1$. This implies that there exists a small neighborhood of the origin where S has non-vanishing Gaussian curvature. This completes the proof.

5 Proof of Theorem 3

We must show that if Φ is as in the estimate (14) above, with k denoting the number of non-vanishing principal curvatures, then the estimate

$$\big(G(\delta)\big)^r \geq CP_\delta \tag{17}$$

can only hold if $r \leq \frac{k}{2} + \frac{n-1-k}{3} = \frac{2n+k-2}{6}$.

Let $\delta = (\delta', \delta'')$, where $\delta' = (\delta_1, \dots, \delta_k)$, and $\delta'' = (\delta_{k+1}, \dots, \delta_{n-1})$.

Let $\delta_j = |\delta''|^{\frac{3}{2}}$. The estimate (16) cannot hold if the inequality

$$|\delta''|^{3r} \geq C|\delta''|^{(\frac{3k}{2}+(n-1-k))} \tag{18}$$

is not satisfied. However, the estimate (17) can only hold if $r \leq \frac{k}{2} + \frac{n-1-k}{3} = \frac{2n+k-2}{6}$. This completes the proof.

6 Proof of Theorem 4

Let $\delta^{-1}y = (\delta_1^{-1}y_1, \dots, \delta_{n-1}^{-1}y_{n-1})$. Let f denote the characteristic function of the rectangle with sides of length $(1, 1, \dots, 1, C)$, C large. Let $\tau_\delta f(x, x_n) = f(\delta x, |\Phi(\delta)|x_n)$, and $\tau_\delta^{-1} f(x, x_n) = f(\delta^{-1}x, |\Phi(\delta)|^{-1}x_n)$. Let $f_\delta(x, x_n) = \tau_\delta^{-1} f(x, x_n)$. Let

$$T_\delta f(x, x_n) = \int f\big(x - y, x_n - \Phi(y)\big)\psi(\delta^{-1}y)\, dy. \tag{19}$$

After making a change of variables we see that

$$T_\delta f_\delta(x, x_n) = P_\delta \tau_\delta^{-1} T_\delta^* f(x, x_n), \tag{20}$$

where

$$T_\delta^* f(x, x_n) = \int f\big(x - y, x_n - \Phi(\delta y)/|\Phi(\delta)|\big)\psi(y)\, dy. \tag{21}$$

It is not hard to see that

$$\|f_\delta\|_p \approx P_\delta^{\frac{1}{p}} \big(|\Phi(\delta)|\big)^{\frac{1}{p}}. \tag{22}$$

Also,

(23)
$$\|T_\delta f_\delta\|_{p'} = \|P_\delta \tau_\delta^{-1} T_\delta^* f\|_{p'} = P_\delta P_\delta^{\frac{1}{p'}} \big(|\Phi(\delta)|\big)^{\frac{1}{p'}} \|T_\delta^* f\|_{p'} \approx P_\delta P_\delta^{\frac{1}{p'}} \big(|\Phi(\delta)|\big)^{\frac{1}{p'}} \|g_\delta\|_{L^{p'}(ds)},$$

where g_δ is defined above.

Comparing the estimates (22) and (23) yields the assertion of the theorem.

7 Proofs of Theorem 5 and Theorem 6

Let $G(\delta) = \sup_{x \in [-1,1]^{n-1}} |\Phi(\delta x)|$. The proof of Theorem 4 shows that if the estimate (5) holds then

(24)
$$\big(G(\delta)\big)^r \geq C P_\delta.$$

The proofs of Theorem 5 and Theorem 6 now follow in the same way as the proofs of Theorem 2 and Theorem 3.

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