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ON BODIES ASSOCIATED WITH A GIVEN CONVEX BODY

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ABSTRACT. Let $d \geq 2$, and $K \subset \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} K$. We consider the intersection body IK, the cross-section body CK and the projection body ΠK of K, which satisfy $IK \subset CK \subset \Pi K$. We prove that $[\operatorname{bd}(IK)] \cap [\operatorname{bd}(CK)] \neq \emptyset$ (a joint observation with R. J. Gardner), while for $d \geq 3$ the relation $[CK] \subset \operatorname{int}(\Pi K)$ holds for K in a dense open set of convex bodies, in the Hausdorff metric. If $IK = c \cdot CK$ for some constant $c > 0$, then K is centred, and if both IK and CK are centred balls, then K is a centred ball. If the chordal symmetral and the difference body of K are constant multiples of each other, then K is centred; if both are centred balls, then K is a centred ball. For $d \geq 3$ we determine the minimal number of facets, and estimate the minimal number of vertices, of a convex d-polytope P having no plane shadow boundary with respect to parallel illumination (this property is related to the inclusion $[CP] \subset \operatorname{int}(\Pi P)$).

1. **Introduction.** For $d \geq 2$, let $K \subset \mathbb{R}^d$ be a convex body. We recall the definition of the *intersection body* IK of K, of the *cross-section body* CK of K, and of the *projection body* ΠK of K.

The body IK, for K a convex body with the origin 0 as an interior point (introduced by Lutwak [Lu], *cf.* also [Ga], Definition 8.1.1), is the star body with (necessarily continuous) radial function $V_{d-1}(K \cap u^{\perp})$ for $u \in S^{d-1}$, where $u^{\perp}$ is the linear $(d-1)$-subspace orthogonal to the unit vector u, and V_{d-1} means $(d-1)$-dimensional Lebesgue measure. In a sense, the history of intersection bodies started with the paper [Bu], where it is proved that, if K is a convex body in $\mathbb{R}^d$ centred at the origin, then IK also is a convex body with centre 0. The term *intersection body* appears later, namely in [Lu]. Intersection bodies are important for considering dual mixed volumes and questions like the famous Busemann-Petty problem (see [Lu] and [Ga], chapter 8). For example, any sufficiently smooth and strictly convex body in $\mathbb{R}^3$ with centre 0 is the intersection body of a star body, where the convex body K in the definition of IK above is replaced by a star-shaped body.

The body CK (introduced by [Ma 92], *cf.* also [Ga], Definition 8.3.1) is the star body with (necessarily continuous) radial function $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$ for $u \in S^{d-1}$. It was shown in [MM], Part II, that the cross-section body CT of a regular tetrahedron $T \subset \mathbb{R}^3$ is a cube, and an interesting open problem is the question whether cross-section bodies are convex, posed in [Ma 94], p. 279.

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The body ΠK, going back to Minkowski, is the convex body with support function $V_{d-1}(K \mid u^\perp)$, where $K \mid u^\perp$ is the orthogonal projection of K to $u^\perp$ (*cf.* [BF], Section 30, p. 50, where it is also shown that $V_{d-1}(K \mid u^\perp)$ is in fact the support function of a centred convex body; see also [Ga], Definition 4.1.1). The set of projection bodies is equal to the set of d-dimensional *zonoids* centred at the origin, *i.e.*, every such zonoid is the projection body of a certain class of convex bodies, and each projection body is such a zonoid. Zonoids are limits of zonotopes with respect to the Hausdorff metric, and zonotopes are vector sums of finitely many line segments. The literature on zonoids is widely covered by the surveys [SW], [GW], and [Ma 94]. For example, one sees easily that for T a regular tetrahedron in 3-space, ΠT is a rhombic dodecahedron, and for special convex double cones $C \subset \mathbb{R}^3$ with axial symmetry, ΠC is a spindle-shaped body formed by revolving a cosine curve around the corresponding coordinate axis.

We have evidently $IK \subset CK$ for $0 \in \operatorname{int} K$, and $CK \subset \Pi K$ by [Pe 52], p. 60, and [Ma 89], the latter stating, equivalently, that the radial function of CK is at most the reciprocal of the distance function of ΠK, *i.e.*, is at most the radial function of ΠK; *cf.* also [Ga], Theorem 8.3.3. For $d = 2$, one easily sees $CK = \Pi K =$ the convex body obtained from $K + (-K)$ by rotating it through $\frac{\pi}{2}$ about the origin, see [MM], Section 1, and [Ga], Theorems 4.1.4 and 8.3.5.

In this paper we prove some theorems related to the bodies IK, CK and ΠK.

Before Theorem 1, we recall some more definitions, *cf.* also [Fe], 2.10.1–2. For a metric space X and $m \geq 0$ the *m-dimensional Hausdorff measure* H^m is an outer measure defined on all subsets of X as follows: for $A \subset X$,

$$H^m(A) = \sup_{\delta>0}\Big(\inf\Big\{\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^m \cdot \pi^{m/2}/\big(2^m\Gamma(1+\frac{m}{2})\big) \;\Big|\; A \subset \bigcup_{i=1}^{\infty} A_i \subset X, \\ \forall i \operatorname{diam}(A_i) \leq \delta\Big\}\Big),$$

where diam means diameter. All closed subsets of X are H^m-measurable (see [Fe], pp. 54, 170). If m is a positive integer, one calls $A \subset X$, with $H^m(A) < \infty$, *(H^m, m)-rectifiable*, if

$$\forall \varepsilon > 0 \quad \exists A_\varepsilon \subset X, \quad H^m(A \setminus A_\varepsilon) < \varepsilon,$$

and A_ε is the image of a bounded subset of $\mathbb{R}^m$ by a Lipschitz map defined on this subset, *cf.* [Fe], pp. 251-252.

If X is a Euclidean space and A is a compact C^1 m-submanifold, then A is (H^m, m)-rectifiable, and $H^m(A)$ coincides with the differential geometric m-volume (Theorems 3.2.26 and 3.2.39 in [Fe]).

It should be noticed that a first version of Theorem 1 was jointly observed by R. J. Gardner and the second named author.

THEOREM 1. *For $d \geq 2$, let $K \subset \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} K$. Then we have* $[\operatorname{bd}(IK)] \cap \operatorname{bd}(CK) \neq \emptyset$. *More generally, even omitting the hypothesis $0 \in \operatorname{int} K$, there is a direction $u \in S^{d-1}$ such that $V_{d-1}(K \cap u^\perp) \geq V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$ holds* *for each* $\lambda \in \mathbb{R}$. *Moreover, for* $d \geq 3$ *the set of these* u*'s is not contained in any* H^{d-2}*-measurable,* $(H^{d-2}, d-2)$*-rectifiable subset* A *of the unit sphere* S^{d-1}*, which satisfies* $H^{d-2}(A) < H^{d-2}(S^{d-2})$ *(in particular, in any compact* $C^1(d-2)$*-submanifold of* S^{d-1} *with* $(d-2)$*-volume less than that of* S^{d-2}*). The estimate is sharp, e.g. for* K *any non-centred ball.*

We note that for $d = 2$ the assertion of Theorem 1 follows from an analogous result of Hammer (*cf.* [Ha 51], [Ha 63], and [PC]) on 1-dimensional sections of convex bodies in $\mathbb{R}^d$, which will be discussed in Section 3.

THEOREM 2. *For* $d \geq 2$*, let* $K \subset \mathbb{R}^d$ *be a convex body with* $0 \in \operatorname{int} K$*. If for some constant* $c > 0$ *we have* $IK = c \cdot CK$*, then* K *is centred. More generally, even omitting the hypothesis* $0 \in \operatorname{int} K$*, if for every* $u \in S^{d-1}$ *we have* $V_{d-1}(K \cap u^\perp) = c \cdot \max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$*, where* $c \geq 0$*, then the same conclusion holds.*

The following theorem is a characterization of balls.

THEOREM 3. *For* $d \geq 2$*, let* $K \subset \mathbb{R}^d$ *be a convex body with* $0 \in \operatorname{int} K$*. If both* IK *and* CK *are centred balls, then* K *is a centred ball. More generally, even omitting the hypothesis* $0 \in \operatorname{int} K$*, if both* $V_{d-1}(K \cap u^\perp)$ *and* $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$ *are constant for* $u \in S^{d-1}$*, then the same conclusion holds.*

In Section 3, we will prove Proposition 1 that is the analogue of Theorems 2 and 3 for 1-dimensional sections and also includes a characterization of balls.

THEOREM 4. *For* $d \geq 3$*, let* $\mathfrak{K}^d$ *denote the set of convex bodies* $K \subset \mathbb{R}^d$*, endowed with the Hausdorff metric. Then* $\{K \in \mathfrak{K}^d \mid CK \subset \operatorname{int}(\Pi K)\}$ *is a dense open set in* $\mathfrak{K}^d$*.*

Related to the proof of Theorem 4 we will prove Proposition 2 (in Section 3), determining the minimal number of facets (vertices) of a convex polytope $P \subset \mathbb{R}^d$, $d \geq 3$ ($P \subset \mathbb{R}^3$) having no plane shadow boundary with respect to parallel illumination. For the question about the minimal number of vertices we give an estimate in $\mathbb{R}^d$, $d > 3$.

2. **Proofs of the Theorems.**

PROOF OF THEOREM 1. A. First we will show that, irrespective whether 0 is an interior, boundary or exterior point of K, any large 1-sphere S^1 of S^{d-1} (*i.e.*, the intersection of S^{d-1} with any linear 2-subspace) contains a point u satisfying $V_{d-1}(K \cap u^\perp) = \max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$. Letting $f_u(\lambda) = V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$, we have to prove that, for some $u \in S^1$, $f_u(\lambda)$ attains its maximum at $\lambda = 0$.

By the Brunn-Minkowski theorem $f_u(\lambda)^{1/(d-1)}$ is concave for $\lambda \in [-h_K(-u), h_K(u)] := [a, b]$, where h_K is the support function of K. Thus, $\big\{\lambda \in [a,b] \mid f_u(\lambda) = \max\{f_u(\mu) \mid \mu \in [a,b]\}\big\}$ is a non-empty closed interval, $I(u)$, say. Evidently, $I(-u) = -I(u)$. We consider the linear hull $\operatorname{lin} S^1$ of S^1, and the orthogonal projection $K \mid \operatorname{lin} S^1$ of K to $\operatorname{lin} S^1$. Now we distinguish three cases: 0 is an interior, boundary, or exterior point of $K \mid \operatorname{lin} S^1$, relative to $\operatorname{lin} S^1$.

First we investigate the case $0 \in \mathrm{int}(K \mid \mathrm{lin}\, S^1)$. Let $u \in S^1$. For $0 \in I(u)$ we are done. Thus, we may suppose that for all $u \in S^1$ either $I(u) \subset (0,\infty)$, or $I(u) \subset (-\infty, 0)$. Hence $S^1 = S^1_+ \cup S^1_-$, where $S^1_+ = \{u \in S^1 \mid I(u) \subset (0,\infty)\}$, $\quad S^1_- = \{u \in S^1 \mid I(u) \subset (-\infty, 0)\}$. Let u_0 be a common accumulation point of S^1_+ and S^1_-. If *e.g.* $I(u_0) \subset (0,\infty)$, then for some $\lambda_0 \in \big(0, h_K(u_0)\big)$ we have $V_{d-1}(K \cap u_0^\perp) < V_{d-1}\big(K \cap (u_0^\perp + \lambda_0 u_0)\big)$. Then, for $u \in S^1_-$ and sufficiently near to u_0 we have $V_{d-1}(K \cap u^\perp) < V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$, where $u^\perp + \lambda u$ is a hyperplane obtained from $u_0^\perp + \lambda_0 u_0$ by rotation about some point of $(\mathrm{int}\, K) \cap (u_0^\perp + \lambda_0 u_0)$. Thus $u \in S^1_+$, a contradiction.

Secondly we investigate the case $0 \notin (K \mid \mathrm{lin}\, S^1)$. Then the set $\{u \in S^1 \mid K \cap u^\perp \neq \emptyset\}$ is the union of two disjoint closed opposite arcs of S^1; one of these we denote by $\widehat{u_1 u_2}$. Then $V_{d-1}(K \cap u^\perp)$ depends continuously on u, for u varying in $\widehat{u_1 u_2}$. We may suppose that for all $u \in \widehat{u_1 u_2}$ we have $V_{d-1}(K \cap u^\perp) < \max_{\lambda \in \mathbf{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$, and therefore either $I(u) \subset (0,\infty)$ or $I(u) \subset (-\infty, 0)$. Using this for $u = u_1, u_2$ we easily see that $I(u_1) \subset (0,\infty)$ and $I(u_2) \subset (-\infty, 0)$, or conversely. Now choosing a common accumulation point of $\{u \in \widehat{u_1 u_2} \mid I(u) \subset (0,\infty)\}$ and $\{u \in \widehat{u_1 u_2} \mid I(u) \subset (-\infty, 0)\}$, we obtain a contradiction like in the previous case.

Third we investigate the case $0 \in \mathrm{bd}(K \mid \mathrm{lin}\, S^1)$. Then there is a supporting hyperplane $u_0^\perp$ of K with $u_0 \in S^1$. Consider a fixed closed arc $\widehat{u_0(-u_0)}$ of S^1. Let the arc $\widehat{u_1 u_2}$ be the closure of $\{u \in \widehat{u_0(-u_0)} \mid (\mathrm{int}\, K) \cap u^\perp \neq \emptyset\}$. Again we may suppose that $V_{d-1}(K \cap u^\perp) < \max_{\lambda \in \mathbf{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$ holds for all $u \in \widehat{u_1 u_2}$, in particular for $u = u_1, u_2$. Thus $V_{d-1}(K \cap u_i^\perp) < V_{d-1}\big(K \cap (u_i^\perp + \lambda_i u_i)\big)$, for some hyperplane $u_i^\perp + \lambda_i u_i$, intersecting $\mathrm{int}\, K$. The function $V_{d-1}(K \cap u^\perp)$, $\quad u \in \widehat{u_1 u_2}$, is continuous for $u \in \mathrm{relint}\, \widehat{u_1 u_2}$, and is upper semicontinuous at u_1 and u_2. Now we choose some $v_1, v_2 \in \mathrm{relint}\, \widehat{u_1 u_2}$, close to u_1 and u_2, respectively. Then $V_{d-1}(K \cap v_i^\perp) < V_{d-1}\big(K \cap (v_i^\perp + \mu_i v_i)\big)$, where $v_i^\perp + \mu_i v_i$ is a hyperplane obtained from $u_i^\perp + \lambda_i u_i$ by rotation about some point of $(\mathrm{int}\, K) \cap (u_i^\perp + \lambda_i u_i)$. Then we easily see that $I(v_1) \subset (0,\infty)$ and $I(v_2) \subset (-\infty, 0)$, or conversely. Now considering the closed subarc $\widehat{v_1 v_2}$ of $\widehat{u_1 u_2}$ and choosing a common accumulation point of $\{u \in \widehat{v_1 v_2} \mid I(u) \subset (0,\infty)\}$ and $\{u \in \widehat{v_1 v_2} \mid I(u) \subset (-\infty, 0)\}$, we obtain a contradiction like above.

B. Now we show that, for $d \geq 3$ and irrespective whether 0 is an interior, boundary or exterior point of K, the set

$$B := \Big\{u \in S^{d-1} \mid V_{d-1}(K \cap u^\perp) = \max_{\lambda \in \mathbf{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)\Big\}$$

is not contained in any set A like in the theorem. Suppose the contrary. By part A of this proof and by central symmetry, B intersects each large 1-sphere S^1 of S^{d-1} in at least two points. Hence this holds for A as well.

Now we recall a special case of an integral geometric kinematic formula on S^{d-1} from [Fe], Theorem 3.2.48. Let $0 \leq k \leq d-1$ be an integer, and let $A \subset S^{d-1}$ be H^k-measurable with $H^k(A) < \infty$, and let A be (H^k, k)-rectifiable. Then, letting S^{d-1-k} be a fixed large $(d-1-k)$-subsphere of S^{d-1}, we have

$$\mathrm{int}_{O(d)} H^0\big(A \cap g(S^{d-1-k})\big)\, d\Theta_n(g) = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{d-k}{2})}{2\pi^{(d+1)/2}} H^k(A) \cdot H^{d-1-k}(S^{d-1-k}).$$

Here $H^0(X)$ is the 0-dimensional Hausdorff measure of X, that equals the cardinality of X if it is finite, and equals ∞ if the cardinality of X is infinite. Moreover, Θ_d is the invariant (Haar) measure on the orthogonal group $O(d)$ of $\mathbb{R}^d$, with $\Theta_d\big(O(d)\big) = 1$. Letting $k = d-2$, and using $H^0\big(A \cap g(S^1)\big) \geq 2$, both sides of the above equality are at least 2, and by a small calculation this yields $H^{d-2}(A) \geq H^{d-2}(S^{d-2})$. This contradicts our indirect assumption, and thus shows the statement of the theorem concerning A.

For K a ball with centre $a \neq 0$ we have $B = \{u \in S^{d-1} \mid \langle u, a\rangle = 0\}$, thus B is a large S^{d-2} of S^{d-1}, and therefore the estimate of $H^{d-2}(A)$ is sharp. ■

PROOF OF THEOREM 2. For each $u \in S^{d-1}$, let there hold the relation $V_{d-1}(K \cap u^\perp) = c \cdot \max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$. Then $c \leq 1$, and $c > 0$. Hence $V_{d-1}(K \cap u^\perp) > 0$ for each $u \in S^{d-1}$, which implies $0 \in K$.

For $0 \in \operatorname{bd} K$, the body K has a supporting hyperplane $u_0^\perp$ at 0, and $V_{d-1}(K \cap u_0^\perp) > 0$ implies that $K \cap u_0^\perp$ is a $(d-1)$-face of K. Now let us move u on a large 1-sphere S^1 of S^{d-1}, containing u_0. Then for $u \longrightarrow u_0$ the sum of the two one-sided limits of $V_{d-1}(K \cap u^\perp)$ is $V_{d-1}(K \cap u_0^\perp)$; hence u_0 is a discontinuity point of $V_{d-1}(K \cap u^\perp)$. Since $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$ is a continuous function of u, this is a contradiction, implying $0 \in \operatorname{int} K$. Therefore from $IK = c \cdot CK$, $c \in [0,1]$, we have by Theorem 1 that $c = 1$ and $IK = CK$. In [MMÓ] it was shown that for $0 \in \operatorname{int} K$ the coincidence of IK and CK implies that K is centred. This implies the result. ■

PROOF OF THEOREM 3. Let both $V_{d-1}(K \cap u^\perp)$ and $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^\perp + \lambda u)\big)$ be constant for $u \in S^{d-1}$. Then by Theorem 2 we have that K is centred. Recall now the section theorem of Funk: a centred convex body K is uniquely determined by IK, *cf.* [Fu], p. 287, [Pe 52], Corollary 1.31, [LP], p. 1144, [Pe 61], Theorem 4.2, and [Fa], Theorem 2; see also [Ga], Corollary 7.2.7. Applying this to K and using that IK is a centred ball, we obtain that K is a centred ball. ■

To prove Theorem 4, we need some preparations. First we recall three definitions, for which we also refer to [BES], [Sh], [GH], and [Pe 83].

DEFINITION 1. Let $K \subset \mathbb{R}^d$ be a convex body, s a direction, and $x \in \mathbb{R}^d \setminus K$ a point. The *shadow boundary* $\operatorname{Sb}_s K$ ($\operatorname{Sb}_x K$) of K with respect to parallel illumination from the direction s (central illumination from the point x) is the intersection of K and the union of all its supporting lines parallel to s (passing through x). ■

DEFINITION 2. A shadow boundary $\operatorname{Sb}_s K$ ($\operatorname{Sb}_x K$) is said to be a *plane shadow boundary* if it lies in a hyperplane. ■

DEFINITION 3. A shadow boundary $\operatorname{Sb}_s K$ ($\operatorname{Sb}_x K$) is called a *generalized plane shadow boundary* provided there exists a hyperplane H not parallel to s (not containing x and intersecting each ray xy, $y \in K$, transversally) such that $H \cap \operatorname{bd}\big(K + l(s)\big) \subset \operatorname{Sb}_s K$, where $l(s)$ is a line of direction s passing through 0 ($H \cap \operatorname{bd} C(x, K) \subset \operatorname{Sb}_x K$, where $C(x, K)$ is the minimal infinite cone with apex x that contains K). ■

One easily sees that plane shadow boundaries can be equivalently defined by replacing the inclusion signs in Definition 3 by equalities.

Now we recall a result from [Pe 52], p. 60, and [Ma 89], *cf.* the Theorem in the latter, in the equivalent reformulation mentioned in our Section 1, fifth paragraph. For $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body, and let $u \in S^{d-1}$. Then the radial function of CK at u is at most that of ΠK at u, with equality if and only if the following condition is satisfied: There is a direction s not orthogonal to u and some $\lambda \in \mathbb{R}$ such that for the shadow boundary $\mathrm{Sb}_s K$ of K with respect to parallel illumination from the direction s we have $(u^\perp + \lambda u) \cap \mathrm{bd}\,\big(K + l(s)\big) \subset \mathrm{Sb}_s K$ (again, $l(s)$ is a line of direction s passing through 0).

An easy consequence of this result is the following

LEMMA 1. *For $d \geq 3$, a convex body $K \subset \mathbb{R}^d$ satisfies $CK \subset \mathrm{int}(\Pi K)$ if and only if it does not have a generalized plane shadow boundary* $\mathrm{Sb}_s K$ *with respect to parallel illumination from any direction s.*

PROOF. By the above result, the following four statements are equivalent.

(1) $CK \not\subset \mathrm{int}(\Pi K)$.

(2) There is some $u \in S^{d-1}$ such that the radial functions of CK and ΠK are equal at u.

(3) There is some $u \in S^{d-1}$ and some direction s (where u and s are not orthogonal) such that there exists some $\lambda \in \mathbb{R}$ with $(u^\perp + \lambda u) \cap \mathrm{bd}\big(K + l(s)\big) \subset \mathrm{Sb}_s K$.

(4) There is some direction s such that $\mathrm{Sb}_s K$ is a generalized plane shadow boundary. ■

Now we turn to the space $\mathfrak{K}^d$ of all convex bodies $K \subset \mathbb{R}^d$, endowed with the Hausdorff metric. This is a locally compact metric space, hence a Baire space, *i.e.*, a countable union of nowhere dense sets (otherwise: a set of first category) cannot have an interior point. One says that *most convex bodies in $\mathbb{R}^d$ have some property* if the set of convex bodies in $\mathbb{R}^d$ not having this property is of first category in $\mathfrak{K}^d$. For this concept and results concerning it we refer to the surveys of Gruber [Gr 85], [Gr 93] and Zamfirescu [Za 91a].

Now we recall two results on most convex bodies. Namely, most convex bodies $K \subset \mathbb{R}^d$ are strictly convex and smooth, by [Kl], Theorems 2.2 and 2.3 (which actually prove a version of this for Banach spaces) and [Gr 77], Satz 1; *cf.* also [Gr 85], pp. 163-164, [Za 91a], Theorem 1, and [Gr 93], Theorem 5. Also most convex bodies $K \subset \mathbb{R}^d$ have no plane shadow boundaries $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$, for s any direction and $x \in \mathbb{R}^d \setminus K$ any point, by [Za 91b], Theorem 1; *cf.* also [Za 91a], Theorem 12.

An easy consequence of these results is the following lemma, whose proof is probably quite similar to the proof of the case $\dim L = d - 1$ of Theorem 31, announced without proof in [Gr 93].

LEMMA 2. *Most convex bodies $K \subset \mathbb{R}^d$ do not have any generalized plane shadow boundaries* $\mathrm{Sb}_s K$ *or* $\mathrm{Sb}_x K$, *for any direction s and any point $x \in \mathbb{R}^d \setminus K$.*

PROOF. By the above recalled two results, most convex bodies $K \subset \mathbb{R}^d$ both are strictly convex and have no plane shadow boundaries $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$, with s, x as above. However, a convex body $K \subset \mathbb{R}^d$ with these two properties cannot have a generalized plane shadow boundary $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$ either. This is easily seen by using the equivalent definition of plane shadow boundary after Definition 3, and that K contains exactly one point of any generator of $K + l(s)$, or of $C(x, K)$, respectively. ■

PROOF OF THEOREM 4. Let $\mathfrak{K}' = \{K \in \mathfrak{K}^d \mid CK \subset \operatorname{int}(\Pi K)\}$. Both CK and ΠK are monotonous functions of K with respect to inclusion, of homogeneity $d-1$, and are translation invariant, that readily implies their continuity. (For ΠK we use the topology of Hausdorff-distance, for CK that of uniform convergence of the radial functions.) Since also

$$\mathfrak{K}' = \{K \in \mathfrak{K}^d \mid \exists \varepsilon > 0, \quad CK \subset (1-\varepsilon)\Pi K\},$$

we have by the above continuity properties that $\mathfrak{K}'$ is open.

By Lemma 1 we have $CK \subset \operatorname{int}(\Pi K)$ provided K does not have any generalized plane shadow boundary $\mathrm{Sb}_s K$, for any direction s. Hence by Lemma 2 most convex bodies $K \subset \mathbb{R}^d$ satisfy $CK \subset \operatorname{int}(\Pi K)$. In particular, this is satisfied for K in a dense set of $\mathfrak{K}^d$. Hence the open set $\mathfrak{K}' \subset \mathfrak{K}^d$ also is dense, and the proof is finished. ■

3. **Proofs of the propositions.** 1. Replacing $(d-1)$-dimensional sections and projections by 1-dimensional sections and projections, one gets the following natural comparison. Let $d \geq 2$ and let $K \subset \mathbb{R}^d$ be a convex body. For $0 \in \operatorname{int} K$ let $\tilde{\Delta} K$ be the star-body whose radial function is given by $V_1(K \cap l_u)/2$, where V_1 is 1-dimensional Lebesgue measure and l_u is the linear 1-subspace with $u \in S^{d-1}$ as its direction vector. The body $\tilde{\Delta} K$ is said to be the *chordal symmetral* of K, see [Ga], Definition 5.1.3, and it is clear that $2\tilde{\Delta} K$ is the analogue (for 1-dimensional sections) of the intersection body IK. On the other hand, the *difference body* $DK = K + (-K)$ (see, *e.g.*, [Ga], Section 3.2) is the analogue of the cross-section body CK, and, at the same time, of the projection body ΠK, depending on the use of the radial function and of the support function of DK, respectively. We have evidently $2\tilde{\Delta} K \subset DK$ for $0 \in \operatorname{int} K$.

It turns out that the statements analogous to our first three theorems hold true, except the statement about A in Theorem 1 (here, for any non-centred ball, u is unique up to sign). Namely, Hammer (see [Ha 63], Theorem 3.1, and [Ha 51], Theorem 1, and also [PC], proof of Theorem 4) has proved that each $x \in \mathbb{R}^d$ belongs to a diametrical chord of K, which is an analogue of our Theorem 1, without the statement about A. (A chord of K is said to be *diametrical* if it has maximal length among all chords of K parallel to it.) This shows $[\operatorname{bd}(2\tilde{\Delta} K)] \cap \operatorname{bd}(DK) \neq \emptyset$, and therefore we obtain the following analogue of Theorems 2 and 3, also containing a characterization of balls.

PROPOSITION 1. *For $d \geq 2$, let $K \subset \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} K$. If for some constant $c > 0$ we have $2\tilde{\Delta} K = c \cdot DK$, then K is centred. In particular, if $2\tilde{\Delta} K$ and DK are centred balls, then K is a centred ball. More generally, even omitting the hypothesis $0 \in \operatorname{int} K$, if $V_1(K \cap l_u)$ is proportional to the radial function of DK, then K is centred. In particular, if these two functions are constant, then K is a centred ball.*

PROOF. We proceed analogously as with respect to Theorems 2 and 3. So we only indicate the differences in the proof. If $V_1(K \cap l_u)$ is positive and continuous for $u \in S^{d-1}$, then $0 \in \operatorname{int} K$. Namely, if $0 \in \operatorname{bd} K$, u_0 is an outer normal of K at 0, and $u_1 \in u_0^{\perp} \cap S^{d-1}$, then for $u \longrightarrow u_1$ (u varying in the large 1-sphere S^1 of S^{d-1} containing u_0 and u_1) the sum of the two one-sided limits of $V_1(K \cap l_u)$ is $V_1(K \cap l_{u_1})$. Then by $[\operatorname{bd}(2\tilde{\Delta}K)] \cap \operatorname{bd}(DK) \neq \emptyset$ the proportionality of $V_1(K \cap l_u)$ and the radial function of DK implies centredness of K, *cf.* [Ha 54], Theorem 1, and also [PC], Theorem 4. The particular cases follow immediately. ∎

It is obvious that, regarding 1-dimensional sections and projections, there is no analogue of Theorem 4.

2. Let $d \geq 3$. By Theorem 4 and denseness of convex polytopes in $\mathfrak{K}^d$ we see that there are convex polytopes $P \subset \mathbb{R}^d$ such that $CP \subset \operatorname{int}(\Pi P)$. (We always will consider non-degenerate convex polytopes, *i.e.*, $\operatorname{int} P \neq \emptyset$.) Thus, by Lemma 1 there are convex polytopes $P \subset \mathbb{R}^d$ that do not have a generalized plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination from any direction s. We ask for the minimal complexity (for example, number of vertices, or facets) of a convex polytope $P \subset \mathbb{R}^d$ having no generalized plane shadow boundary, or no plane shadow boundary $\mathrm{Sb}_s P$, with respect to parallel illumination from any direction s.

Some of these questions for plane shadow boundaries are answered by

PROPOSITION 2. *Let $d \geq 3$ be an integer. The minimal number of facets (vertices) of a non-degenerate convex polytope $P \subset \mathbb{R}^d$, having no plane shadow boundary* $\mathrm{Sb}_s P$ *with respect to parallel illumination from any direction s, is $d+2$ (at most $2d$, with equality for $d = 3$).*

PROOF. A. First we deal with the number of facets. Since a simplex has plane shadow boundaries $\mathrm{Sb}_s P$, it suffices to give an example of a (non-degenerate) convex polytope P with $d+2$ facets having no plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination. Such a polytope will be a prism P over a $(d-1)$-simplex, whose bases and lateral facets will be denoted by B_1, B_2 and $L_1, \ldots, L_d$, respectively. Namely, if the direction s is parallel to the generator or to a basis of P, then $\mathrm{Sb}_s P$ contains some facets of P, and hence $\mathrm{Sb}_s P$ is not planar. Otherwise, a unit vector of direction s is of the form $x + y$, $x \neq 0$ parallel to a base, $y \neq 0$ parallel to the generator of P. Like above, we may suppose that $\mathrm{Sb}_s P$ does not contain any facet of P. Then the illuminated facets are one of the bases (B_1, say) and a non-empty proper subset ($\{L_1, \ldots, L_k\}$, say) of the lateral facets, where L_i is a prism over a facet l_i of B_1. Then $\mathrm{Sb}_s P \supset (B_1 \cap L_{k+1}) \cup \cdots \cup (B_1 \cap L_d) \cup (B_2 \cap L_1) \cup \cdots \cup (B_2 \cap L_k)$, that is the union of $d-k$ facets of B_1 and k facets of B_2. Since $\max\{d-k, k\} \geq 2$ and $\min\{d-k, k\} \geq 1$, the shadow boundary $\mathrm{Sb}_s P$ contains all vertices of B_1 and all but one vertices of B_2, or conversely. Hence $\mathrm{Sb}_s P$ is not planar.

B. Secondly we deal with the number of vertices and begin by giving an example of a convex polytope $P \subset \mathbb{R}^d$, $d \geq 3$, with $2d$ vertices having no plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination.

First, let P_0 denote the cross-polytope with vertices $\pm e_i$ (e_i is the i-th basic unit vector). We claim that its only plane shadow boundaries $\mathrm{Sb}_s P_0$ with respect to parallel illumination are the intersections of $\operatorname{bd} P_0$ with the d coordinate hyperplanes. Let $\mathrm{Sb}_s P_0$ be a plane shadow boundary of P_0 with respect to parallel illumination from some direction s. (The following considerations in this paragraph will be valid for any convex d-polytope rather than P_0.) Choose a new orthogonal coordinate system $y_1, \ldots, y_d$, where y_d has direction s. Let π denote the projection of $\mathbb{R}^d$ to the $(y_1, \ldots, y_{d-1})$-hyperplane, and let $P_0' = \pi(P_0)$. Then P_0' is a convex $(d-1)$-polytope, and $\mathrm{Sb}_s P_0$ is the graph of the restriction of an affine function $f\colon \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ to $\mathrm{relbd}\, P_0'$ ($\mathbb{R}^{d-1}$ denoting the $(y_1, \ldots, y_{d-1})$-hyperplane, and $\mathbb{R}$ denoting the y_d-axis). Let us denote the facets of P_0' by F_i'. Then $F_i = P_0 \cap \pi^{-1}(F_i') \subset \mathrm{Sb}_s P_0$ are non-degenerate affine images of F_i', and by $F_i' = P_0' \cap \mathrm{aff}\, F_i'$ also the sets $F_i = P_0 \cap \pi^{-1}(\mathrm{aff}\, F_i')$ are faces of P_0, of dimension $d-2$. Conversely, each $(d-2)$-face F of P_0, contained in $\mathrm{Sb}_s P_0$, also is a $(d-2)$-face of $\mathrm{conv}\big(\mathrm{Sb}_s(P_0)\big)$, which is affinely equivalent under the map π to P_0'. Thus, each such F is of the above form F_i. Each $(d-3)$-face of P_0' is a common face of exactly two facets F_i' of P_0'. Therefore, for each $(d-2)$-face of P_0, of the form F_i, and each $(d-3)$-face G of F_i there is another $(d-2)$-face of P_0, of the form F_{i_1}, also having G as a $(d-3)$-face.

From now on we will exploit the combinatorial and affine structure of P_0. The $(d-2)$-faces ($(d-3)$-faces) of P_0 are given by equations $\sum_{i \neq k} \varepsilon_i x_i = 1, x_k = 0$ ($\sum_{i \neq k,l} \varepsilon_i x_i = 1, x_k = x_l = 0$, where $k \neq l$, here and later) with $\varepsilon_i \in \{-1, 1\}$. (Writing such sums we always will assume that the summands are non-negative, but will not repeat it in the following.) If two different $(d-2)$-faces F_1^{d-2}, F_2^{d-2} have a common $(d-3)$-face F^{d-3}, and if F^{d-3} is given by $\sum_{i \neq k,l} \varepsilon_i x_i = 1, \quad x_k = x_l = 0$, then each of the faces F_1^{d-2}, F_2^{d-2} is of the form

$$\Big(\sum_{i \neq k,l} \varepsilon_i x_i\Big) \pm x_k = 1, \quad x_l = 0, \text{ or} \tag{1}$$

$$\Big(\sum_{i \neq k,l} \varepsilon_i x_i\Big) \pm x_l = 1, \quad x_k = 0. \tag{2}$$

Thus, the sets F_1^{d-2}, F_2^{d-2} are two of four $(d-2)$-faces. They lie in the same facet of P_0 if and only if one of them is of the form (1) and the other is of the form (2). In this case, if moreover F_1^{d-2}, F_2^{d-2} are $(d-2)$-faces of P_0 contained in $\mathrm{Sb}_s P_0$, then, by planarity of $\mathrm{Sb}_s P_0$, we see that $\mathrm{Sb}_s P_0$ is a subset of this facet of P_0, F^{d-1}, say. Since $\mathrm{Sb}_s P_0$ also is a union of $(d-2)$-faces of P_0, of the above form F_i, and is affinely equivalent to $\mathrm{relbd}\, P_0'$, thus is homeomorphic to S^{d-2}, then $\mathrm{Sb}_s P_0 = \mathrm{relbd}\, F^{d-1}$. Therefore $P_0' = \mathrm{conv}\, \mathrm{relbd}\, P_0') = \mathrm{conv}\big(\pi(\mathrm{Sb}_s P_0)\big) = \pi\big(\mathrm{conv}(\mathrm{Sb}_s P_0)\big) = \pi(F^{d-1})$ is a $(d-1)$-simplex. For the opposite parallel facet $-F^{d-1}$ of P_0 we have $P_0' \supset \pi(-F^{d-1})$, and therefore $\pi(F^{d-1}) \supset \pi(-F^{d-1})$, a contradiction.

Thus, if a $(d-2)$-face F_1^{d-2} of P_0 is contained in $\mathrm{Sb}_s P_0$, then at any of its $(d-3)$-faces F^{d-3} it is neighbourly to a $(d-2)$-face F_2^{d-2} of P_0, contained in $\mathrm{Sb}_s P_0$, both F_1^{d-2}, F_2^{d-2} being of the form (1), or both being of the form (2) above. If F_1^{d-2} is given by $\sum_{i \neq k} \varepsilon_i x_i = 1, \quad x_k = 0$, then, for fixed F^{d-3}, the face F_2^{d-2} is unique and has the form

$$\Big(\sum_{i \neq k,l} \varepsilon_i x_i\Big) - \varepsilon_l x_l = 1, \quad x_k = 0,$$

where $l \in \{1, \ldots, d\} \setminus \{k\}$ can be arbitrary, for arbitrary F^{d-3}. Repeating this consideration at most $d-1$ times, we see that each $(d-2)$-face of P_0 of the form $\sum_{i \neq k}(\pm x_i) = 1, \quad x_k = 0$, is contained in $\mathrm{Sb}_s P_0$. Hence $\mathrm{Sb}_s P_0$ contains, and therefore is equal to, the intersection of $\mathrm{bd} P_0$ with the k-th coordinate hyperplane.

Now let us perturb the vertices of P_0 sufficiently little, so that the combinatorial type of their convex hull P remains the same as that of P_0, and no $d+1$ vertices lie in a hyperplane. Supposing that P has a plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination, we can repeat, for sufficiently small perturbations, the above considerations. In the first case observe that P_0' contains and is contained in some centred balls of fixed radii, and then we obtain a contradiction like above. In the second case the above existing d plane shadow boundaries (each containing $2d-2 \geq d+1$ vertices of P) will become non-planar, a contradiction once more.

Still we have to prove that convex polyhedra P in $\mathbb{R}^3$ with at most 5 vertices have plane shadow boundaries $\mathrm{Sb}_s P$ with respect to parallel illumination. For a tetrahedron this is evident. If P has 5 vertices, then it is either a double triangular pyramid, or a quadrangular pyramid. Then $\mathrm{Sb}_s P$ is planar for s parallel to the diagonal of P, or to the segment joining the apex of P with a relative interior point of its base, respectively. ∎

4. **Concluding Remarks.** Remark 1. For $d = 2$, the convex body $K \subset \mathbb{R}^2$ has a cross-section body CK which is the convex body obtained from $K + (-K)$ by rotating it through $\frac{\pi}{2}$ about the origin, *cf.* Section 1. Thus, CK is a centred ball if and only if K is of constant width. For $d \geq 3$, no example of a convex body $K \subset \mathbb{R}^d$, different from a ball and having as CK a centred ball, is known. However, CK does not seem to give full information about K, since the informations obtained by u and $-u$ are the same, like also in the case of the difference body $K + (-K)$, or IK (this for $0 \in \mathrm{int} K$), or ΠK. (Roughly: we have 'half the information that is needed'.) In these latter cases, there is given complete information on the even parts of the support function, of the $(d-1)$-st power of the radial function (*cf.* [Ga], Theorem 8.1.3), and of the surface area measure (see [Ga], Theorem 3.3.2), respectively; however, no information is given about the odd parts. Therefore we can well imagine that there exists a convex body $K \subset \mathbb{R}^d$, $d \geq 3$, with CK a centred ball but not having itself a center of symmetry. For the history of this problem (the Bonnesen-Klee problem, also posed in [Ga], Problem 8.12) we refer to [Ga], Note 8.6.

Remark 2. Theorem 4 shows that, for $d \geq 3$, there is a number $c(d) < 1$ such that for some convex body $K \subset \mathbb{R}^d$ we have $CK \subset c(d) \cdot \Pi K$. What is the minimal number with this property? Surely it is positive, since by [MM], Part I, there is a constant $c'(d)$ such that for all convex bodies $K \subset \mathbb{R}^d$ we have $\Pi K \subset c'(d) \cdot CK$.

Remark 3. Let $1 < k < d-1$, and let $K \subset \mathbb{R}^d$ be a convex body. We ask if there is a linear k-subspace L_k such that

$$V_k(K \cap L_k) = \max\{V_k\big(K \cap (L_k + x)\big) \mid x \in \mathbb{R}^d\},$$

where V_k denotes k-dimensional Lebesgue measure. Moreover, if we consider the set of linear k-subspaces L_k of $\mathbb{R}^d$ with some natural metric, then is the set of L_k's with this maximum property large enough? For example, cannot it be included in a compact C^1 $(k-1)(d-k)$-submanifold, of measure less than in the case of a non-centred ball. (For K a non-centred ball, the set of these L_k's is homeomorphic to the set of linear $(k-1)$-subspaces of $\mathbb{R}^{d-1}$, which is a $(k-1)(d-k)$-manifold.) Probably some algebraic topology would be necessary to answer this question.

REMARK. (Added 11 December 1995): Meanwhile the authors proved the existence of L_k in Remark 3, and moreover that the set of all such L_k's cannot be included in a compact C^∞ $(k-1)(d-k)$-submanifold, of measure less than some positive constant depending on d and k, when the set of linear k-subspaces of $\mathbb{R}^d$ is given in a natural $O(d)$-invariant Riemann-metric, and measure is taken relative to this. Thus we have the analogues of Theorems 2 and 3 as well, for $1 < k < d-1$. For Theorem 4 there is a natural generalisation of the relation $CK \subset \operatorname{int}(\Pi K)$ for k-dimensional sections and projections: for K in a dense open set of $\mathfrak{R}^d$ there exists $c(K) < 1$ such that $\max\{V_k([K \cap (L_k+x)] \mid L_k')\, V_k(K \mid L_k') \mid L_k, L_k' \subset \mathbb{R}^d$ are linear k-subspaces, $x \in \mathbb{R}^d\} \leq c(K)$. (The set $A \mid L_k'$ is the orthogonal projection of the set A to L_k'.) This is connected with problems of illumination from a $(d-k-1)$-dimensional projective subspace of the infinite hyperplane.

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